Practice exam paper

1 **a** The directrix lies at $x = 6$, therefore $\pm\frac{p}{e} = 6$

The focus lies at (3, 0), therefore $\pm pe = 3$

$pe \times \frac{p}{e} = 3 \times 6$

$p^2 = 18$

$p = 3\sqrt{2}$ as required

b (0, 3) lies on the ellipse, therefore:

$0 + \frac{9}{q^2} = 1 \Rightarrow q = \pm 3$

2 **a** $1 + 2\sinh^2 x = 1 + 2\left(\frac{e^x - e^{-x}}{2}\right)^2$

$= 1 + 2\left(\frac{e^{2x} - 2 + e^{-2x}}{4}\right)$

$= 1 + \frac{1}{2}e^{2x} - 1 + \frac{1}{2}e^{-2x}$

$= \frac{1}{2}\left(e^{2x} + e^{-2x}\right)$

$= \cosh 2x$ as required

2 b $\cosh 2x - 2\sinh x - 16 = 0$

$1 + \sinh^2 x - 2\sinh x - 16 = 0$

$\sinh^2 x - 2\sinh x - 15 = 0$

$(\sinh x - 5)(\sinh x + 3) = 0$

$\sinh x = -3$ or $\sinh x = 5$

When $\sinh x = -3$

$\left(\frac{e^x - e^{-x}}{2}\right) = -3$

$e^x - e^{-x} = -6$

$e^{2x} - 1 = -6e^x$

$e^{2x} + 6e^x - 1 = 0$

Let $y = e^x$

$y^2 + 6y - 1 = 0$

$(y+3)^2 - 10 = 0$

$y + 3 = \pm\sqrt{10}$

$y = -3 \pm \sqrt{10}$

Since $y = e^x$

$e^x = -3 \pm \sqrt{10}$

$x = \ln\left(-3 + \sqrt{10}\right)$

When $\sinh x = 5$

$\left(\frac{e^x - e^{-x}}{2}\right) = 5$

$e^x - e^{-x} = 10$

$e^{2x} - 1 = 10e^x$

$e^{2x} - 10e^x - 1 = 0$

Let $y = e^x$

$y^2 - 10y - 1 = 0$

$(y-5)^2 - 26 = 0$

$y - 5 = \pm\sqrt{26}$

$y = 5 \pm \sqrt{26}$

Since $y = e^x$

$e^x = 5 \pm \sqrt{26}$

$x = \ln\left(5 + \sqrt{26}\right)$

3 a $\frac{d}{dx}\left[\left(9 - x^2\right)^{\frac{1}{2}}\right] = -2x \cdot \frac{1}{2}\left(9 - x^2\right)^{-\frac{1}{2}}$

$= -\frac{x}{\left(9 - x^2\right)^{\frac{1}{2}}}$

$\int \frac{x}{\sqrt{9 - x^2}}\,dx = -\left(9 - x^2\right)^{\frac{1}{2}} + c$

3 b $I_n = \int_0^3 \frac{x^n}{\sqrt{9-x^2}}\,dx$

$= \int_0^3 \frac{x^{n-1}\cdot x}{\sqrt{9-x^2}}\,dx$

Let $u = x^{n-1} \Rightarrow \frac{du}{dx} = (n-1)x^{n-2}$

Let $\frac{dv}{dx} = \frac{x}{\sqrt{9-x^2}} \Rightarrow v = -(9-x^2)^{\frac{1}{2}}$

$I_n = \left[-x^{n-1}(9-x^2)^{\frac{1}{2}}\right]_0^3 + (n-1)\int_0^3 x^{n-2}(9-x^2)^{\frac{1}{2}}\,dx$

$\left[-x^{n-1}(9-x^2)^{\frac{1}{2}}\right]_0^3 = 0$

Therefore:

$I_n = (n-1)\int_0^3 x^{n-2}(9-x^2)^{\frac{1}{2}}\,dx$

$= (n-1)\int_0^3 \frac{x^{n-2}(9-x^2)}{(9-x^2)^{\frac{1}{2}}}\,dx$

$= (n-1)\int_0^3 \frac{9x^{n-2}-x^n}{(9-x^2)^{\frac{1}{2}}}\,dx$

$= 9(n-1)\int_0^3 \frac{x^{n-2}}{(9-x^2)^{\frac{1}{2}}}\,dx - (n-1)\int_0^3 \frac{x^n}{(9-x^2)^{\frac{1}{2}}}\,dx$

$= 9(n-1)I_{n-2} - (n-1)I_n$

$I_n + (n-1)I_n = 9(n-1)I_{n-2}$

$nI_n = 9(n-1)I_{n-2}$

$I_n = \frac{9(n-1)}{n}I_{n-2}$

3 c $I_4 = \frac{9(4-1)}{4} I_2$

$= \frac{27}{4} I_2$

$= \frac{27}{4}\left(\frac{9}{2} I_0\right)$

$= \frac{243}{8} I_0$

$I_0 = \int_0^3 \frac{x^0}{\sqrt{9-x^2}} \, \mathrm{d}x$

$= \int_0^3 \frac{1}{\sqrt{9-x^2}} \, \mathrm{d}x$

$= \left[\arcsin\left(\frac{x}{3}\right)\right]_0^3$

$= \arcsin(1) - \arcsin(0)$

$= \frac{\pi}{2}$

Hence:

$I_4 = \frac{243}{8} \times \frac{\pi}{2}$

$= \frac{243}{16}\pi$

4 a Let $y = \text{arsinh}\, 2x$

$\sinh y = 2x$

$\cosh y \frac{\mathrm{d}y}{\mathrm{d}x} = 2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\cosh y}$

$= \frac{2}{\sqrt{1+\sinh^2 y}}$

$= \frac{2}{\sqrt{1+4x^2}}$

$\frac{\mathrm{d}}{\mathrm{d}x}(x \text{arsinh}\, 2x) = \text{arsinh}\, 2x \frac{\mathrm{d}}{\mathrm{d}x}(x) + x \frac{\mathrm{d}}{\mathrm{d}x}(\text{arsinh}\, 2x)$

$= \text{arsinh}\, 2x + \frac{2x}{\sqrt{1+4x^2}}$

4 b $\int_0^{\sqrt{2}} x\text{arsinh}2x\,\text{d}x = \left[\text{arsinh}2x + \frac{2x}{\sqrt{1+4x^2}}\right]_0^{\sqrt{2}}$

$$= \left[\ln\left(2x+\sqrt{4x^2+1}\right)+\frac{2x}{\sqrt{1+4x^2}}\right]_0^{\sqrt{2}}$$

$$= \left[\ln\left(2\sqrt{2}+3\right)+\frac{2\sqrt{2}}{3}\right]-\left[\ln(1)+0\right]$$

$$= \ln\left(2\sqrt{2}+3\right)+\frac{2\sqrt{2}}{3}$$

5 $x=\cosh 2\theta \Rightarrow \frac{\text{d}x}{\text{d}\theta}=2\sinh 2\theta \Rightarrow \frac{\text{d}x}{\text{d}\theta}=4\sinh\theta\cosh\theta$

$y=4\sinh\theta \Rightarrow \frac{\text{d}y}{\text{d}\theta}=4\cosh\theta$

Using $S=2\pi\int_{\theta_1}^{\theta_2} y\sqrt{\left(\frac{\text{d}x}{\text{d}\theta}\right)^2+\left(\frac{\text{d}y}{\text{d}\theta}\right)^2}\,\text{d}\theta$ gives:

$$S=2\pi\int_0^1 4\sinh\theta\sqrt{(4\sinh\theta\cosh\theta)^2+(4\cosh\theta)^2}\,\text{d}\theta$$

$$=8\pi\int_0^1 \sinh\theta\sqrt{16\sinh^2\theta\cosh^2\theta+16\cosh^2\theta}\,\text{d}\theta$$

$$=8\pi\int_0^1 \sinh\theta\sqrt{16\cosh^2\theta\left(\sinh^2\theta+1\right)}\,\text{d}\theta$$

$$=32\pi\int_0^1 \sinh\theta\cosh\theta\sqrt{\sinh^2\theta+1}\,\text{d}\theta$$

$$=32\pi\int_0^1 \sinh\theta\cosh^2\theta\,\text{d}\theta$$

$$=32\pi\left[\frac{1}{3}\cosh^3\theta\right]_0^1$$

$$=\frac{32}{3}\pi\left(\cosh^3(1)-1\right)$$

6 a $A\begin{pmatrix}1\\-2\\-2\end{pmatrix}$, $B\begin{pmatrix}7\\0\\-3\end{pmatrix}$ and $C\begin{pmatrix}4\\4\\0\end{pmatrix}$

$$\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}$$
$$=\begin{pmatrix}4\\4\\0\end{pmatrix}-\begin{pmatrix}1\\-2\\-2\end{pmatrix}$$
$$=\begin{pmatrix}3\\6\\2\end{pmatrix}$$
$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}$$
$$=\begin{pmatrix}4\\4\\0\end{pmatrix}-\begin{pmatrix}7\\0\\-3\end{pmatrix}$$
$$=\begin{pmatrix}-3\\4\\3\end{pmatrix}$$
$$\overrightarrow{AC}\times\overrightarrow{BC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\3 & 6 & 2\\-3 & 4 & 3\end{vmatrix}$$
$$=\mathbf{i}(18-8)-\mathbf{j}(9+6)+\mathbf{k}(12+18)$$
$$=10\mathbf{i}-15\mathbf{j}+30\mathbf{k}$$

b Area ABC $=\frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{BC}\right|$
$$=\frac{1}{2}\left|10\mathbf{i}-15\mathbf{j}+30\mathbf{k}\right|$$
$$=\frac{1}{2}\sqrt{10^2+(-15)^2+30^2}$$
$$=\frac{35}{2}$$

6 c $\mathbf{r}\cdot\begin{pmatrix}10\\-15\\30\end{pmatrix}=\begin{pmatrix}1\\-2\\-2\end{pmatrix}\cdot\begin{pmatrix}10\\-15\\30\end{pmatrix}$

$=10+30-60$

$\mathbf{r}\cdot\begin{pmatrix}10\\-15\\30\end{pmatrix}=-20$

$\mathbf{r}\cdot\begin{pmatrix}2\\-3\\6\end{pmatrix}=-4$

7 i $\displaystyle\int_0^5\frac{1}{\sqrt{15+2x-x^2}}\,\mathrm{d}x$

$15+2x-x^2=15-(1-x)^2+1$

$=16-(1-x)^2$

Therefore:

$$\int_0^5\frac{1}{\sqrt{15+2x-x^2}}\,\mathrm{d}x=\int_0^5\frac{1}{\sqrt{16-(1-x)^2}}\,\mathrm{d}x$$

Let $u=1-x\Rightarrow \mathrm{d}u=-\mathrm{d}x$

When $x=0$, $u=1$

When $x=5$, $u=-4$

$$\int_0^5\frac{1}{\sqrt{16-(1-x)^2}}\,\mathrm{d}x=-\int_1^{-4}\frac{1}{\sqrt{4^2-u^2}}\,\mathrm{d}u$$

$$=-\left[\arcsin\left(\frac{u}{4}\right)\right]_1^{-4}$$

$$=\left[\arcsin\left(\frac{u}{4}\right)\right]_{-4}^{1}$$

$$=\arcsin\left(\frac{1}{4}\right)-\arcsin(1)$$

$$=\arcsin\left(\frac{1}{4}\right)-\frac{\pi}{2}$$

ii a $5\cosh x-4\sinh x=5\left(\frac{e^x+e^{-x}}{2}\right)-4\left(\frac{e^x-e^{-x}}{2}\right)$

$$=\frac{5}{2}e^x+\frac{5}{2}e^{-x}-2e^x+2e^{-x}$$

$$=\frac{1}{2}e^x+\frac{9}{2}e^{-x}$$

$$=\frac{e^{2x}+9}{2e^x}$$

7 b $\displaystyle\int \frac{1}{5\cosh x - 4\sinh x}\,dx = \int \frac{2e^x}{e^{2x}+9}\,dx$

$\displaystyle= 2\int \frac{e^x}{e^{2x}+9}\,dx$

Let $u = e^x \Rightarrow du = e^x dx$

$\displaystyle 2\int \frac{e^x}{e^{2x}+9}\,dx = 2\int \frac{1}{u^2+3^2}\,du$

$\displaystyle = 2\left(\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right)+c$

$\displaystyle = \frac{2}{3}\arctan\left(\frac{e^x}{3}\right)+c$

8 a $\mathbf{M} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$ is an eigenvector of **M**

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ k & 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix} = \lambda\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 6+0+18 \\ 0-2+6 \\ 6k+0+6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix}$$

$$\begin{pmatrix} 24 \\ 4 \\ 6k+6 \end{pmatrix} = \begin{pmatrix} 6\lambda \\ \lambda \\ 6\lambda \end{pmatrix}$$

Equating the middle elements gives $\lambda = 4$

Hence 4 is an eigenvalue corresponding to $\begin{pmatrix} 6 \\ 1 \\ 6 \end{pmatrix}$

b Equating the lower elements gives:
$6k + 6 = 24 \Rightarrow k = 3$

8 c $\mathbf{M}=\begin{pmatrix}1 & 0 & 3\\ 0 & -2 & 1\\ 3 & 0 & 1\end{pmatrix}$

$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 0 & 3\\ 0 & -2-\lambda & 1\\ 3 & 0 & 1-\lambda\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{M}-\lambda\mathbf{I}) &= (1-\lambda)\left[(-2-\lambda)(1-\lambda)-0\right]-0\left[0(1-\lambda)-3\right]+3\left[0-3(-2-\lambda)\right]\\ &= -(1-\lambda)(1-\lambda)(2+\lambda)+9(2+\lambda)\\ &= (2+\lambda)\left[-(1-\lambda)(1-\lambda)+9\right]\\ &= (2+\lambda)\left[-(1-2\lambda+\lambda^2)+9\right]\\ &= (2+\lambda)(8+2\lambda-\lambda^2)\\ &= (2+\lambda)(2+\lambda)(4-\lambda)\end{aligned}$$

$\det(\mathbf{M}-\lambda\mathbf{I})=0$

$(2+\lambda)(2+\lambda)(4-\lambda)=0$

$\lambda=-2$ or $\lambda=4$

Hence **M** has only two eigenvalues, −2 and 4

8 d $l_1: \frac{x-2}{1} = \frac{y}{-3} = \frac{z+1}{4}$

Written in vector form this is:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2+0-3 \\ 0+0-1 \\ 6+0-1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+0+12 \\ 0+6+4 \\ 3+0+4 \end{pmatrix} = \begin{pmatrix} 13 \\ 10 \\ 7 \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 13 \\ 10 \\ 7 \end{pmatrix}$$

or

$$l_2: \frac{x+1}{13} = \frac{y+1}{10} = \frac{z-5}{7}$$