Practice exam paper

1 $\frac{1}{x+1} < \frac{x}{x+3}$

For $x < -3$ and $x > -1$, $(x+1)(x+3) > 0$, so:

$x+3 < x(x+1)$

$3 < x^2$

$x < -\sqrt{3}$ or $x > \sqrt{3}$

But after intersecting with the ranges $x < -3$ and $x > -1$, this becomes:

$x < -3$ and $x > \sqrt{3}$

The remaining case is for the range $-3 < x < -1$, where $(x+1)(x+3) < 0$, so:

$x+3 > x(x+1)$

$3 > x^2$

So within this range, $-\sqrt{3} < x < -1$

Collecting all of these ranges:

$x < -3$, $-\sqrt{3} < x < -1$ and $x > \sqrt{3}$

2 $\sum_{r=1}^{n} \frac{1}{(r+2)(r+4)}$

$\frac{1}{(r+2)(r+4)} = \frac{A}{r+2} + \frac{B}{r+4}$

$1 = A(r+4) + B(r+2)$

Comparing coefficients:
For r:
$A + B = 0$
$A = -B$
For constant terms:
$4A + 2B = 1$

$A = \frac{1}{2}$

$B = -\frac{1}{2}$

$\frac{1}{(r+2)(r+4)} = \frac{1}{2(r+2)} - \frac{1}{2(r+4)}$

Using the method of differences:

When $r = 1$ $\frac{1}{6} - \frac{1}{10}$

$r = 2$ $\frac{1}{8} - \frac{1}{12}$

$r = 3$ $\frac{1}{10} - \frac{1}{14}$

$r = 4$ $\frac{1}{12} - \frac{1}{16}$

$r = n-1$ $\frac{1}{2(n+1)} - \frac{1}{2(n+3)}$

$r = n$ $\frac{1}{2(n+2)} - \frac{1}{2(n+4)}$

So:

$$\sum_{r=1}^{n} \frac{1}{(r+2)(r+4)} = \frac{1}{6} + \frac{1}{8} - \frac{1}{2(n+3)} - \frac{1}{2(n+4)}$$

$$= \frac{7(n+3)(n+4) - 12(n+4) - 12(n+3)}{24(n+3)(n+4)}$$

$$= \frac{7n^2 + 49n + 84 - 12n - 48 - 12n - 36}{24(n+3)(n+4)}$$

$$= \frac{7n^2 + 25n}{24(n+3)(n+4)}$$

$$= \frac{n(7n+25)}{24(n+3)(n+4)}$$

So $p = 7$ and $q = 25$

3 a $z = \cos\theta - \mathrm{i}\sin\theta$

$z^n = (\cos\theta - \mathrm{i}\sin\theta)^n$

$z^n = \cos n\theta + \mathrm{i}\sin n\theta$

$\frac{1}{z^n} = \cos n\theta - \mathrm{i}\sin n\theta$

$z^n - \frac{1}{z^n} = 2\mathrm{i}\sin n\theta$ as required

b Solution to come

4 a $|z+12+5\mathrm{i}| = 13$

$|x+y\mathrm{i}+12+5\mathrm{i}| = 13$

$|(x+12)+\mathrm{i}(y+5)| = 13$

$|(x+12)+\mathrm{i}(y+5)|^2 = 13^2$

$(x+12)^2+(y+5)^2 = 169$

$(r\cos\theta+12)^2+(r\sin\theta+5)^2 = 169$

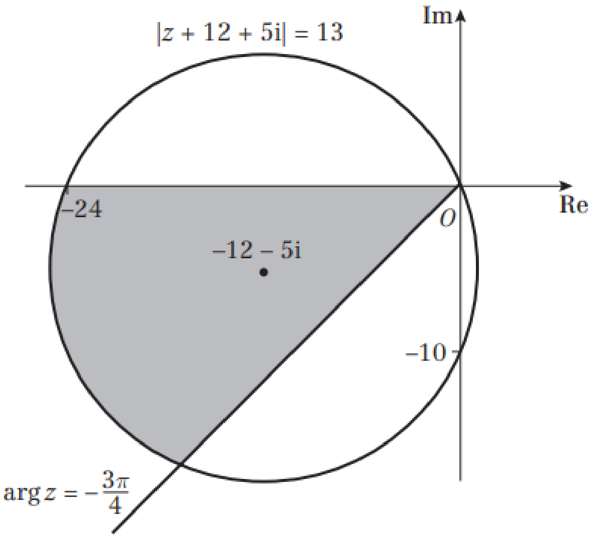
$r^2\cos^2\theta+24r\cos\theta+144+r^2\sin^2\theta+10r\sin\theta+25 = 169$

$r^2+24r\cos\theta+10r\sin\theta+169 = 169$

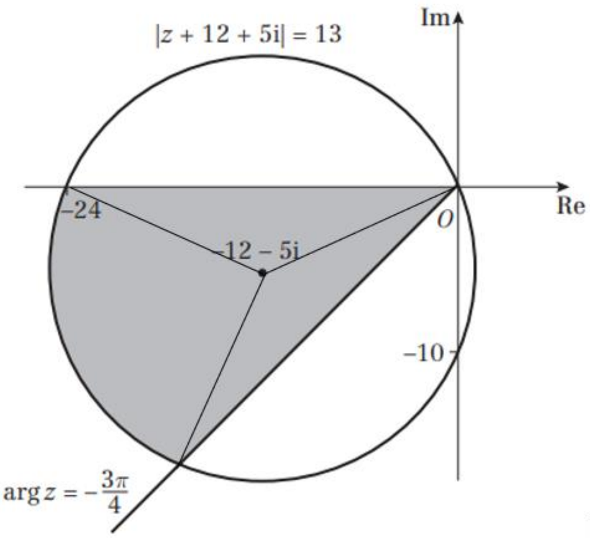
$r^2 = -24r\cos\theta-10r\sin\theta$

$r = -2(12\cos\theta+5\sin\theta)$ as required.

b



4 c



Each triangle has area $\frac{1}{2}\times 24\times 5 = 60$

The angle at the centre is twice the angle at the origin $\frac{\pi}{4}$

Therefore the sector has area $\frac{1}{4}\pi r^2 = \frac{169}{4}\pi$

Total shaded area = $\left(120+\frac{169}{4}\pi\right)$

= 252.73…

= 253 (3 s.f.)

5 a $\cos x\frac{dy}{dx}+y\sin x=\cos^3 x$

$\frac{dy}{dx}+y\tan x=\cos^2 x$

The integrating factor is:

$e^{\int \tan x\,dx}=e^{\ln \sec x}=\sec x$

$\sec x\frac{dy}{dx}+y\tan x\sec x=\sec x\cos^2 x$

$\frac{d}{dx}(y\sec x)=\cos x$

$y\sec x=\int \cos x\,dx$

$=\sin x+c$

$y=\cos x(\sin x+c)$

b when $x = 0$, $y = 3$

$3=\cos 0(\sin 0+c)$

$c = 3$

$y=\cos x(\sin x+3)$

6 a $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+8x\frac{\mathrm{d}y}{\mathrm{d}x}+12y=0$ **(1)**

Let $x=\mathrm{e}^t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{1}{\mathrm{e}^t}$$

$$=\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}$$

Substituting into (1) gives:

$$x^2\left(-\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{1}{x^2}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)+8x\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}\right)+12y=0$$

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+7\frac{\mathrm{d}y}{\mathrm{d}t}+12y=0$ as required

b $m^2+7m+12=0$

$(m+3)(m+4)=0$

$m=-3$ or $m=-4$

Therefore:

$y=A\mathrm{e}^{-3t}+B\mathrm{e}^{-4t}$

Since $x=\mathrm{e}^t$

$$y=Ax^{-3}+Bx^{-4}$$

$$=\frac{A}{x^3}+\frac{B}{x^4}$$

7 $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+2y=0$

When $x = 0$, $y = 1$ and $\frac{dy}{dx}=1$

$$\left.\frac{d^2y}{dx^2}\right|_0=-1-2$$
$$=-3$$
$$\frac{d^3y}{dx^3}+2\frac{dy}{dx}\times\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$
$$\frac{d^3y}{dx^3}=-2\frac{dy}{dx}\frac{d^2y}{dx^2}-2\frac{dy}{dx}$$
$$\left.\frac{d^3y}{dx^3}\right|_0=-2\times1\times-3-2\times1$$
$$=4$$

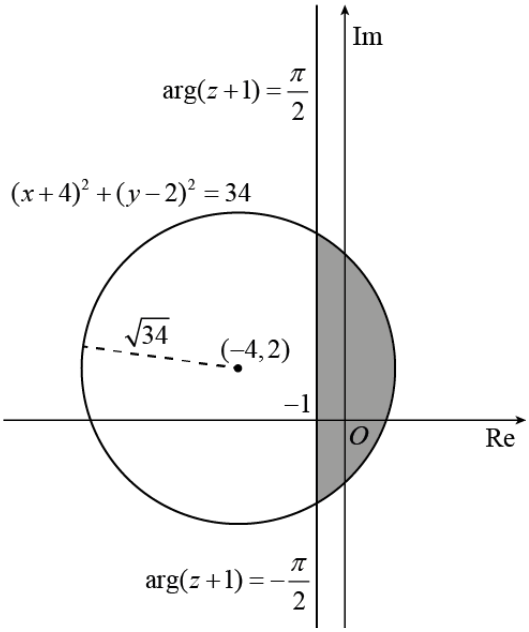
Substituting into:

$$y=y_0+x\left.\frac{dy}{dx}\right|_0+\frac{x^2}{2!}\left.\frac{d^2y}{dx^2}\right|_0+\frac{x^3}{3!}\left.\frac{d^3y}{dx^3}\right|_0$$
$$=1+x-\frac{3}{2}x^2+\frac{2}{3}x^3$$

8 a $\sqrt{2}|z-\mathrm{i}|=|z-4|$

$$\sqrt{2}|x+y\mathrm{i}-\mathrm{i}|=|x+y\mathrm{i}-4|$$
$$\sqrt{2}|x+\mathrm{i}(y-1)|=|(x-4)+y\mathrm{i}|$$
$$\left(\sqrt{2}|x+\mathrm{i}(y-1)|\right)^2=|(x-4)+y\mathrm{i}|^2$$
$$2\left(x^2+(y-1)^2\right)=(x-4)^2+y^2$$
$$2x^2+2y^2-4y+2=x^2-8x+16+y^2$$
$$x^2+8x+y^2-4y=14$$
$$(x+4)^2-16+(y-2)^2-4=14$$
$$(x+4)^2+(y-2)^2=34$$

8 b, c $(x+4)^2+(y-2)^2=34$ is the circle with centre $(-4, 2)$ and radius $\sqrt{34}$



d when $x=-1$

$(-1+4)^2+(y-2)^2=34$

$9+y^2-4y+4-34=0$

$y^2-4y-21=0$

$(y+3)(y-7)=0$

$y=-3$ or $y=7$

Therefore:

$-1-3\text{i}$ and $-1+7\text{i}$