Review exercise 2

1

$$(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 1\\ -1 & 1 & -1\end{vmatrix}$$

$$=\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ -1 & 1\end{vmatrix}\mathbf{k}$$

$$=\left((-1\times-1)-(1\times1)\right)\mathbf{i}-\left((-1\times-1)-(-1\times1)\right)\mathbf{j}+\left((-1\times1)-(-1\times-1)\right)\mathbf{k}$$

$$=(1-1)\mathbf{i}-(1-(-1))\mathbf{j}+(-1-1)\mathbf{k}=-2\mathbf{j}-2\mathbf{k}$$

Formulae for finding the vector product are given in the Edexcel formulae booklet which is provided for the examination. One form it gives is $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ and that has been used here.

Hence

$$\left|(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})\right|=\sqrt{\left((-2)^2+(-2)^2\right)}$$

$$=\sqrt{8}=2\sqrt{2}$$

You use the formula for the magnitude of a vector $|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}|=\sqrt{(x^2+y^2+z^2)}$

2 $\mathbf{p}=3\mathbf{i}+\mathbf{k}$ and $\mathbf{q}=\mathbf{i}+3\mathbf{j}+c\mathbf{k}$

$$\mathbf{p}\times\mathbf{q}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 0 & 1\\ 1 & 3 & c\end{vmatrix}$$

$$=\mathbf{i}(0-3)-\mathbf{j}(3c-1)+\mathbf{k}(9-0)$$

$$=-3\mathbf{i}-(3c-1)\mathbf{j}+9\mathbf{k}$$

Hence:

$$(\mathbf{p}\times\mathbf{q})+\mathbf{p}=-3\mathbf{i}-(3c-1)\mathbf{j}+9\mathbf{k}+3\mathbf{i}+\mathbf{k}$$

$$=-(3c-1)\mathbf{j}+10\mathbf{k}$$

If $(\mathbf{p}\times\mathbf{q})+\mathbf{p}$ is parallel to the vector $\mathbf{k}$ then:

$$3c-1=0\Rightarrow c=\frac{1}{3}$$

3

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$

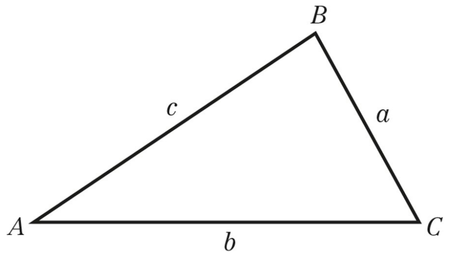
$= \mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{a}$

As $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, $\mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{a} = 0$

$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}$

$= \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$

You multiply out the brackets using the usual rules of algebra. You must take care with the order in which the vectors are multiplied as the vector product is not commutative. For a vector product $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$.



The area of the triangle, Δ, say, is given by

$\Delta = \frac{1}{2} bc \sin A$

$= \frac{1}{2} AC \times AB \sin A$

$= \frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$= \frac{1}{2} \left| \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} \right|$, as required.

The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between the vectors. The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

4 a

$\overrightarrow{AB} = (5\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

$= 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\overrightarrow{AC} = (6\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) - (2\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

$= 4\mathbf{i} + 5\mathbf{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$ and obtain the negative of the correct answer.

$\overrightarrow{AB} \times \overrightarrow{AC} = (3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} + 5\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 3 \\ 4 & 0 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} \mathbf{k}$$

$= 5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

4 b

$\overrightarrow{AD} = (12\mathbf{i} + \mathbf{j} - 9\mathbf{k}) - (2\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

$= 10\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}$

$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = (10\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}) \cdot (5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$

$= 10 \times 5 + (-6) \times (-3) + (-8) \times (-4)$

$= 50 + 18 + 32 = 100$

$10\mathbf{i} - 6\mathbf{j} - 9\mathbf{k} = 2(5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$ so $\overrightarrow{AD}$ and $\overrightarrow{AB} \times \overrightarrow{AC}$ are parallel. As the vector product is perpendicular to AB and AC, it follows that the line AD is perpendicular to the plane of the triangle ABC.

c The volume of the prism, P say, is given by

$$P = \frac{1}{2}\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \frac{1}{2} \times 100 = 50$$

In this case, the volume of the prism is the area of the triangle ABC, which is half the magnitude of $\overrightarrow{AB} \times \overrightarrow{AC}$, multiplied by the distance AD. (Even if the line AD is not perpendicular to the plane of the triangle ABC, the triple scalar product is still twice the volume of the prism.)

5 a

$$n_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$$

If the equation of a plane is given to you in the form $\mathbf{r} = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$, then you can find a normal to the plane by finding $\mathbf{b} \times \mathbf{c}$.

b $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

The Cartesian equation $3x + y - z = 3$ can be written in the vector form $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
Comparison with the standard form, $\mathbf{r} \cdot \mathbf{n} = p$, gives you that $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to Π_2

c

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 8 & -4 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & -4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -4 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 8 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$

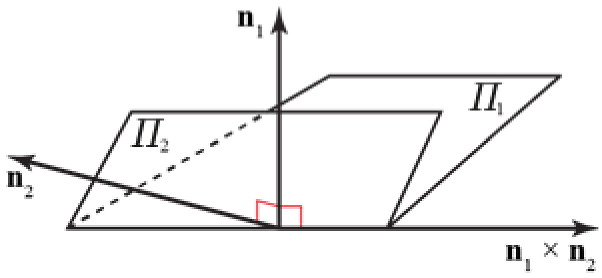
$$= -4\mathbf{i} - 13\mathbf{j} - 25\mathbf{k} = -1(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$$

The scalar product $\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$
So to show that a vector, $\mathbf{r}$ say, is perpendicular to two other vectors, you can show that $\mathbf{r}$ is parallel to the vector product of the two other vectors. An alternative method is to show that the scalar product of $\mathbf{r}$ with each of the other two vectors is zero.

$\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to the plane containing $\mathbf{n}_1$ and $\mathbf{n}_2$, and $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is a multiple of $\mathbf{n}_1 \times \mathbf{n}_2$

Hence $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$

5 d $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$



This diagram illustrates that the line of intersection of the planes Π_1 and Π_2 lies in the direction of $\mathbf{n}_1 \times \mathbf{n}_2$ In this case, $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k} = -\mathbf{n}_1 \times \mathbf{n}_2$ and can be used as the direction of the line, as can any other multiple of this vector.

6 a

$$\overrightarrow{AB} = -\mathbf{i} + 2\mathbf{j} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$
$$= -4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$$
$$\overrightarrow{AC} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$
$$= 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$$
$$\overrightarrow{AB} \times \overrightarrow{AC} = (-4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \times (2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix}$$
$$= \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -4 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 3 \\ 2 & -2 \end{vmatrix}\mathbf{k}$$
$$= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$ and obtain the negative of the correct answer.

b An equation of Π is

$$\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = (3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$
$$= 3 \times 1 + (-1) \times 4 + 4 \times 2$$
$$= 3 - 4 + 8 = 7$$

So $\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = 7$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of A has been used but the position vectors of B and C would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here 7

c

$$\overrightarrow{AD} = 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$
$$= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$
$$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k})$$
$$= 2 \times 1 + 3 \times 4 + (-1) \times 2$$
$$= 2 + 12 - 2 = 12$$

The volume, V say, of the tetrahedron is given by

$$V = \frac{1}{6}\left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right| = \frac{1}{6} \times 12 = 2$$

7 a $A: \mathbf{r} = 4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$

$B: \mathbf{r} = 2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$

Let the angle AOB be θ, then:

$$\cos\theta = \frac{(4\mathbf{i} + \mathbf{j} - 7\mathbf{k})\cdot(2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k})}{|4\mathbf{i} + \mathbf{j} - 7\mathbf{k}||2\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}|}$$

$$= \frac{8 + 6 - 14}{\sqrt{4^2 + 1^2 + (-7)^2} \times \sqrt{2^2 + 6^2 + 2^2}}$$

$$= \frac{0}{\sqrt{66} \times \sqrt{44}}$$

$\theta = 90°$

b Let M be the midpoint of OB, then:

$$\overrightarrow{AM} = \overrightarrow{AO} + \frac{1}{2}\overrightarrow{OB}$$

$$= \begin{pmatrix} -4 \\ -1 \\ 7 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 2 \\ 6 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ 2 \\ 8 \end{pmatrix}$$

Hence a vector equation of the median is:

$$\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -7 \end{pmatrix} + \lambda\begin{pmatrix} -3 \\ 2 \\ 8 \end{pmatrix}$$

c

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & -7 \\ 2 & 6 & 2 \end{vmatrix}$$

$$= \mathbf{i}(2 + 42) - \mathbf{j}(8 + 14) + \mathbf{k}(24 - 2)$$

$$= 44\mathbf{i} - 22\mathbf{j} + 22\mathbf{k}$$

$$= 22(2\mathbf{i} - \mathbf{j} + \mathbf{k})$$

Since $\mathbf{r}\cdot\mathbf{n} = 0$

$\mathbf{r}\cdot(2\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0$

8 a $A: \mathbf{r} = a(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})$

$\Pi: \mathbf{r}\cdot(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) = 5a$

$$a(4\mathbf{i} + \mathbf{j} + 2\mathbf{k})\cdot(\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}) = a(4 - 5 + 6)$$
$$= 5a$$

Hence A lies in the plane Π

8 b $B: \mathbf{r} = a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$

$$\overrightarrow{BA} = a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k}) - a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})$$
$$= a(-2\mathbf{i}+10\mathbf{j}-6\mathbf{k})$$
$$= -2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$$

Hence $\overrightarrow{AB}$ is parallel to the vector $\mathbf{i}-5\mathbf{j}+3\mathbf{k}$ and is, therefore, perpendicular to the plane.

c The angle OBA lies between the lines AB and OB.

$$\overrightarrow{AB} = -2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$$

Let the angle OBA be θ, then:

$$\cos\theta = \frac{-2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})\cdot a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})}{|-2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})||a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})|}$$
$$= \frac{-2(2-55-12)}{\sqrt{4(1^2+(-5)^2+3^2)}\times\sqrt{2^2+11^2+(-4)^2}}$$
$$= \frac{130}{\sqrt{140}\times\sqrt{141}}$$

$\theta = 22.3°$ (3 s.f.)

9 a

$$\overrightarrow{AC} = \begin{pmatrix}6\\4\\5\end{pmatrix} - \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}$$

$$\overrightarrow{AD} = \begin{pmatrix}-7\\6\\-3\end{pmatrix} - \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}-10\\5\\-5\end{pmatrix}$$

$$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{pmatrix}3\\3\\3\end{pmatrix}\times\begin{pmatrix}-10\\5\\-5\end{pmatrix} = \begin{pmatrix}3\times(-5)-3\times5\\3\times(-10)-3\times(-5)\\3\times5-3\times(-10)\end{pmatrix}$$
$$= \begin{pmatrix}-30\\-15\\45\end{pmatrix}$$

For writing vectors, you can use either the form with **i**s, **j**s and **k**s, or column vectors, which are used in this solution. Sometimes it may even be appropriate to use a mixture of the two. The form using **i**, **j** and **k** usually gives a more compact solution but many find column vectors quicker to write. The choice is entirely up to you and you may choose to vary it from question to question.

b $\mathbf{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-1\\3\end{pmatrix}$

The vector $\begin{pmatrix}-30\\-15\\45\end{pmatrix}$ is perpendicular to both $\overrightarrow{AC}$ and $\overrightarrow{AD}$

This vector or any multiple of it may be used for the equation of the line.

9 c For B to lie on the line there must be a value of λ for which

$$\begin{pmatrix}5\\2\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\2\end{pmatrix}+\lambda\begin{pmatrix}-2\\-1\\3\end{pmatrix}$$

Equating the x components of the vectors

$5=3-2\lambda \Rightarrow \lambda=-1$

Checking this value of λ for the other components

y component:

$1+\lambda\times(-1)=1+(-1)\times(-1)=2$, as required.

z component:

$2+\lambda\times 3=2+(-1)\times 3=-1$, as required.

Hence, B lies on the line.

d

$$\overrightarrow{AB}=\begin{pmatrix}5\\2\\-1\end{pmatrix}-\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}2\\1\\-3\end{pmatrix}$$

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})=\begin{pmatrix}2\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}-30\\-15\\45\end{pmatrix}=2\times(-30)+1\times(-15)+(-3)\times 45$$

$$=-60-15-135=-210$$

The volume of the tetrahedron, V say, is given by

$$V=\frac{1}{6}\left|\overrightarrow{AB}\cdot(\overrightarrow{AC}\times\overrightarrow{AD})\right|=\frac{1}{6}|-210|=\frac{1}{6}\times 210=35$$

The volume of the tetrahedron is one sixth of the triple scalar product.

10 a A vector **n** perpendicular to l_1 and l_2 is given by

$$n=(2\mathbf{i}+3\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&0&3\\1&-2&1\end{vmatrix}$$

$$=\begin{vmatrix}0&3\\-2&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&3\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&0\\1&-2\end{vmatrix}\mathbf{k}$$

$$=6\mathbf{i}+\mathbf{j}-4\mathbf{k}$$

10 b An equation for Π_1 has the form

$\mathbf{r}\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=p$

$p=(\mathbf{i}+6\mathbf{j}-\mathbf{k})\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})$

$=6+6+4=16$

A vector equation of Π_1 is

$\mathbf{r}\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=16$

A Cartesian equation of Π_1 is given by

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=16$

$6x+y-4z=16$, as required.

To obtain a Cartesian equation of a plane when you have a vector equation in the form $\mathbf{r.n}=p$, replace $\mathbf{r}$ by $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and work out the scalar product.

c The point with coordinates $(3, p, 0)$ lies on l_1 and, hence, must lie on Π_1

Substituting $(3, p, 0)$ into the result of part **b**

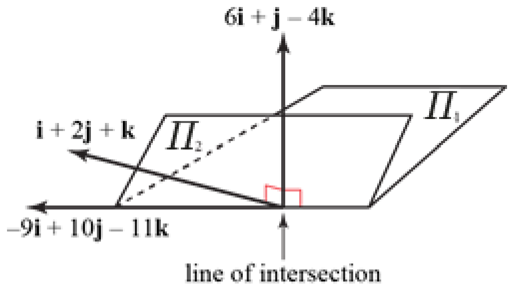
$18+p=16 \Rightarrow p=-2$

d The line of intersection lies in the direction given by

$$(\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 6 & 1 & -4\end{vmatrix}$$

$$=\begin{vmatrix}2 & 1\\ 1 & -4\end{vmatrix}\mathbf{i}-\begin{vmatrix}1 & 1\\ 6 & -4\end{vmatrix}\mathbf{j}+\begin{vmatrix}1 & 2\\ 6 & 1\end{vmatrix}\mathbf{k}$$

$$=-9\mathbf{i}+10\mathbf{j}-11\mathbf{k}$$



To find one point that lies on both Π_1 and Π_2

Π_1: $6x+y-4z=16$ **(1)**

Π_2: $x+2y+z=2$ **(2)**

(1) $+4\times$ **(2)** gives $10x+9y=24$

Choose $x=-3$, $y=6$

Substitute into **(2)**

$-3+12+z=2 \Rightarrow z=-7$

One point on the line is $(-3, 6, -7)$

An equation of the line is

$\big(\mathbf{r}-(-3\mathbf{i}+6\mathbf{j}-7\mathbf{k})\big)\times(-9\mathbf{i}+10\mathbf{j}-11\mathbf{k})=0$

You need to find just one point that is on both planes and there are infinitely many possibilities. Here you can choose any pair of values of x and y which fit this equation. A choice here has been made which gives whole numbers but you may find, for example, $y=0$, $x=2.4$ easier to see.

The form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, for the equation of a straight line, represents a line that passes through the point with position vector $\mathbf{a}$ and is parallel to the vector $\mathbf{b}$.

11 a Π passes through $A(-2, 3, 5)$, $B(1, -3, 1)$ and $C(4, -6, -7)$

$$\overrightarrow{AC}=\begin{pmatrix}4\\-6\\-7\end{pmatrix}-\begin{pmatrix}-2\\3\\5\end{pmatrix}=\begin{pmatrix}6\\-9\\-12\end{pmatrix} \text{ and } \overrightarrow{BC}=\begin{pmatrix}4\\-6\\-7\end{pmatrix}-\begin{pmatrix}1\\-3\\1\end{pmatrix}=\begin{pmatrix}3\\-3\\-8\end{pmatrix}$$

$$\overrightarrow{AC}\times\overrightarrow{BC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 6 & -9 & -12\\ 3 & -3 & -8\end{vmatrix}$$

$$=\mathbf{i}(72-36)-\mathbf{j}(-48+36)+\mathbf{k}(-18+27)$$

$$=36\mathbf{i}+12\mathbf{j}+9\mathbf{k}$$

11 b $\mathbf{n} = 36\mathbf{i} + 12\mathbf{j} + 9\mathbf{k}$

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\mathbf{r} \cdot \begin{pmatrix} 36 \\ 12 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 36 \\ 12 \\ 9 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix}$$

$$= 12 - 12 + 3$$

$$\mathbf{r} \cdot \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = 3$$

c Let D be the point (25, 5, 7) then:

$$\overrightarrow{DF} = \begin{pmatrix} 25 \\ 5 \\ 7 \end{pmatrix} - k \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix}$$

Therefore:

$$\overrightarrow{DF} = \begin{pmatrix} 25 - 12k \\ 5 - 4k \\ 7 - 3k \end{pmatrix}$$

and hence:

$$(25 - 12k,\ 5 - 4k,\ 7 - 3k) \text{ lies on } \mathbf{r} \cdot \begin{pmatrix} 12 \\ 4 \\ 3 \end{pmatrix} = 3$$

$$12(25 - 12k) + 4(5 - 4k) + 3(7 - 3k) = 3$$

$$300 - 144k + 20 - 16k + 21 - 9k = 3$$

$$169k = 338$$

$$k = 2$$

Therefore, F is the point $\begin{pmatrix} 25 - 24 \\ 5 - 8 \\ 7 - 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

12 a Π passes through $P(-1, 3, -2)$, $Q(4, -1, -1)$ and $R(3, 0, c)$

$$\overrightarrow{RP} = \begin{pmatrix}-1\\3\\-2\end{pmatrix} - \begin{pmatrix}3\\0\\c\end{pmatrix} = \begin{pmatrix}-4\\3\\-2-c\end{pmatrix} \text{ and } \overrightarrow{RQ} = \begin{pmatrix}4\\-1\\-1\end{pmatrix} - \begin{pmatrix}3\\0\\c\end{pmatrix} = \begin{pmatrix}1\\-1\\-1-c\end{pmatrix}$$

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -4 & 3 & -c-2\\ 1 & -1 & -1-c\end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix}3 & -(c+2)\\ -1 & -(c+1)\end{vmatrix} - \mathbf{j}\begin{vmatrix}-4 & -(c+2)\\ 1 & -(c+1)\end{vmatrix} + \mathbf{k}\begin{vmatrix}-4 & 3\\ 1 & -1\end{vmatrix}$$

$$= \mathbf{i}\left[-3(c+1) - 1(c+2)\right] - \mathbf{j}\left[4(c+1) + 1(c+2)\right] + \mathbf{k}(4-3)$$

$$= \mathbf{i}(-3c-3-c-2) - \mathbf{j}(4c+4+c+2) + \mathbf{k}$$

$$= (-4c-5)\mathbf{i} - (5c+6)\mathbf{j} + \mathbf{k}$$

b $\overrightarrow{RP} \times \overrightarrow{RQ} = 3\mathbf{i} + d\mathbf{j} + \mathbf{k}$

and

$\overrightarrow{RP} \times \overrightarrow{RQ} = (-4c-5)\mathbf{i} - (5c+6)\mathbf{j} + \mathbf{k}$

Therefore:

$3\mathbf{i} + d\mathbf{j} + \mathbf{k} = (-4c-5)\mathbf{i} - (5c+6)\mathbf{j} + \mathbf{k}$

Comparing coefficients for **i** gives:

$-4c - 5 = 3 \Rightarrow c = -2$

Comparing coefficients for **j** gives:

$d = -(5c+6) \Rightarrow d = 4$ as required

c $\mathbf{r} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}$

$$\mathbf{r} \cdot \begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\-2\end{pmatrix} \cdot \begin{pmatrix}3\\4\\1\end{pmatrix}$$

$$= -3 + 12 - 2$$

$$\mathbf{r} \cdot \begin{pmatrix}3\\4\\1\end{pmatrix} = 7$$

12 d $S : \mathbf{r} = \mathbf{i} + 5\mathbf{j} + 10\mathbf{k}$

Let T be the point where the perpendicular from S crosses the plane.

$$\overrightarrow{ST} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} - k\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Therefore:

$$\overrightarrow{ST} = \begin{pmatrix} 1-3k \\ 5-4k \\ 10-k \end{pmatrix}$$

and hence:

$$(1-3k,\ 5-4k,\ 10-k) \text{ lies on } \mathbf{r}\cdot\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} = 7$$

$3(1-3k)+4(5-4k)+1(10-k)=7$
$3-9k+20-16k+10-k=7$
$26k=26$
$k=1$

$$\overrightarrow{SS'} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} - 2\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Hence:

$$S' = \begin{pmatrix} -5 \\ -3 \\ 8 \end{pmatrix}$$

13 a

$\mathbf{b}-\mathbf{a} = 3\mathbf{i}+3\mathbf{j}-4\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k}) = 2\mathbf{i}-3\mathbf{k}$
$\mathbf{c}-\mathbf{a} = 5\mathbf{i}-2\mathbf{j}-2\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k}) = 4\mathbf{i}-5\mathbf{j}-\mathbf{k}$

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -3 \\ 4 & -5 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -3 \\ -5 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -3 \\ 4 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 0 \\ 4 & -5 \end{vmatrix}\mathbf{k}$$

$$= -15\mathbf{i}-10\mathbf{j}-10\mathbf{k}$$

The vector $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ is perpendicular to both AB and AC and, so, is perpendicular to the plane Π_1
You can use this vector, or any parallel vector, as the $\mathbf{n}$ in the equation $\mathbf{r}.\mathbf{n} = p$ in part **b**. Here each coefficient has been divided by –5
This eases later working and avoids negative signs.

b A vector perpendicular to Π_1 is $3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

A vector equation of Π_1 is

$\mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+2\mathbf{k}) = (\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})$
$= 3+6-2=7$

13 c The line l is parallel to the vector

$$(\mathbf{i}+\mathbf{k})\times(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 1\\ 3 & 2 & 2\end{vmatrix}$$

$$=-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

To find one point on both Π_1 and Π_2

For Π_1 $x+z=3$

Let $z=0$, then $x=3$

> The form $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=0$ is that of a line passing through a point with position vector **p**, parallel to the vector **q**. So you need to find one point on the line; that is any point which is on both Π_1 and Π_2 As there are infinitely many such points, there are many possible answers to this question. The choice of $z = 0$ here is an arbitrary one.

Substituting $z = 0$, $x = 3$ into a Cartesian equation of Π_2

$3x+2y+z=7$

$9+2y+0=7 \Rightarrow y=-1$

One point on Π_1 and Π_2 and, hence on l is (3, –1, 0)

Hence, a vector equation of l is $(\mathbf{r}-(3\mathbf{i}-\mathbf{j}))\times(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$

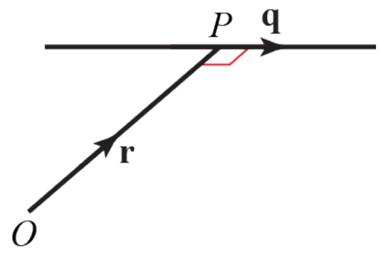
d A vector equation of l is

$\mathbf{r}=(3\mathbf{i}-\mathbf{j})+t(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})$

$=(3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}$

At P, **r** is perpendicular to l

$((3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k})\cdot(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$



$-6+4t-1+t+4t=0 \Rightarrow 9t=7 \Rightarrow t=\frac{7}{9}$

The coordinates of P are

$(3-2t,\ -1+t,\ 2t)=\left(\frac{13}{9},\ -\frac{2}{9},\ \frac{14}{9}\right)$

> At the point P which is nearest to the origin O, the position vector of P, **r**, is perpendicular to the direction of the line, **q**. Forming the scalar product **r**.**q** and equating to zero gives you an equation in t.

14 a

$\mathbf{a}\times\mathbf{b}=(2\mathbf{i}-\mathbf{k})\times(4\mathbf{i}+3\mathbf{j}+\mathbf{k})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -1\\ 4 & 3 & 1\end{vmatrix}=3\mathbf{i}-6\mathbf{j}+6\mathbf{k}$$

b Substituting (0, 0, 0) into $x - 2y + 2z$

$0-2\times0+2\times0=0$

So the plane with equation $x - 2y + 2z = 0$ contains O.

Similarly as

$2-2\times0+2\times(-1)=2-2=0$

and $4-2\times3+2\times1=4-6+2=0$,

the plane with equation $x - 2y + 2z = 0$ contains $A(2, 0, -1)$ and $B(4, 3, 1)$

> 'Verify' means check that the equation is satisfied by the data of this particular question. To do this you can just show that the coordinates of O, A and B satisfy $x - 2y + 2z = 0$
> You do not have to show any general methods.

14 c For B to lie on the plane with equation

$$\mathbf{r}.\mathbf{n}=d$$

$(4\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k})=d$

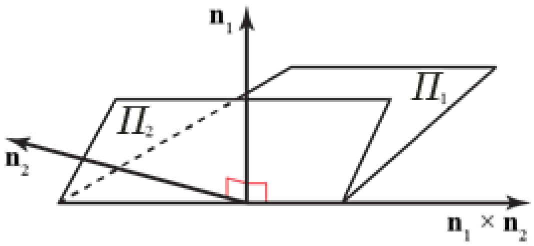
$$d=4\times3+3\times1+1\times(-1)=12+3-1=14$$

14 d The line of intersection L lies in the direction given by

$$(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -2 & 2\\ 3 & 1 & -1\end{vmatrix}$$

$$=0\mathbf{i}+7\mathbf{j}+7\mathbf{k}$$

A vector parallel to $7\mathbf{j}+7\mathbf{k}$ is $\mathbf{j}+\mathbf{k}$ and this is parallel to the line L.
The point B, which has position vector $4\mathbf{i}+3\mathbf{j}+\mathbf{k}$, lies on both Π_1 and Π_2 and, hence, on L.
A vector equations of L is
$\mathbf{r}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}+t(\mathbf{j}+\mathbf{k})$



The vector $\mathbf{n}_1=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ is perpendicular to Π_1 and the vector $\mathbf{n}_2=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ is perpendicular to Π_2
This diagram illustrates the line of intersection of the planes is parallel to $n_1\times n_2$
This gives you the direction of L. To find the equation of L, you also need one point on L. In this case, the information given in the question shows you that you already have such a point, B.

e Rearranging the answer to part **d**
$\mathbf{r}=4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}$
At the point X on L where OX is perpendicular to L

$$\mathbf{r}.(\mathbf{j}+\mathbf{k})=0$$

$$\left(4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}\right)\cdot(\mathbf{j}+\mathbf{k})=3+t+1+t=0$$

$$2t=-4\Rightarrow t=-2$$

The positive vector of X is
$4\mathbf{i}+(3-2)\mathbf{j}+(1-2)\mathbf{k}=4\mathbf{i}+\mathbf{j}-\mathbf{k}$

15 a

$$\overrightarrow{AB}=\begin{pmatrix}1\\2\\3\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}2\\3\\2\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}=\begin{pmatrix}0\\4\\2\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-1\\3\\3\end{pmatrix}\times\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}-6\\2\\-4\end{pmatrix}$$

15 b A vector equation of Π is $\mathbf{r}\cdot\begin{pmatrix}-6\\2\\-4\end{pmatrix}=\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}-6\\2\\-4\end{pmatrix}=-12-2=-14$

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$\mathbf{r}\cdot\begin{pmatrix}-6\\2\\-4\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}-6\\2\\-4\end{pmatrix}=-6x+2y-4z=-14$

A Cartesian equation of Π is

$-6x+2y-4z=-14$

Divide throughout by –2

$3x-y+2z=7$, as required.

> Once you have a vector **n** perpendicular to the plane, you can find a vector equation of the plane using $r.n=a.n$, where **a** is the position vector of any point on the plane. Here the position vector of A has been used but the position vectors of B and C would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here –14

c A vector equation of the line l is

$\mathbf{r}=\begin{pmatrix}5\\5\\3\end{pmatrix}+t\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

> The two vector forms of a straight line $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ and $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ are equivalent and you can always interchange one with the other. Here
>
> $\mathbf{a}=\begin{pmatrix}5\\5\\3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

Parametric equations of l are

$x=5+2t,\ y=5-t,\ z=3-2t$

Substituting the parametric equations into

$$3x-y+2z=7$$

$$3(5+2t)-(5-t)+2(3-2t)=7$$

$$15+6t-5+t+6-4t=7$$

$$3t=-9\Rightarrow t=-3$$

The coordinates of T are

$(5+2t,\ 5-t,\ 3-2t)=(5-6,\ 5+3,\ 3+6)$

$=(-1,\ 8,\ 9\ )$

d $\overrightarrow{BT}=\overrightarrow{OT}-\overrightarrow{OB}=\begin{pmatrix}-1\\8\\9\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}-2\\6\\6\end{pmatrix}$

> When A, B and T lie on the same straight line, AB and BT are in the same direction. So you show that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BT}$ are parallel.

From part **a**

$\overrightarrow{AB}=\begin{pmatrix}-1\\3\\3\end{pmatrix}$

Hence

$\overrightarrow{AB}=\frac{1}{2}\overrightarrow{BT}$ and AB is parallel to BT.

Hence A, B and T lie in the same straight line.

> Points which lie on the same straight line are called **collinear** points.

16 a Π passes through $A(-1, -1, 1)$, $B(4, 2, 1)$ and $C(2, 1, 0)$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} \text{ and } \overrightarrow{BC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AC} \times \overrightarrow{BC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ -2 & -1 & -1 \end{vmatrix}$$

$$= \mathbf{i}(-2-1) - \mathbf{j}(-3-2) + \mathbf{k}(-3+4)$$

$$= -3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$

$D : \mathbf{r} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

Therefore, the vector equation of the perpendicular to the plane that passes through S is:

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - t\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

b $$\overrightarrow{DC} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$V = \frac{1}{6}\left|\overrightarrow{AC} \cdot \left(\overrightarrow{BC} \times \overrightarrow{DC}\right)\right|$$

$$V = \frac{1}{6}\left|\begin{pmatrix} 3 & 2 & -1 \\ -2 & -1 & -1 \\ 1 & -1 & -3 \end{pmatrix}\right|$$

$$= \frac{1}{6}\left|3(3-1) - 2(6+1) - 1(2+1)\right|$$

$$= \frac{1}{6}\left|6 - 14 - 3\right|$$

$$= \frac{11}{6}$$

c $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

$$= 3 - 5 + 1$$

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = -1$$

16 d $\overrightarrow{OE} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - k\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$

Therefore:

$\overrightarrow{OE} = \begin{pmatrix} 1+3k \\ 2-5k \\ 3-k \end{pmatrix}$

and hence:

$(1+3k,\ 2-5k,\ 3-k)$ lies on $\mathbf{r}\cdot\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix} = -1$

$-3(1+3k)+5(2-5k)+1(3-k)=-1$
$-3-9k+10-25k+3-k=-1$
$35k=11$
$k=\frac{11}{35}$

Therefore, E is the point $\begin{pmatrix} 1+\frac{33}{35} \\ 2-\frac{55}{35} \\ 3-\frac{11}{35} \end{pmatrix} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix}$

e $\overrightarrow{DE} = -\frac{11}{35}\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$

$\left|\overrightarrow{DE}\right| = \frac{11}{35}\sqrt{(-3)^2+5^2+1^2} = \frac{11\sqrt{35}}{35}$ as required

16 f The reflection of D in Π is given by:

$$\overrightarrow{OD}' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{22}{35}\begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+\frac{66}{35} \\ 2-\frac{110}{35} \\ 3-\frac{22}{35} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{101}{35} \\ -\frac{40}{35} \\ \frac{83}{35} \end{pmatrix}$$

17 a Let $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\mathbf{b} - \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\mathbf{c} - \mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + \mathbf{k}$

A vector which is perpendicular to Π is

$$(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

The vector product $(\mathbf{b}-\mathbf{c}) \times (\mathbf{c}-\mathbf{a})$ is, by definition, perpendicular to both $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ and, so, it is perpendicular to both AB and AC. It will also be perpendicular to the plane containing AB and AC.

b

$$\Delta ABC = \frac{1}{2}\left|(\mathbf{b}-\mathbf{a}) \times (\mathbf{c}-\mathbf{a})\right|$$

$$= \frac{1}{2}\left|2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}\right|$$

$$= \frac{1}{2}\sqrt{(2^2 + (-3)^2 + (-2)^2)}$$

$$= \frac{\sqrt{17}}{2}$$

The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

c A vector equation of Π is

$$\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 0 - 3 - 4$$

$$= -7$$

d A Cartesian equation of Π is $2x - 3y - 2z = -7$

The vector equation $\mathbf{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = p$ and the Cartesian equation $ax + by + cz = p$ are equivalents.

17 e The distance from a point (α, β, γ) to a plane

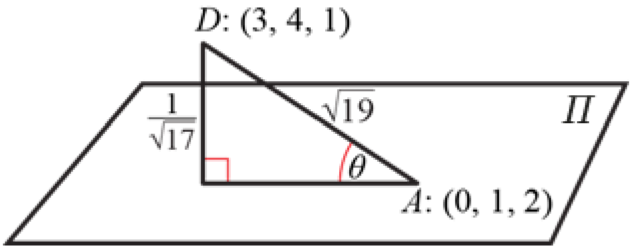
$n_1x + n_2y + n_3z + d = 0$ is $\left|\dfrac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{(n_1^2 + n_2^2 + n_3^2)}}\right|$

Hence the distance from (0, 0, 0) to $2x - 3y - 2z = -7$

is $\left|\dfrac{7}{\sqrt{(2^2 + (-3)^2 + (-2)^2)}}\right| = \dfrac{7}{\sqrt{17}}$

This formula is given in the Edexcel formulae booklet. If you use a formula from the booklet, it is sensible to quote it in your solution. The distance of a point from a plane is defined to be the shortest distance from the point to the plane; that is the perpendicular distance from the point to the plane.

f



Let the angle between AD and Π be θ

$AD^2 = (3-0)^2 + (4-1)^2 + (1-2)^2 = 9 + 9 + 1 = 19$

$AD = \sqrt{19}$

$\sin\theta = \dfrac{\left(\frac{1}{\sqrt{17}}\right)}{\sqrt{19}} = 0.055641\ldots$

$\theta = 3.2^\circ$ (1 d.p.)

18 a Equating the x components

$-1 - 2s = -t$ **(1)**

Equating the y components

$2 + s = -1 + t$ **(2)**

(1) + **(2)** $1 - s = -1 \Rightarrow s = 2$

Substitute $s = 2$ into **(2)** $4 = -1 + t \Rightarrow t = 5$

Checking the z components

For l_1: $-4 + 3s = -4 + 6 = 2$

For l_2: $7 - t = 7 - 5 = 2$

These are the same, so l_1 and l_2 intersect.

To show that two lines intersect, you find the values of the two parameters, here s and t, which make two of the components equal and then you show that these values make the third components equal.

The lines l_1 and l_2 are parallel to

$-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $-\mathbf{i} + \mathbf{j} - \mathbf{k}$ respectively.

$(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \cdot (-\mathbf{i} + \mathbf{j} - \mathbf{k}) = 2 + 1 - 3 = 0$

Hence l_1 is perpendicular to l_2

As the scalar product $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors, if, for non-zero vectors, the scalar produce is zero then $\cos\theta = 0$ and $\theta = 90^\circ$

b Substituting $s = 2$ into the equation for l_1, the common point has position vector

$-\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + 2(-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = -5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

Using $\mathbf{r} = \mathbf{a} + \mu(\mathbf{b} - \mathbf{a})$, an equation of l_3 is

$\mathbf{r} = -5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(4\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k} - (-5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}))$

$\quad = -5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(9\mathbf{i} + (\lambda - 4)\mathbf{j} - 5\mathbf{k})$

$\mathbf{r} = \mathbf{a} + u(\mathbf{b} - \mathbf{a})$ is the appropriate form of the equation of a straight line going through two points with position vectors $\mathbf{a}$ and $\mathbf{b}$

Here $\mathbf{a} = -5\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$\mathbf{b} = 4\mathbf{i} + \lambda\mathbf{j} - 3\mathbf{k}$

18 c A vector **n** perpendicular to the plane, Π say, containing l_1 and l_2 is

$$\mathbf{n} = (-\mathbf{i}+\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = 4\mathbf{i}+5\mathbf{j}+\mathbf{k}$$

Let the angle between l_3 and Π be θ

$$|\mathbf{n}|^2 = 4^2+5^2+1^2 = 42$$

$$|9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k}| = 9^2+(\lambda-4)^2+(-5)^2$$

$$= 81+\lambda^2-8\lambda+16+25 = \lambda^2-8\lambda+122$$

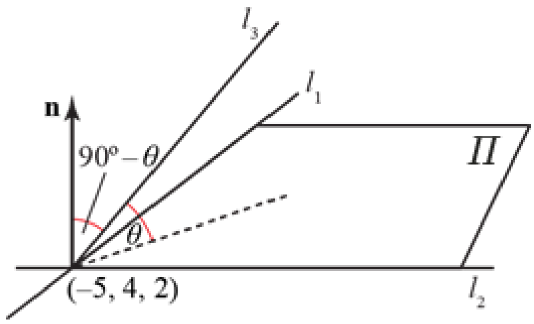
$$\mathbf{n}\cdot(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k}) = |\mathbf{n}||(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k})|\cos(90^\circ-\theta)$$

$$(4\mathbf{i}+5\mathbf{j}+\mathbf{k})\cdot(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k})$$

$$= \sqrt{42}\times\sqrt{(\lambda^2-8\lambda+122)}\sin\theta$$

$$\sin\theta = \frac{4\times9+5(\lambda-4)+1\times(-5)}{\sqrt{42}\sqrt{(\lambda^2-8\lambda+122)}}$$

$$= \frac{5\lambda+11}{\sqrt{42}\sqrt{(\lambda^2-8\lambda+122)}}$$



The cosine of the angle between **n** and l_3 can be found using the scalar product of **n** and a vector parallel to l_3

This cosine is $\sin\theta$

d If l_1, l_2 and l_3 are coplanar then

$\theta = 0$ and $\sin\theta = 0$

Hence $5\lambda+11 = 0 \Rightarrow \lambda = -\frac{11}{5}$

Looking at the diagram in part **b** above, if l_3 lies in the plan Π, then $\theta = 0$

19 a The Cartesian equations of the planes are

$P_1: 2x - y + 2z = 9$ **(1)**

$P_2: 4x + 3y - z = 8$ **(2)**

(1) $+ 2\times$ **(2)**

$10x + 5y = 25$

$2x + y = 5$

Let $x = t$, then $y = 5 - 2x = 5 - 2t$

From **(2)**

$z = 4x + 3y - 8$

$= 4t + 3(5 - 2t) - 8 = 7 - 2t$

The general point on the line of intersection of the planes has coordinates $(t, 5-2t, 7-2t)$

The distance, y say, from O to this general point is given by

$y^2 = t^2 + (5-2t)^2 + (7-2t)^2$

$= t^2 + 25 - 20t + 4t^2 + 49 - 28t + 4t^2$

$= 9t^2 - 48t + 74$ **(3)**

Differentiating both sides of **(3)** with respect to t

$2y\frac{dy}{dt} = 18t - 48$

At a minimum distance $\frac{dy}{dt} = 0$

$18t - 48 = 0 \Rightarrow t = \frac{48}{18} = \frac{8}{3}$

Substituting into **(3)**

$y^2 = 9\times\left(\frac{8}{3}\right)^2 - 48\times\frac{8}{3} + 74$

$= 64 - 128 + 74 = 10$

The shortest distance from O to the line of intersection of the planes is $\sqrt{10}$

Points on the line of intersection of the two planes can be found by solving simultaneous the Cartesian equations of the two planes. As there are 2 equations in 3 unknowns, there are infinitely many solutions. A free choice can be made for one variable, here x is given the parameter t, and the other variables cab then be found in terms of t.

This is the equivalent of the parametric equations of the common line $x = t,\ y = 5 - 2t,\ z = 7 - 2t$
The equivalent vector equation of this line is $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

A calculus method of finding the minimum distance is shown here. You could instead use the property that, at the shortest distance, the position vector of the point is perpendicular to the common line. This method is illustrated in Question **13**.

b The line of intersection of P_1 and P_2 has vector equation $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

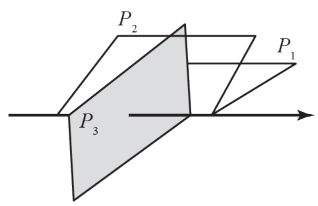
Hence the vector $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is perpendicular to Π_3

An equation of P_3 is

$\mathbf{r}\cdot(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = (2\mathbf{j} + \mathbf{k})\cdot(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

$= -4 - 2 = -6$

The common on line of P_1 and P_2 is a normal to the plane P_3 which is perpendicular to P_1 and P_2



c Substituting $(t, 5-2t, 7-2t)$ into $x - 2y - 2z = -6$

$t - 2(5-2t) - 2(7-2t) = -6$

$t - 10 + 4t - 14 + 4t = -6 \Rightarrow 9t = 18 \Rightarrow t = 2$

The position vector of the common point is

$t\mathbf{i} + (5-2t)\mathbf{j} + (7-2t)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

The point that lies on the three planes is given by substituting the general point on the line of intersection of P_1 and P_2 into the Cartesian equation of P_3

20 a $l:\mathbf{r}=\mathbf{j}+3\mathbf{k}+t(2\mathbf{i}+\mathbf{j}-\mathbf{k})$

$m:\mathbf{r}=\mathbf{i}+\mathbf{j}-\mathbf{k}+u(-2\mathbf{i}+\mathbf{j}+\mathbf{k})$

Equating the x-components gives:

$2t=1-2u$ **(1)**

Equating the y-components gives:

$1+t=1+u\Rightarrow t=u$ **(2)**

Substituting **(2)** into **(1)** gives:

$2u=1-2u\Rightarrow u=\frac{1}{4}$

Checking the z-components:

For l:

$3-t=3-\frac{1}{4}=\frac{11}{4}$

For m:

$-1+u=-1+\frac{1}{4}=-\frac{3}{4}$

For l:

$\frac{11}{4}\neq-\frac{3}{4}$ therefore the lines do not intersect

b $\overrightarrow{AB}=\begin{pmatrix}1-2u_1\\1+u_1\\-1+u_1\end{pmatrix}-\begin{pmatrix}2t_1\\1+t_1\\3-t_1\end{pmatrix}$

$=\begin{pmatrix}1-2u_1-2t_1\\u_1-t_1\\-4+u_1+t_1\end{pmatrix}$

$=(1-2u_1-2t_1)\mathbf{i}+(u_1-t_1)\mathbf{j}+(-4+u_1+t_1)\mathbf{k}$

20 c Since $\overrightarrow{AB}$ is perpendicular to the direction vectors of both l and m, form an equation for each line:

(1) $\overrightarrow{AB}\cdot(2\mathbf{i}+\mathbf{j}-\mathbf{k})=0$

(2) $\overrightarrow{AB}\cdot(-2\mathbf{i}+\mathbf{j}+\mathbf{k})=0$

Substituting the expression from part **b** and simplifying gives:

(1) $\begin{pmatrix}1-2u_1-2t_1\\ u_1+t_1\\ -4+u_1+t_1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-1\end{pmatrix}=0$

$(2-4u_1-4t_1)+(u_1-t_1)+(4-u_1-t_1)=0$

$6-4u_1-6t_1=0$

(2) $\begin{pmatrix}1-2u_1-2t_1\\ u_1+t_1\\ -4+u_1+t_1\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\1\end{pmatrix}=0$

$(-2+4u_1+4t_1)+(u_1-t_1)+(-4+u_1+t_1)=0$

$-6+6u_1+4t_1=0$

Adding 1.5 × **(1)** to **(2)** gives $3+0-5t_1=0$

$t_1=\frac{3}{5}$

Substituting back into **(2)** gives $-6+6u_1+\frac{12}{5}=0$

$6u_1=\frac{18}{5}$

$u_1=\frac{3}{5}$

Then $\overrightarrow{AB}=\begin{pmatrix}1-2\times\frac{3}{5}-2\times\frac{3}{5}\\ \frac{3}{5}-\frac{3}{5}\\ -4+\frac{3}{5}+\frac{3}{5}\end{pmatrix}=\frac{1}{5}\begin{pmatrix}5-6-6\\0-0\\-20+6+6\end{pmatrix}=\frac{1}{5}\begin{pmatrix}-7\\0\\-14\end{pmatrix}=-\frac{7}{5}\begin{pmatrix}1\\0\\2\end{pmatrix}$

$\left|\overrightarrow{AB}\right|=\frac{7}{5}\sqrt{1^2+0^2+2^2}=\frac{7\sqrt{5}}{5}$ or equivalently $\frac{7}{\sqrt{5}}$ as required.

21 $\mathbf{A}=\begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix}$

$$\mathbf{A}^n=\begin{pmatrix}1&n&\frac{1}{2}(n^2+3n)\\0&1&n\\0&0&1\end{pmatrix}$$

Let $n=1$

$$\mathbf{A}^1=\begin{pmatrix}1&1&\frac{1}{2}(1^2+3\times1)\\0&1&1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix}$$

$$=\mathbf{A}$$

The formula is true for $n=1$
Assume the formula is true for $n=k$
That is:

$$\mathbf{A}^k=\begin{pmatrix}1&k&\frac{1}{2}(k^2+3k)\\0&1&k\\0&0&1\end{pmatrix}$$

$\mathbf{A}^{k+1}=\mathbf{A}^k\mathbf{A}$

$$=\begin{pmatrix}1&k&\frac{1}{2}(k^2+3k)\\0&1&k\\0&0&1\end{pmatrix}\begin{pmatrix}1&1&2\\0&1&1\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&1+k&2+k+\frac{1}{2}(k^2+3k)\\0&1&1+k\\0&0&1\end{pmatrix}$$

$$2+k+\frac{1}{2}(k^2+3k)=\frac{1}{2}k^2+\frac{3k}{2}+k+2$$

$$=\frac{1}{2}(k^2+5k+4)$$

$$=\frac{1}{2}(k^2+2k+1+3k+3)$$

$$=\frac{1}{2}\left((k+1)^2+3(k+1)\right)$$

Therefore:

$$\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 1+k & \frac{1}{2}\left((k+1)^2 + 3(k+1)\right) \\ 0 & 1 & 1+k \\ 0 & 0 & 1 \end{pmatrix}$$

This is the formula with $k + 1$ substituted for n.
Hence, if the formula is true for $n = k$, then it is true for $n = k + 1$.
As the formula is true for $n = 1$, by mathematical induction the formula is true for all positive integers n.

22 a $\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$

$$\det(\mathbf{A}) = k\begin{vmatrix} -1 & k \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 0 & k \\ 9 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 \\ 9 & 1 \end{vmatrix}$$
$$= k(0-k) - (0-9k) - 2(0+9)$$
$$= -k^2 + 9k - 18$$

If the matrix is singular, then $\det(\mathbf{A}) = 0$

$-k^2 + 9k - 18 = 0$

$k^2 - 9k + 18 = 0$

$(k-3)(k-6) = 0$

$k = 3$ or $k = 6$

22 b $\mathbf{A} = \begin{pmatrix} k & 1 & -2 \\ 0 & -1 & k \\ 9 & 1 & 0 \end{pmatrix}$

Step 1:
$\det(\mathbf{A}) = -k^2 + 9k - 18$

Step 2:

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} -1 & k \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 0 & k \\ 9 & 0 \end{vmatrix} & \begin{vmatrix} 0 & -1 \\ 9 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} k & -2 \\ 9 & 0 \end{vmatrix} & \begin{vmatrix} k & 1 \\ 9 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ -1 & k \end{vmatrix} & \begin{vmatrix} k & -2 \\ 0 & k \end{vmatrix} & \begin{vmatrix} k & 1 \\ 0 & -1 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0-k & 0-9k & 0+9 \\ 0+2 & 0+18 & k-9 \\ k-2 & k^2-0 & -k-0 \end{pmatrix}$$

$$= \begin{pmatrix} -k & -9k & 9 \\ 2 & 18 & k-9 \\ k-2 & k^2 & -k \end{pmatrix}$$

Step 3:

$$\mathbf{C} = \begin{pmatrix} -k & 9k & 9 \\ -2 & 18 & 9-k \\ k-2 & -k^2 & -k \end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & 9-k & -k \end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$= \frac{1}{-k^2+9k-18}\begin{pmatrix} -k & -2 & k-2 \\ 9k & 18 & -k^2 \\ 9 & 9-k & -k \end{pmatrix}$$

23 a $\mathbf{M} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}$

$$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix}$$

$$\mathbf{MM}^{\mathrm{T}} = \begin{pmatrix} 1 & 4 & -1 \\ 3 & 0 & p \\ a & b & c \end{pmatrix}\begin{pmatrix} 1 & 3 & a \\ 4 & 0 & b \\ -1 & p & c \end{pmatrix}$$

$$= \begin{pmatrix} 1+16+1 & 3+0-p & a+4b-c \\ 3+0-p & 9+0+p^2 & 3a+0+cp \\ a+4b-c & 3a+0+cp & a^2+b^2+c^2 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+cp \\ a+4b-c & 3a+cp & a^2+b^2+c^2 \end{pmatrix}$$

Since $\mathbf{MM}^{\mathrm{T}} = k\mathbf{I}$

$$\begin{pmatrix} 18 & 3-p & a+4b-c \\ 3-p & 9+p^2 & 3a+cp \\ a+4b-c & 3a+cp & a^2+b^2+c^2 \end{pmatrix} = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{pmatrix}$$

Hence:

$3-p=0 \Rightarrow p=3$

b $k = 18$

c $a+4b-c=0$ **(1)**

$3a+cp=0 \Rightarrow 3a+3c=0 \Rightarrow c=-a$ **(2)**

$a^2+b^2+c^2=18$ **(3)**

Substituting $c=-a$ into **(1)** gives:

$$a+4b+a=0 \Rightarrow a=-2b \Rightarrow b=-\frac{a}{2}$$

Substituting $c=-a$ and $b=-\frac{a}{2}$ into **(3)** gives:

$$a^2+\left(-\frac{a}{2}\right)^2+(-a)^2=18$$

$$\frac{9a^2}{4}=18$$

$a^2=8$

$a=2\sqrt{2}$ (as it was given that $a>0$)

$$b=-\frac{2\sqrt{2}}{2}=-\sqrt{2}$$

$c=-2\sqrt{2}$

23 d $\mathbf{M}=\begin{pmatrix}1 & 4 & -1\\ 3 & 0 & 3\\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2}\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{M}) &= 1\begin{vmatrix}0 & 3\\ -\sqrt{2} & -2\sqrt{2}\end{vmatrix}-4\begin{vmatrix}3 & 3\\ 2\sqrt{2} & -2\sqrt{2}\end{vmatrix}-1\begin{vmatrix}3 & 0\\ 2\sqrt{2} & -\sqrt{2}\end{vmatrix}\\ &= 1\left(0+3\sqrt{2}\right)-4\left(-6\sqrt{2}-6\sqrt{2}\right)-1\left(-3\sqrt{2}-0\right)\\ &= 3\sqrt{2}+48\sqrt{2}+3\sqrt{2}\\ &= 54\sqrt{2}\end{aligned}$$

24 a $\mathbf{A}=\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}$

$$\begin{aligned}\mathbf{A}^2 &= \begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}\\ &= \begin{pmatrix}1+0+2 & 1+2+0 & 2+1+4\\ 0+0+1 & 0+4+0 & 0+2+2\\ 1+0+2 & 1+0+0 & 2+0+4\end{pmatrix}\\ &= \begin{pmatrix}3 & 3 & 7\\ 1 & 4 & 4\\ 3 & 1 & 6\end{pmatrix}\end{aligned}$$

b $$\begin{aligned}5\mathbf{A}^2-6\mathbf{A}+\mathbf{I} &= 5\begin{pmatrix}3 & 3 & 7\\ 1 & 4 & 4\\ 3 & 1 & 6\end{pmatrix}-6\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}+\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}15 & 15 & 35\\ 5 & 20 & 20\\ 15 & 5 & 30\end{pmatrix}-\begin{pmatrix}6 & 6 & 12\\ 0 & 12 & 6\\ 6 & 0 & 12\end{pmatrix}+\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}15-6+1 & 15-6+0 & 35-12+0\\ 5-0+0 & 20-12+1 & 20-6+0\\ 15-6+0 & 5-0+0 & 30-12+1\end{pmatrix}\\ &= \begin{pmatrix}10 & 9 & 23\\ 5 & 9 & 14\\ 9 & 5 & 19\end{pmatrix}=\mathbf{A}^3\end{aligned}$$

Therefore:

$5\mathbf{A}^2-6\mathbf{A}+\mathbf{I}=\mathbf{A}^3$

So:

$\mathbf{A}^3-5\mathbf{A}^2+6\mathbf{A}-\mathbf{I}=\mathbf{0}$ as required

24 c $\mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} - \mathbf{I} = \mathbf{0}$

$\mathbf{A}^3 - 5\mathbf{A}^2 + 6\mathbf{A} = \mathbf{I}$

$\mathbf{A}\left(\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}\right) = \mathbf{I}$

$\mathbf{A}(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 3\mathbf{I}) = \mathbf{I}$ as required

d $\mathbf{A}\left(\mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}\right) = \mathbf{I}$ and $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

Therefore:

$$\mathbf{A}^{-1} = \mathbf{A}^2 - 5\mathbf{A} + 6\mathbf{I}$$

$$= \begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix} - 5\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & 7 \\ 1 & 4 & 4 \\ 3 & 1 & 6 \end{pmatrix} - \begin{pmatrix} 5 & 5 & 10 \\ 0 & 10 & 5 \\ 5 & 0 & 10 \end{pmatrix} + \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 3-5+6 & 3-5 & 7-10 \\ 1 & 4-10+6 & 4-5 \\ 3-5 & 1 & 6-10+6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 1 & 2 \end{pmatrix}$$

25 a $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+4+0 & 0+2+1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

25 b $\mathbf{A}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+8+0 & 0+4+3 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 8 & 7 \\ 0 & 0 & 1 \end{pmatrix}$$

25 c $\mathbf{A}^n = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^n & 2^n - 1 \\ 0 & 0 & 1 \end{pmatrix}$

Let $n = 1$

$\mathbf{A}^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^1 & 2^1 - 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \mathbf{A}$

This formula is true for $n = 1$

Assume the formula is true for $n = k$, that is:

$\mathbf{A}^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 2^k - 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\mathbf{A}^{k+1} = \mathbf{A}^k \cdot \mathbf{A}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^k & 2^k - 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+2\times 2^k+0 & 0+2^k+2^k-1 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{k+1} & 2^{k+1} - 1 \\ 0 & 0 & 1 \end{pmatrix}$

This is the formula with $k + 1$ substituted for n.

Hence, if the formula is true for $n = k$, then it is true for $n = k + 1$.

As the formula is true for $n = 1$, by mathematical induction it is true for all positive integers n.

25 d $\mathbf{A}^{k+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{k+1} & 2^{k+1}-1 \\ 0 & 0 & 1 \end{pmatrix}$

To find the inverse of $\mathbf{A}^n$ let $k = -n-1$

$$\mathbf{A}^{-n-1+1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n-1+1} & 2^{-n-1+1}-1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A}^{-n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2^{-n} & 2^{-n}-1 \\ 0 & 0 & 1 \end{pmatrix}$$

26 a $\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 5 & 3 & u \end{pmatrix}$

$$\det(\mathbf{A}) = 3\begin{vmatrix} 1 & 1 \\ 3 & u \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 5 & u \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix}$$
$$= 3(u-3) - 1(u-5) - 1(3-5)$$
$$= 3u - 9 - u + 5 + 2$$
$$= 2u - 2$$
$$= 2(u-1) \text{ as required}$$

26 b $\mathbf{A}=\begin{pmatrix}3 & 1 & -1\\ 1 & 1 & 1\\ 5 & 3 & u\end{pmatrix}$

Step 1:

$\det(\mathbf{A})=2(u-1)$

Step 2:

$$\mathbf{M}=\begin{pmatrix}\begin{vmatrix}1 & 1\\ 3 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\ 5 & u\end{vmatrix} & \begin{vmatrix}1 & 1\\ 5 & 3\end{vmatrix}\\ \begin{vmatrix}1 & -1\\ 3 & u\end{vmatrix} & \begin{vmatrix}3 & -1\\ 5 & u\end{vmatrix} & \begin{vmatrix}3 & 1\\ 5 & 3\end{vmatrix}\\ \begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}3 & -1\\ 1 & 1\end{vmatrix} & \begin{vmatrix}3 & 1\\ 1 & 1\end{vmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}u-3 & u-5 & 3-5\\ u+3 & 3u+5 & 9-5\\ 1+1 & 3+1 & 3-1\end{pmatrix}$$

$$=\begin{pmatrix}u-3 & u-5 & -2\\ u+3 & 3u+5 & 4\\ 2 & 4 & 2\end{pmatrix}$$

Step 3:

$$\mathbf{C}=\begin{pmatrix}u-3 & 5-u & -2\\ -u-3 & 3u+5 & -4\\ 2 & -4 & 2\end{pmatrix}$$

Step 4:

$$\mathbf{C}^{\mathrm{T}}=\begin{pmatrix}u-3 & -u-3 & 2\\ 5-u & 3u+5 & -4\\ -2 & -4 & 2\end{pmatrix}$$

Step 5:

$$\mathbf{A}^{-1}=\frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$

$$=\frac{1}{2(u-1)}\begin{pmatrix}u-3 & -u-3 & 2\\ 5-u & 3u+5 & -4\\ -2 & -4 & 2\end{pmatrix}$$

26 c When $u = 6$

$$\mathbf{A}^{-1} = \frac{1}{2(6-1)}\begin{pmatrix} 6-3 & -6-3 & 2 \\ 5-6 & 3\times 6+5 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

Hence:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 3 & -9 & 2 \\ -1 & 23 & -4 \\ -2 & -4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 9-9+12 \\ -3+23-24 \\ -6-4+12 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 12 \\ -4 \\ 2 \end{pmatrix}$$

Therefore:

$a = 1.2$, $b = -0.4$, $c = 0.2$

27 a $\mathbf{M} = \begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}$ and $\mathbf{M}^{-1} = \begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}$

$\mathbf{MM}^{-1} = \mathbf{I}$

$$\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & a & 0 \\ 2 & b & 0 \\ c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9+2a+0 & 3a+ab+0 & 0+0+0 \\ 6+2b+0 & 2a+b^2+0 & 0+0+0 \\ 3c+0+c & ac+0+0 & 0+0+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 9+2a & 3a+ab & 0 \\ 6+2b & 2a+b^2 & 0 \\ 4c & ac & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$9+2a = 1 \Rightarrow a = -4$
$6+2b = 0 \Rightarrow b = -3$
$4c = 0 \Rightarrow c = 0$

b $\mathbf{M} = \begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\begin{aligned}\det(\mathbf{M}) &= 3\begin{vmatrix} -3 & 0 \\ 0 & 1 \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} \\ &= 3(-3-0)+4(2-0) \\ &= -1\end{aligned}$$

c If there is a line of invariant points then **M** must have an eigenvalue of 1.
Find the corresponding eigenvector:

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x-4y \\ 2x-3y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements gives:
$3x - 4y = x \Rightarrow x = 2y$
The points which remain invariant under R are satisfied by $x = 2y$

28 a $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2a-b \\ 2d-e \\ 2g-h \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -b+2c \\ -e+2f \\ -h+2i \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$

$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} \alpha \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\alpha+1 \\ 5 \\ 2\alpha+2 \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha a+c \\ \alpha d+f \\ \alpha g+i \end{pmatrix} = \begin{pmatrix} -\alpha+1 \\ 5 \\ 2\alpha+2 \end{pmatrix}$

Equating the bottom elements gives:

$2g-h=0 \Rightarrow h=2g$

$-h+2i=0 \Rightarrow -2g+2i=0 \Rightarrow g=i$

$\alpha g+i=2\alpha+2 \Rightarrow \alpha i+i=2\alpha+2 \Rightarrow i=2$

When $i = 2$, $g = 2$ and $h = 4$

Equating the middle elements gives:

$\alpha d+f=5$

$-e+2f=9 \Rightarrow -e=9-2f$

$2d-e=-1 \Rightarrow 2d+9-2f=-1 \Rightarrow 2d-2f=-10 \Rightarrow f=d+5$

Substituting $f=d+5$ into $\alpha d+f=5$ gives:

$\alpha d+d+5=5$

$d(\alpha+1)=0$

$d=0$

When $d = 0$, $e = 1$ and $f = 5$

Equating the top elements gives:

$2a-b=-5 \Rightarrow -b=-5-2a$

$-b+2c=-1 \Rightarrow -5-2a+2c=-1 \Rightarrow a-c=-2$

$\alpha a+c=-\alpha+1$

Subtracting $a-c=-2$ from $\alpha a+c=-\alpha+1$ gives:

$\alpha a+a=-\alpha-1$

$a(\alpha+1)=-(\alpha+1)$

$a=-1$

When $a = -1$, $b = 3$ and $c = 1$

Therefore:

$\mathbf{M} = \begin{pmatrix} -1 & 3 & 1 \\ 0 & 1 & 5 \\ 2 & 4 & 2 \end{pmatrix}$

28 b Π_1: $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$

These are the three vectors given in the first part of the question, with α = 3
Therefore the images of each vector are already known:

$$\mathbf{M}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3+1 \\ 5 \\ 2\times 3+2 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ 8 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$$

Substituting into the vector equation:

$$\Pi_2: \mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 9 \\ 0 \end{pmatrix}$$

Hence a Cartesian equation of the plane is $z = 8$

29 a **(1)** $x + y - z = a$
(2) $y + z = b$
(3) $z = c$
Substituting the value of z into **(2)** gives $y = b - c$
Substituting the values of y and z into **(1)** gives $x + (b - c) - c = a$
$x = a - b + 2c$
Therefore:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

So: $\mathbf{A}^{-1} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

29 b $\mathbf{B} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$

$\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

$\mathbf{BB}^{\mathrm{T}} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1+4+4 & 2+2-4 & 2-4+2 \\ 2+2-4 & 4+1+4 & 4-2-2 \\ 2-4+2 & 4-2-2 & 4+4+1 \end{pmatrix}$

$= \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$

Since $\mathbf{BB}^{\mathrm{T}} = k\mathbf{I}$

$k = 9$

29 c $\mathbf{BB}^{\mathrm{T}} = 9\mathbf{I}$ and $\mathbf{BB}^{-1} = \mathbf{I}$

Hence:

$$\mathbf{B}^{-1} = \frac{1}{9}\mathbf{B}^{\mathrm{T}}$$

$$= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}$$

$$\mathbf{A}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0+2 \\ 0+0-1 \\ 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\mathbf{B}\begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \mathbf{B}^{-1}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$$

$$\frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 3-2+2 \\ -6+1+2 \\ 6+2+1 \end{pmatrix}$$

$$= \frac{1}{9}\begin{pmatrix} 3 \\ -3 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

30 a $\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$

$$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{pmatrix}$$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (4-\lambda)(-9-\lambda) + 30 = -36 - 4\lambda + 9\lambda + \lambda^2 + 30 = \lambda^2 + 5\lambda - 6 = (\lambda - 1)(\lambda + 6)$$

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

$(\lambda - 1)(\lambda + 6) = 0$

$\lambda = 1$ or $\lambda = -6$

Therefore, the eigenvalues of **M** are 1 and −6

b For the eigenvalue 1

$$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 4x - 5y \\ 6x - 9y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating the upper elements gives:

$$4x - 5y = x \Rightarrow 3x = 5y \Rightarrow y = \frac{3}{5}x$$

The points which remain invariant under R are satisfied by $y = \frac{3}{5}x$

31 a $\mathbf{A} = \begin{pmatrix} k & 2 \\ 2 & -1 \end{pmatrix}$ and $y = 2x + 1$

When $k = -4$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ 2x+1 \end{pmatrix} = \begin{pmatrix} -4x + 4x + 2 \\ 2x - 2x - 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

So the image of the line $y = 2x$ is the point (2, –1)

31 b When $k = 2$

$$\mathbf{A} = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= -(2-\lambda)(1+\lambda) - 4 \\ &= -(2+\lambda-\lambda^2) - 4 \\ &= \lambda^2 - \lambda - 6 \\ &= (\lambda-3)(\lambda+2) \end{aligned}$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

$(\lambda-3)(\lambda+2) = 0$

$\lambda = -2$ or $\lambda = 3$

Therefore, the eigenvalues of **A** are −2 and 3

c For the eigenvalue −2

$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

Equating the upper elements gives:

$2x + 2y = -2x \Rightarrow 4x = -2y \Rightarrow y = -2x$

For the eigenvalue 3

$$\begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y \\ 2x-y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

Equating the upper elements gives:

$2x + 2y = 3x \Rightarrow x = 2y \Rightarrow y = \frac{1}{2}x$

The required equations are $y = -2x$ and $y = \frac{1}{2}x$

32 a $\mathbf{M}=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}$

$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{M}-\lambda\mathbf{I}) &= (4-\lambda)(1-\lambda)+2\\ &= 4-5\lambda+\lambda^2+2\\ &= \lambda^2-5\lambda+6\\ &= (\lambda-2)(\lambda-3)\end{aligned}$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0$

$(\lambda-2)(\lambda-3)=0$

$\lambda=2$ or $\lambda=3$

Therefore, the eigenvalues of **M** are 2 and 3

Therefore:

$\lambda_1=2$ and $\lambda_2=3$

b $\mathbf{M}=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{M}) &= 4+2\\ &= 6\end{aligned}$$

$\mathbf{M}^{-1}=\frac{1}{6}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}$

32 c $\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{2}{6} \\ -\frac{1}{6} & \frac{4}{6} \end{pmatrix}$

$\mathbf{M}^{-1} - \lambda\mathbf{I} = \begin{pmatrix} \frac{1}{6}-\lambda & \frac{2}{6} \\ -\frac{1}{6} & \frac{4}{6}-\lambda \end{pmatrix}$

$$\begin{aligned}\det\left(\mathbf{M}^{-1} - \lambda\mathbf{I}\right) &= \left(\frac{1}{6}-\lambda\right)\left(\frac{4}{6}-\lambda\right)+\frac{2}{36} \\ &= \frac{4}{36}-\frac{5}{6}\lambda+\lambda^2+\frac{2}{36} \\ &= \lambda^2-\frac{5}{6}\lambda+\frac{6}{36} \\ &= \left(\lambda-\frac{2}{6}\right)\left(\lambda-\frac{3}{6}\right)\end{aligned}$$

$\det\left(\mathbf{A} - \lambda\mathbf{I}\right) = 0$

$\left(\lambda-\frac{2}{6}\right)\left(\lambda-\frac{3}{6}\right) = 0$

$\lambda = \frac{2}{6}$ or $\lambda = \frac{3}{6}$

Therefore, the eigenvalues of $\mathbf{M}^{-1}$ are $\frac{1}{3}$ and $\frac{1}{2}$

Hence the eigenvalues of $\mathbf{M}^{-1}$ are λ_1^{-1} and λ_2^{-1} as required.

d For eigenvalue 2

$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 4x-2y \\ x-y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

Equating the upper elements gives:

$4x-2y = 2x \Rightarrow y = x$

For eigenvalue 3

$\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} 4x-2y \\ x-y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

Equating the upper elements gives:

$4x-2y = 3x \Rightarrow 2y = x \Rightarrow y = \frac{1}{2}x$

The required equations are $y = x$ and $y = \frac{1}{2}x$

33 $\mathbf{M} = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{pmatrix}$

$$\det(\mathbf{M} - \lambda\mathbf{I}) = (2-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 2 & -4-\lambda \end{vmatrix} + 3\begin{vmatrix} 3 & 3 \\ -5 & -4-\lambda \end{vmatrix} + 1\begin{vmatrix} 3 & 1-\lambda \\ -5 & 2 \end{vmatrix}$$
$$= (2-\lambda)\left[(1-\lambda)(-4-\lambda)-6\right] + 3\left[3(-4-\lambda)+15\right] + 1\left[6+5(1-\lambda)\right]$$
$$= (2-\lambda)(-4+3\lambda+\lambda^2-6) + 3(-12-3\lambda+15) + 1(6+5-5\lambda)$$
$$= (2-\lambda)(\lambda^2+3\lambda-10) + 3(3-3\lambda) + 11 - 5\lambda$$
$$= 2\lambda^2 + 6\lambda - 20 - \lambda^3 - 3\lambda^2 + 10\lambda + 9 - 9\lambda + 11 - 5\lambda$$
$$= -\lambda^3 - \lambda^2 + 2\lambda$$
$$= -\lambda(\lambda^2 + \lambda - 2)$$
$$= -\lambda(\lambda-1)(\lambda+2)$$

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

$-\lambda(\lambda-1)(\lambda+2) = 0$

$\lambda = -2$ or $\lambda = 0$ or $\lambda = 1$

For eigenvalue -2

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

Equating the upper elements gives:

$2x - 3y + z = -2x \Rightarrow 3y = 4x + z$

Equating the middle elements gives:

$3x + y + 3z = -2y \Rightarrow 3y = -3x - 3z$

$4x + z = -3x - 3z \Rightarrow 7x = -4z$

Setting $z = -7$ gives $x = 4$

Substituting $z = -7$ and $x = 4$ into $4x - 3y + z = 0$ gives:

$16 - 3y - 7 = 0 \Rightarrow y = 3$

Hence the eigenvector corresponding to eigenvalue -2 is $\begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix}$

For eigenvalue 0

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the upper elements gives:

$2x-3y+z=0 \Rightarrow z=3y-2x$

Equating the middle elements gives:

$3x+y+3z=0 \Rightarrow 3z=-y-3x$

$3(3y-2x)=-y-3x$

$9y-6x=-y-3x \Rightarrow 10y=3x$

Setting $x = 10$ gives $y = 3$

Substituting $x = 10$ and $y = 3$ into $2x-3y+z=0$ gives:

$2\times10-3\times3+z=0 \Rightarrow z=-11$

Hence the eigenvector corresponding to eigenvalue 0 is $\begin{pmatrix} 10 \\ 3 \\ -11 \end{pmatrix}$

For eigenvalue 1

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-3y+z \\ 3x+y+3z \\ -5x+2y-4z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the middle elements gives:

$3x+y+3z=y \Rightarrow x=-z$

Setting $x = 1$ gives $z=-1$

Equating the top elements and substituting $x = 1$ and $z=-1$ gives:

$2-3y-1=1 \Rightarrow y=0$

Hence the eigenvector corresponding to eigenvalue 1 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

34 a $\mathbf{A}=\begin{pmatrix}3&4&p\\-1&q&-4\\1&1&3\end{pmatrix}$

$$\begin{pmatrix}3&4&p\\-1&q&-4\\1&1&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=k\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3&4&p\\-1&q&-4\\1&1&3\end{pmatrix}\begin{pmatrix}0\\1\\-1\end{pmatrix}=k\begin{pmatrix}0\\1\\-1\end{pmatrix}$$

$$\begin{pmatrix}4-p\\q+4\\1-3\end{pmatrix}=\begin{pmatrix}0\\k\\-k\end{pmatrix}$$

Equating the lower elements gives:
$1-3=-k\Rightarrow k=2$

b $\begin{pmatrix}4-p\\q+4\\1-3\end{pmatrix}=\begin{pmatrix}0\\2\\-2\end{pmatrix}$

Equating the upper elements gives:
$4-p=0\Rightarrow p=4$
Equating the middle elements gives:
$q+4=2\Rightarrow q=-2$

c $\mathbf{A}=\begin{pmatrix}3&4&4\\-1&-2&-4\\1&1&3\end{pmatrix}$

$$\begin{pmatrix}3&4&4\\-1&-2&-4\\1&1&3\end{pmatrix}\begin{pmatrix}l\\m\\n\end{pmatrix}=\begin{pmatrix}10\\-4\\3\end{pmatrix}$$

$$\begin{pmatrix}3l+4m+4n\\-l-2m-4n\\l+m+3n\end{pmatrix}=\begin{pmatrix}10\\-4\\3\end{pmatrix}$$

$3l+4m+4n=10$ **(1)**
$-l-2m-4n=-4$ **(2)**
$l+m+3n=3$ **(3)**
Adding **(2)** and **(3)** gives:
$-m-n=-1$ **(4)**
Adding **(1)** and 3 × **(2)** gives:
$-2m-8n=-2$ **(5)**
Adding −2 × **(4)** and **(5)** gives:
$-6n=0\Rightarrow n=0$
When $n=0$, $m=1$ and $l=2$

35 a $\mathbf{A}=\begin{pmatrix}5 & 1 & -2\\ -1 & 6 & 1\\ 0 & 1 & 3\end{pmatrix}$

$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}5-\lambda & 1 & -2\\ -1 & 6-\lambda & 1\\ 0 & 1 & 3-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=(5-\lambda)\begin{vmatrix}6-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix}-1\begin{vmatrix}-1 & 1\\ 0 & 3-\lambda\end{vmatrix}-2\begin{vmatrix}-1 & 6-\lambda\\ 0 & 1\end{vmatrix}$

If $\lambda=3$ is an eigenvalue of **A**, then:

$2\begin{vmatrix}3 & 1\\ 1 & 0\end{vmatrix}-1\begin{vmatrix}-1 & 1\\ 0 & 0\end{vmatrix}-2\begin{vmatrix}-1 & 3\\ 0 & 1\end{vmatrix}=0$

$2(0-1)-1(0-0)-2(-1-0)=0$

$-2+2=0$

Therefore, $\lambda=3$ is an eigenvalue of **A**.

b $\det(\mathbf{A}-\lambda\mathbf{I})=(5-\lambda)\begin{vmatrix}6-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix}-1\begin{vmatrix}-1 & 1\\ 0 & 3-\lambda\end{vmatrix}-2\begin{vmatrix}-1 & 6-\lambda\\ 0 & 1\end{vmatrix}$

$=(5-\lambda)[(6-\lambda)(3-\lambda)-1]-1[-1(3-\lambda)-0]-2(-1-0)$

$=(5-\lambda)[(6-\lambda)(3-\lambda)-1]-1(\lambda-3)+2$

$=(5-\lambda)[(6-\lambda)(3-\lambda)-1]+(5-\lambda)$

$=(5-\lambda)[(6-\lambda)(3-\lambda)-1+1]$

$=(5-\lambda)(6-\lambda)(3-\lambda)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0$

$(5-\lambda)(6-\lambda)(3-\lambda)=0$

$\lambda=3$ or $\lambda=5$ or $\lambda=6$

c $\begin{pmatrix}5 & 1 & -2\\ -1 & 6 & 1\\ 0 & 1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=3\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

The bottom row gives: $y+3z=3z\Rightarrow y=0$

After substituting $y=0$, the middle row gives: $-x+0+z=0\Rightarrow x=z$

Setting $x=1$ gives $z=1$

So $\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$ is an eigenvector, and has magnitude $\sqrt{1^2+0^2+1^2}=\sqrt{2}$

So a normalised eigenvector of **A** for the eigenvalue 3 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

36 a $\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & k \end{pmatrix}$

$$\det(\mathbf{A}) = 3\begin{vmatrix} 0 & 2 \\ 2 & k \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 4 & k \end{vmatrix} + 4\begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix}$$
$$= 3(0-4) - 2(2k-8) + 4(4-0)$$
$$= -12 - 4k + 16 + 16$$
$$= 20 - 4k \text{ as required}$$

b Step 1

$\det(\mathbf{A}) = 20 - 4k$

Step 2

$$\mathbf{M} = \begin{pmatrix} \begin{vmatrix} 0 & 2 \\ 2 & k \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 4 & k \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ 2 & k \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 4 & k \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$
$$= \begin{pmatrix} 0-4 & 2k-8 & 4-0 \\ 2k-8 & 3k-16 & 6-8 \\ 4-0 & 6-8 & 0-4 \end{pmatrix}$$
$$= \begin{pmatrix} -4 & 2k-8 & 4 \\ 2k-8 & 3k-16 & -2 \\ 4 & -2 & -4 \end{pmatrix}$$

Step 3

$$\mathbf{C} = \begin{pmatrix} -4 & 8-2k & 4 \\ 8-2k & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

Step 4

$$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -4 & 8-2k & 4 \\ 8-2k & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

Step 5

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathbf{C}^{\mathrm{T}}$$
$$= \frac{1}{20-4k}\begin{pmatrix} -4 & 8-2k & 4 \\ 8-2k & 3k-16 & 2 \\ 4 & 2 & -4 \end{pmatrix}$$

36 c When $k = 3$ and $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of **A**:

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 0+4-4 \\ 0+0-2 \\ 0+4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2\lambda \\ -\lambda \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2k \\ -k \end{pmatrix}$$

Equating the elements of the lower row gives $\lambda = -1$

d To find eigenvector of **A** for the eigenvalue 8

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x+2y+4z \\ 2x+2z \\ 4x+2y+3z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

$3x+2y+4z = 8x \Rightarrow -5x+2y+4z = 0$ **(1)**

$2x+2z = 8y \Rightarrow 2x-8y+2z = 0$ **(2)**

$4x+2y+3z = 8z \Rightarrow 4x+2y-5z = 0$ **(3)**

Subtracting 2 × **(2)** from **(3)** gives:

$18y - 9z = 0 \Rightarrow 2y = z$

Setting $y = 1$ gives $z = 2$

Substituting into **(3)** gives:

$4x + 2\times 1 - 5\times 2 = 0 \Rightarrow 4x = 8 \Rightarrow x = 2$

Hence an eigenvector of **A** for the eigenvalue 8 is $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

37 a $\mathbf{A}=\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}$ and $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is an eigenvector

$$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}=k\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

$$\begin{pmatrix}2+0+4\\0-10+4\\8-8+3\end{pmatrix}=\begin{pmatrix}2k\\-2k\\k\end{pmatrix}$$

$$\begin{pmatrix}6\\-6\\3\end{pmatrix}=\begin{pmatrix}2k\\-2k\\k\end{pmatrix}$$

Equating the lower elements gives $k=3$
Hence 3 is an eigenvalue of **A**

b To check that $\lambda=9$ is an eigenvalue of **A**, check that det(**A** – 9**I**) = 0:

$$\mathbf{A}-9\mathbf{I}=\begin{pmatrix}1-9&0&4\\0&5-9&4\\4&4&3-9\end{pmatrix}=\begin{pmatrix}-8&0&4\\0&-4&4\\4&4&-6\end{pmatrix}$$

$$\begin{aligned}\det(\mathbf{A}-9\mathbf{I})&=-8\begin{vmatrix}-4&4\\4&-6\end{vmatrix}-0\begin{vmatrix}0&4\\4&-6\end{vmatrix}+4\begin{vmatrix}0&-4\\4&4\end{vmatrix}\\&=-8(24-16)+4(0+16)\\&=-64+64\\&=0\end{aligned}$$

Hence 9 is an eigenvalue of **A**.
To find an eigenvector of **A** for the eigenvalue 9

$$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=9\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+4z\\5y+4z\\4x+4y+3z\end{pmatrix}=\begin{pmatrix}9x\\9y\\9z\end{pmatrix}$$

Equating the top elements gives:
$x+4z=9x\Rightarrow z=2x$
Setting $x=1$ gives $z=2$ and $y=2$

Hence an eigenvector of **A** for the eigenvalue 9 is $\begin{pmatrix}1\\2\\2\end{pmatrix}$

37 c $\mathbf{A}=\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}$ and $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ is an eigenvector

$$\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}2\\1\\-2\end{pmatrix}=k\begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$$\begin{pmatrix}2+0-8\\0+5-8\\8+4-6\end{pmatrix}=\begin{pmatrix}2k\\k\\-2k\end{pmatrix}$$

$$\begin{pmatrix}-6\\-3\\6\end{pmatrix}=\begin{pmatrix}2k\\k\\-2k\end{pmatrix}$$

Equating the lower elements gives $k=-3$
Hence -3 is an eigenvalue of $\mathbf{A}$.

The eigenvectors of $\mathbf{A}$ are $\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}1\\2\\2\end{pmatrix}$

$\begin{pmatrix}2\\-2\\1\end{pmatrix}$ has magnitude $\sqrt{2^2+(-2)^2+1^2}=3$

Hence a normalised eigenvector of $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ is $\begin{pmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{1}{3}\end{pmatrix}$

$\begin{pmatrix}2\\1\\-2\end{pmatrix}$ has magnitude $\sqrt{2^2+1^2+(-2)^2}=3$

Hence a normalised eigenvector of $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ is $\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{pmatrix}$

$\begin{pmatrix}1\\2\\2\end{pmatrix}$ has magnitude $\sqrt{1^2+2^2+2^2}=3$

Hence a normalised eigenvector of $\begin{pmatrix}1\\2\\2\end{pmatrix}$ is $\begin{pmatrix}\frac{1}{3}\\\frac{2}{3}\\\frac{2}{3}\end{pmatrix}$

$$\mathbf{P}=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}=\begin{pmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\end{pmatrix}$$

$\mathbf{P}^{\mathrm{T}}\mathbf{AP}=\mathbf{D}$

$$\mathbf{D}=\begin{pmatrix}\frac{2}{3}&-\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&\frac{2}{3}\\\frac{2}{3}&\frac{1}{3}&-\frac{2}{3}\end{pmatrix}\begin{pmatrix}1&0&4\\0&5&4\\4&4&3\end{pmatrix}\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{2}{3}+0+\frac{4}{3}&0-\frac{10}{3}+\frac{4}{3}&\frac{8}{3}-\frac{8}{3}+1\\\frac{1}{3}+0+\frac{8}{3}&0+\frac{10}{3}+\frac{8}{3}&\frac{4}{3}+\frac{8}{3}+2\\\frac{2}{3}+0-\frac{8}{3}&0+\frac{5}{3}-\frac{8}{3}&\frac{8}{3}+\frac{4}{3}-2\end{pmatrix}\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}2&-2&1\\3&6&6\\-2&-1&2\end{pmatrix}\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{4}{3}+\frac{4}{3}+\frac{1}{3}&\frac{2}{3}-\frac{4}{3}+\frac{2}{3}&\frac{4}{3}-\frac{2}{3}-\frac{2}{3}\\2-4+2&1+4+4&2+2-4\\-\frac{4}{3}+\frac{2}{3}+\frac{2}{3}&-\frac{2}{3}-\frac{2}{3}+\frac{4}{4}&-\frac{4}{3}-\frac{1}{3}-\frac{4}{3}\end{pmatrix}=\begin{pmatrix}3&0&0\\0&9&0\\0&0&-3\end{pmatrix}$$

38 a $\mathbf{A}=\begin{pmatrix}6 & 2 & -3\\ 2 & 0 & 0\\ -3 & 0 & 2\end{pmatrix}$

$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}6-\lambda & 2 & -3\\ 2 & -\lambda & 0\\ -3 & 0 & 2-\lambda\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{A}-\lambda\mathbf{I}) &= (6-\lambda)\left[-\lambda(2-\lambda)-0\right]-2\left[2(2-\lambda)-0\right]-3(0-3\lambda)\\ &= -\lambda(2-\lambda)(6-\lambda)-4(2-\lambda)+9\lambda\\ &= -\lambda\left(12-8\lambda+\lambda^2\right)-8+4\lambda+9\lambda\\ &= -12\lambda+8\lambda^2-\lambda^3-8+13\lambda\\ &= -\lambda^3+8\lambda^2+\lambda-8\end{aligned}$$

$\mathbf{A}-\lambda\mathbf{I}=0$

$-\lambda^3+8\lambda^2+\lambda-8=0$

$(\lambda+1)$ and $(\lambda-8)$ are factors of $\lambda^3-8\lambda^2-\lambda+8$

By inspection $(\lambda-1)$ is a factor of $\lambda^3-8\lambda^2-\lambda+8$

Hence the third eigenvalue of **A** is 1

38 b To find the eigenvector of eigenvalue 8:

$$\begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x+2y-3z \\ 2x \\ -3x+2z \end{pmatrix} = \begin{pmatrix} 8x \\ 8y \\ 8z \end{pmatrix}$$

Equating the middle elements gives:

$2x = 8y \Rightarrow x = 4y$

Setting $y = 1$ gives $x = 4$

Equating the lower elements and substituting $x = 4$ gives:

$-12 + 2z = 8z \Rightarrow z = -2$

Hence the eigenvector of eigenvalue 8 is $\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ has magnitude $\sqrt{4^2 + 1^2 + (-2)^2} = \sqrt{21}$

Hence the normalised eigenvector of eigenvalue 8 is $\begin{pmatrix} \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ -\frac{2}{\sqrt{21}} \end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \end{pmatrix}$

38 d $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{21}} \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} & -\frac{2}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{\sqrt{14}}+\frac{4}{\sqrt{14}}-\frac{9}{\sqrt{14}} & \frac{2}{\sqrt{14}}+0+0 & -\frac{3}{\sqrt{14}}+0+\frac{6}{\sqrt{14}} \\ \frac{6}{\sqrt{6}}-\frac{4}{\sqrt{6}}-\frac{3}{\sqrt{6}} & \frac{2}{\sqrt{6}}+0+0 & -\frac{3}{\sqrt{6}}+0+\frac{2}{\sqrt{6}} \\ \frac{24}{\sqrt{21}}+\frac{2}{\sqrt{21}}+\frac{6}{\sqrt{21}} & \frac{8}{\sqrt{21}}+0+0 & -\frac{12}{\sqrt{21}}+0-\frac{4}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{32}{\sqrt{21}} & \frac{8}{\sqrt{21}} & -\frac{16}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{14}+\frac{4}{14}+\frac{9}{14} & \frac{1}{\sqrt{84}}-\frac{4}{\sqrt{84}}+\frac{3}{\sqrt{84}} & \frac{4}{\sqrt{294}}+\frac{2}{\sqrt{294}}-\frac{6}{\sqrt{294}} \\ -\frac{1}{\sqrt{84}}+\frac{4}{\sqrt{84}}-\frac{3}{\sqrt{84}} & -\frac{1}{6}-\frac{4}{6}-\frac{1}{6} & -\frac{4}{\sqrt{126}}+\frac{2}{\sqrt{126}}+\frac{2}{\sqrt{126}} \\ \frac{32}{\sqrt{294}}+\frac{16}{\sqrt{294}}-\frac{48}{\sqrt{294}} & \frac{32}{\sqrt{126}}-\frac{16}{\sqrt{126}}-\frac{16}{\sqrt{126}} & \frac{128}{21}+\frac{8}{21}+\frac{32}{21} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

39 a $\mathbf{M}=\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 4 & 3 & 1\end{pmatrix}$

$\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}1-\lambda & 0 & 1\\ 0 & 2-\lambda & 0\\ 4 & 3 & 1-\lambda\end{pmatrix}$

$$\begin{aligned}\det(\mathbf{M}-\lambda\mathbf{I}) &= (1-\lambda)\left[(2-\lambda)(1-\lambda)-0\right]-0\left[0(1-\lambda)-0\right]+1\left[0-4(2-\lambda)\right]\\ &=(1-\lambda)(1-\lambda)(2-\lambda)-4(2-\lambda)\\ &=(2-\lambda)\left[(1-\lambda)(1-\lambda)-4\right]\\ &=(2-\lambda)(\lambda^2-2\lambda-3)\\ &=(2-\lambda)(\lambda-3)(\lambda+1)\end{aligned}$$

$\det(\mathbf{M}-\lambda\mathbf{I})=0$

$(2-\lambda)(\lambda-3)(\lambda+1)=0$

$\lambda=-1$ or $\lambda=2$ or $\lambda=3$

Hence the eigenvalues of **M** are −1, 2 and 3

To find an eigenvector corresponding to eigenvalue −1:

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 4 & 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-1\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}x+z\\ 2y\\ 4x+3y+z\end{pmatrix}=\begin{pmatrix}-x\\ -y\\ -z\end{pmatrix}$$

Equating the middle elements gives:

$2y=-y\Rightarrow y=0$

Equating the upper elements gives:

$x+z=-x\Rightarrow 2x=-z$

Setting $x=1$ gives $z=-2$

Hence the eigenvector corresponding to eigenvalue −1 is $\begin{pmatrix}1\\ 0\\ -2\end{pmatrix}$

To find an eigenvector corresponding to eigenvalue 2:

$$\begin{pmatrix}1 & 0 & 1\\ 0 & 2 & 0\\ 4 & 3 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}x+z\\ 2y\\ 4x+3y+z\end{pmatrix}=\begin{pmatrix}2x\\ 2y\\ 2z\end{pmatrix}$$

Equating the upper elements gives:

$x+z=2x\Rightarrow x=z$

Set $x=1$ so $z=1$

Equating the lower elements and substituting $x=1$ and $z=1$ gives:

$4+3y+1=2\Rightarrow y=-1$

Hence the eigenvector corresponding to eigenvalue 2 is $\begin{pmatrix}1\\-1\\1\end{pmatrix}$

To find an eigenvector corresponding to eigenvalue 3:

$$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+z\\2y\\4x+3y+z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}$$

Equating the middle elements gives:

$2y=3y\Rightarrow y=0$

Equating the upper elements gives:

$x+z=3x\Rightarrow 2x=z$

Setting $x = 1$ gives $z = 2$

Hence the eigenvector corresponding to eigenvalue 3 is $\begin{pmatrix}1\\0\\2\end{pmatrix}$

b $\frac{x}{2}=y=\frac{z}{-1}$

Written in vector form this is the equation:

$$\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$\mathbf{M}\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}2\\1\\-1\end{pmatrix}$$

$$=\begin{pmatrix}2+0-1\\0+2-0\\8+3-1\end{pmatrix}$$

$$=\begin{pmatrix}1\\2\\10\end{pmatrix}$$

Hence:

$$\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\10\end{pmatrix}$$

Written in Cartesian form this is the equation:

$$x=\frac{y}{2}=\frac{z}{10}$$

40 a $\mathbf{M} = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}$

$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 6-\lambda & -2 & 2 \\ -2 & 5-\lambda & 0 \\ 2 & 0 & 7-\lambda \end{pmatrix}$

$\det(\mathbf{M} - \lambda\mathbf{I}) = (6-\lambda)[(5-\lambda)(7-\lambda)-0] + 2[-2(7-\lambda)-0] + 2[0-2(5-\lambda)]$

For $\lambda = 9$:

$\det(\mathbf{M} - 9\mathbf{I}) = (6-9)[(5-9)(7-9)-0] + 2[-2(7-9)-0] + 2[0-2(5-9)]$

$= -24 + 8 + 16$

$= 0$

Therefore, 9 is an eigenvalue of **M**.

b $\det(\mathbf{M} - \lambda\mathbf{I}) = (6-\lambda)[(5-\lambda)(7-\lambda)-0] + 2[-2(7-\lambda)-0] + 2[0-2(5-\lambda)]$

$= (5-\lambda)(6-\lambda)(7-\lambda) - 4(7-\lambda) - 4(5-\lambda)$

$= (5-\lambda)(6-\lambda)(7-\lambda) - 28 + 4\lambda - 20 + 4\lambda$

$= (5-\lambda)(6-\lambda)(7-\lambda) - 48 + 8\lambda$

$= (5-\lambda)(6-\lambda)(7-\lambda) - 8(6-\lambda)$

$= (6-\lambda)[(5-\lambda)(7-\lambda) - 8]$

$= (6-\lambda)(\lambda^2 - 12\lambda + 27)$

$= (6-\lambda)(\lambda-3)(\lambda-9)$

$\det(\mathbf{M} - \lambda\mathbf{I}) = 0$

$(6-\lambda)(\lambda-3)(\lambda-9) = 0$

$\lambda = 3$ or $\lambda = 6$ or $\lambda = 9$

Hence the eigenvalues of **M** are 3, 6 and 9

40 c To find an eigenvector corresponding to eigenvalue 3:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the middle elements gives:

$-2x+5y=3y \Rightarrow x=y$

Setting $x=1$ gives $y=1$

Equating the lower elements and substituting $x=1$ gives:

$2+7z=3z \Rightarrow z=-\frac{1}{2}$

Hence the eigenvector corresponding to eigenvalue 3 is $\begin{pmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$ or $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ has magnitude $\sqrt{2^2+2^2+(-1)^2}=3$

Hence the normalised eigenvector corresponding to eigenvalue 3 is $\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$

To find an eigenvector corresponding to eigenvalue 6:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the middle elements gives:

$-2x+5y=6y \Rightarrow 2x=-y$

Setting $x=1$ gives $y=-2$:

Equating the lower elements and substituting $x=1$ gives

$2+7z=6z \Rightarrow z=-2$

Hence the eigenvector corresponding to eigenvalue 6 is $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ has magnitude $\sqrt{1^2+(-2)^2+(-2)^2}=3$

Hence the normalised eigenvector corresponding to eigenvalue 6 is $\begin{pmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$

To find an eigenvector corresponding to eigenvalue 9:

$$\begin{pmatrix} 6 & -2 & 2 \\ -2 & 5 & 0 \\ 2 & 0 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 6x-2y+2z \\ -2x+5y \\ 2x+7z \end{pmatrix}=\begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$$

Equating the middle elements gives:
$-2x+5y=9y \Rightarrow x=-2y$
Setting $y=1$ gives $x=-2$:
Equating the lower elements and substituting $x=-2$ gives:
$-4+7z=9z \Rightarrow z=-2$

Hence the eigenvector corresponding to eigenvalue 9 is $\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$

$\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$ has magnitude $\sqrt{(-2)^2+1^2+(-2)^2}=3$

Hence the normalised eigenvector corresponding to eigenvalue 9 is $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$

40 d

$$\mathbf{P}=\begin{pmatrix}\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ -\frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2 & 1 & -2\\ 2 & -2 & 1\\ -1 & -2 & -2\end{pmatrix}$$

If $\mathbf{PP}^{\mathrm{T}}=\mathbf{I}$ then $\mathbf{P}$ is an orthogonal matrix.

$$\mathbf{PP}^{\mathrm{T}}=\frac{1}{3}\times\frac{1}{3}\begin{pmatrix}2 & 1 & -2\\ 2 & -2 & 1\\ -1 & -2 & -2\end{pmatrix}\begin{pmatrix}2 & 2 & -1\\ 1 & -2 & -2\\ -2 & 1 & -2\end{pmatrix}$$

$$=\frac{1}{9}\begin{pmatrix}4+1+4 & 4-2-2 & -2-2+4\\ 4-2-2 & 4+4+1 & -2+4-2\\ -2-2+4 & -2+4-2 & 1+4+4\end{pmatrix}$$

$$=\frac{1}{9}\begin{pmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{pmatrix}$$

$$=\mathbf{I}$$

Therefore $\mathbf{P}$ is indeed an orthogonal matrix.

Challenge

1 a Let α be the angle between the vector and the x-axis.

$$\cos\alpha=\frac{\mathbf{v}_x}{|\mathbf{v}|}$$

$$=\frac{3\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)}{3}$$

$$=\frac{\sqrt{6}}{4}$$

Let β be the angle between the vector and the y-axis.

$$\cos\beta=\frac{\mathbf{v}_y}{|\mathbf{v}|}$$

$$=\frac{3\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)}{3}$$

$$=\frac{\sqrt{6}}{4}$$

Let γ be the angle between the vector and the y-axis.

$$\cos\gamma=\frac{\mathbf{v}_z}{|\mathbf{v}|}$$

$$=\frac{3\cos\left(\frac{\pi}{3}\right)}{3}$$

$$=\frac{1}{2}$$

1 b To convert from spherical to cylindrical coordinates i.e. from $(\rho,\theta,\phi)\mapsto(r,\theta,z)$:

$z=\rho\cos\phi$
$\theta=\theta$
$r=\rho\sin\phi$

To convert from spherical to cylindrical coordinates i.e. from $(r,\theta,z)\mapsto(x,y,z)$:

$x=r\cos\theta\Rightarrow x=r\sin\varphi\cos\theta$
$y=r\sin\theta\Rightarrow r\sin\phi\sin\theta$
$z=r\cos\varphi$

Since the required cosines are in the direction of (ρ,θ,ϕ)

$l=\sin\varphi\cos\theta$
$m=\sin\phi\sin\theta$
$n=\cos\varphi$

2 a $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$$\mathbf{AB}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$
$$=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$\text{tr}(\mathbf{AB})=ae+bg+cf+dh$

$$\mathbf{BA}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$=\begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

$\text{tr}(\mathbf{BA})=ae+bg+cf+dh$

So tr(**AB**) = tr(**BA**) as required

b $\text{tr}(\mathbf{P}^{-1}\mathbf{MP})=\mathbf{tr}(\mathbf{P}^{-1}(\mathbf{MP}))$
$=\mathbf{tr}((\mathbf{MP})\mathbf{P}^{-1})$
$=\mathbf{tr}(\mathbf{M})$

Since $\text{tr}(\mathbf{P}^{-1}\mathbf{MP})=p+q$ then $\text{tr}(\mathbf{M})=p+q$ as required.