Review Exercise 2

1 **a** $\mathrm{E}(X) = \dfrac{a+b}{2} = 31 \Rightarrow a = 62 - b$

$\mathrm{Var}(X) = \dfrac{(b-a)^2}{12} = 192$

Substituting gives:

$\dfrac{(b-(62-b))^2}{12} = 192$

$(2b-62)^2 = 2304$

$2b - 62 = 48$

$b = 55$

$a = 7$

b $\mathrm{P}(10 < X < 30) = \dfrac{20}{48}$

$\mathrm{P}(X < 22 \cap 10 < X < 30) = \mathrm{P}(10 < X < 22) = \dfrac{12}{48}$

$\mathrm{P}(X < 22 \mid 10 < X < 30) = \dfrac{\mathrm{P}(X < 22 \cap 10 < X < 30)}{\mathrm{P}(10 < X < 30)}$

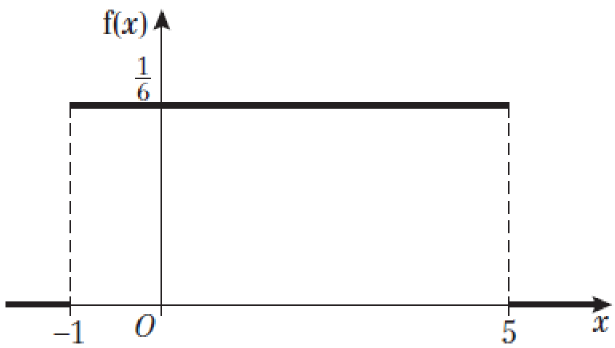
$= \dfrac{\frac{12}{48}}{\frac{20}{48}}$

$= \dfrac{3}{5}$

2 **a** $\dfrac{1}{b-a} = \dfrac{1}{5-(-1)} = \dfrac{1}{6}$

So between $\left(-1, \dfrac{1}{6}\right)$ and $\left(5, \dfrac{1}{6}\right)$, f($x$) is a horizontal straight line, for $x < -1$ and $x > 5$, f(x) = 0.

The graph is:



2 b $\text{E}(X)=\dfrac{b+a}{2}=\dfrac{5+(-1)}{2}=2$

c $\text{Var}(X)=\dfrac{(b-a)^2}{12}=\dfrac{(5+1)^2}{12}=3$

d $\text{P}(-0.3<X<3.3)=(3.3-(-0.3))\times\dfrac{1}{6}=\dfrac{3.6}{6}=0.6$

3 a $\dfrac{1}{b-a}=\dfrac{1}{6-2}=\dfrac{1}{4}$

So:

$$\text{f}(x)=\begin{cases}\dfrac{1}{4} & 2\leqslant x\leqslant 6\\ 0 & \text{otherwise}\end{cases}$$

b $\text{E}(X)=\dfrac{b+a}{2}=\dfrac{6+2}{2}=4$

c $\text{Var}(X)=\dfrac{(b-a)^2}{12}=\dfrac{(6-2)^2}{12}=\dfrac{16}{12}=\dfrac{4}{3}$

d $\text{F}(x)=\displaystyle\int_2^x \frac{1}{4}\,\text{d}t=\left[\frac{1}{4}t\right]_2^x=\frac{1}{4}(x-2)$

So:

$$\text{F}(x)=\begin{cases}0 & x<2\\ \dfrac{1}{4}(x-2) & 2\leqslant x\leqslant 6\\ 1 & x>6\end{cases}$$

e $\text{P}(2.3<X<3.4)=\dfrac{1}{4}(3.4-2.3)=\dfrac{11}{40}=0.275$

Alternative method:

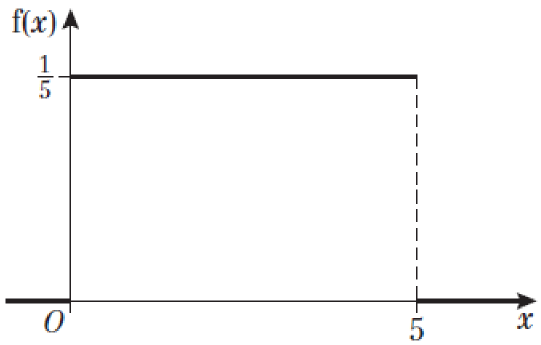
$\text{P}(2.3<X<3.4)=\text{F}(3.4)-\text{F}(2.3)=\dfrac{1}{4}(3.4-2)-\dfrac{1}{4}(2.3-2)=0.275$

4 a The probability distribution of X is a continuous uniform distribution

$$\frac{1}{b-a}=\frac{1}{5-0}=\frac{1}{5}$$

So between $\left(0,\frac{1}{5}\right)$ and $\left(5,\frac{1}{5}\right)$, f($x$) is a horizontal straight line, for $x < 0$ and $x > 5$, f(x) = 0.

The graph is:



b $\mathrm{E}(X)=\frac{b+a}{2}=\frac{5+0}{2}=2.5\,\text{cm}$

$$\mathrm{Var}(X)=\frac{(b-a)^2}{12}=\frac{5^2}{12}=\frac{25}{12}=2.083 \text{ (3 d.p.)}$$

c $\mathrm{P}(X>3)=(5-3)\times\frac{1}{5}=\frac{2}{5}$

d $\mathrm{P}(X=3)=0$

5 a $$\mathrm{f}(x)=\begin{cases}\frac{1}{\beta-\alpha} & \alpha\leqslant x\leqslant\beta\\ 0 & \text{otherwise}\end{cases}$$

5 b $\mathrm{E}(X)=2 \Rightarrow \dfrac{\beta+\alpha}{2}=2 \Rightarrow \beta+\alpha=4 \Rightarrow \beta=4-\alpha$

$\mathrm{P}(X<3)=\dfrac{5}{8} \Rightarrow \dfrac{3-\alpha}{\beta-\alpha}=\dfrac{5}{8} \Rightarrow 24-8\alpha=5\beta-5\alpha \Rightarrow 5\beta=24-3\alpha$

Substituting for β from the first equation gives:

$5(4-\alpha)=24-3\alpha \Rightarrow 20-24=2\alpha \Rightarrow \alpha=-2$

So $\beta=4-\alpha=4-(-2)=6$

Solution is $\alpha=-2,\ \beta=6$

6 a $\mathrm{E}(X)=\dfrac{150+0}{2}=75\,\mathrm{cm}$

b $\mathrm{Var}(X)=\dfrac{150^2}{12} \Rightarrow \text{Standard deviation}=\dfrac{150}{\sqrt{12}}=43.3 \text{ (3 s.f.)}$

c If the piece of string with the ring on it is longer than 150 cm, then the shorter piece of wire is at most 30 cm, and if the piece of string with the ring on it is shorter than or equal to 30 cm, then by definition the shorter length of wire is at most 30 cm long. So the required probability is:

$\mathrm{P}(X\leqslant 30)+\mathrm{P}(X\geqslant 120)=\dfrac{30}{150}+\dfrac{30}{150}=\dfrac{60}{150}=\dfrac{2}{5}$

7 a $\mathrm{E}(X)=12$

therefore

$a=7$ and $b=17$

b $\mathrm{E}(kX-3)=0$

$k\mathrm{E}(X)-3=0$

$12k=3$

$k=0.25$

c $$\mathrm{E}\left(X^2\right)=\frac{1}{10}\int_7^{17} x^2\,\mathrm{d}x$$

$$=\frac{1}{10}\left[\frac{1}{3}x^3\right]_7^{17}$$

$$=\frac{1}{30}\left(17^3-7^3\right)$$

$$=\frac{457}{3}$$

d $$\mathrm{P}(5X-b>a)=\mathrm{P}(5X-17>7)$$

$$=\mathrm{P}(5X>24)$$

$$=\mathrm{P}(X>4.8)$$

$$=1$$

8 a i A named or numbered list of all members of the population.

ii A random variable consisting of any function of the observations and no other quantities.

b i A statistic as it contains only observations.

ii Not a statistic as it contains a parameter μ.

9 a (R, R), (R, B), (R, Y), (B, R), (B, B), (B, Y), (Y, R), (Y, B), (Y, Y)

b The sample space diagram of combinations is

	R	R	R	Y	B	B
R	R & R	R & R	R & R	R & Y	R & B	R & B
R	R & R	R & R	R & R	R & Y	R & B	R & B
R	R & R	R & R	R & R	R & Y	R & B	R & B
Y	R & Y	R & Y	R & Y	Y & Y	B & Y	B & Y
B	R & B	R & B	R & B	B & Y	B & B	B & B
B	R & B	R & B	R & B	B & Y	B & B	B & B

The sample space diagram of points awarded is

	R	R	R	Y	B	B
R	5	5	5	1	1	1
R	5	5	5	1	1	1
R	5	5	5	1	1	1
Y	1	1	1	5	1	1
B	1	1	1	1	5	5
B	1	1	1	1	5	5

The sample space distribution for the points awarded X is

x	1	5
$P(X = x)$	$\frac{11}{18}$	$\frac{7}{18}$

c $E(X) = 1 \times \frac{11}{18} + 5 \times \frac{7}{18}$

$= \frac{23}{9}$

Therefore in 18 trials the total number of points would be $\frac{23}{9} \times 18 = 46$

10 Possible combinations are

(1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1), (2, 2, 2)
(1, 1, 5), (1, 5, 1), (5, 1, 1), (1, 5, 5), (5, 1, 5), (5, 5, 1), (5, 5, 5)
(2, 2, 5), (2, 5, 2), (5, 2, 2), (2, 5, 5), (5, 2, 5), (5, 5, 2)
(1, 2, 5), (1, 5, 2), (2, 1, 5), (2, 5, 1), (5, 1, 2), (5, 2, 1)

The median can only take the values 1, 2 and 5.

Let $\text{P}(1) = p = 0.3$
Let $\text{P}(2) = q = 0.3$
Let $\text{P}(5) = r = 0.4$

$\text{P}(N = 1) = \text{P}(1, 1, 1) + \text{P}(1, 1, 2) + \text{P}(1, 2, 1) + \text{P}(2, 1, 1) + \text{P}(1, 1, 5) + \text{P}(1, 5, 1) + \text{P}(5, 1, 1)$
$= ppp + ppq + pqp + qpp + ppr + prp + rpp$
$= 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.4 + 0.3 \times 0.4 \times 0.3 + 0.4 \times 0.3 \times 0.3$
$= 0.216$

$\text{P}(N = 2) = \text{P}(1, 2, 2) + \text{P}(2, 1, 2) + \text{P}(2, 2, 1) + \text{P}(2, 2, 2) + \text{P}(2, 2, 5) + \text{P}(2, 5, 2) + \text{P}(5, 2, 2) + \text{P}(1, 2, 5) + \text{P}(1, 5, 2) + \text{P}(2, 1, 5) + \text{P}(2, 5, 1) + \text{P}(5, 1, 2) + \text{P}(5, 2, 1)$
$= pqq + qpq + qqp + qqq + qqr + rqr + rqq + pqr + prq + qpr + qrp + rpq + rqp$
$= 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.4 + 0.3 \times 0.4 \times 0.3 + 0.4 \times 0.3 \times 0.3 + 0.3 \times 0.3 \times 0.4 + 0.3 \times 0.4 \times 0.3 + 0.3 \times 0.3 \times 0.4 + 0.3 \times 0.4 \times 0.3 + 0.4 \times 0.3 \times 0.3 + 0.4 \times 0.3 \times 0.3$
$= 0.432$

$\text{P}(N = 5) = \text{P}(1, 5, 5) + \text{P}(5, 1, 5) + \text{P}(5, 5, 1) + \text{P}(2, 5, 5) + \text{P}(5, 2, 5) + \text{P}(5, 5, 2) + \text{P}(5, 5, 5)$
$= ppp + ppq + pqp + qpp + ppr + prp + rpp$
$= 0.3 \times 0.4 \times 0.4 + 0.4 \times 0.3 \times 0.4 + 0.4 \times 0.4 \times 0.3 + 0.3 \times 0.4 \times 0.4 + 0.4 \times 0.3 \times 0.4 + 0.4 \times 0.4 \times 0.3 + 0.4 \times 0.4 \times 0.4$
$= 0.352$

m	1	2	5
$\text{P}(N = m)$	0.216	0.432	0.352

11 a i A hypothesis test is where the value of a population parameter (whose assumed value is given by the null hypothesis H_0) is tested against what value it takes if H_0 is rejected (this could be an increase, a decrease or a change).

ii A range of values of a test statistic that would lead to the rejection of the null hypothesis (H_0).

11 b Let the random variable X be the number of incoming calls received in a 20-minute interval,

so $X \sim \text{Po}(20 \times 0.45)$, i.e. $X \sim \text{Po}(9)$
H0: $\lambda = 9$ H1: $\lambda \neq$ p
Assume H_0, so that $X \sim \text{Po}(9)$

Let $X = c_1$ be the upper boundary of the lower critical region
Require $\text{P}(X \leqslant c_1)$ to be as close as possible to 2.5%
From the tables: $\text{P}(X \leqslant 3) = 0.0212$ and $\text{P}(X \leqslant 4) = 0.0550$
0.0212 is closer to 0.025, so $c_1 = 3$ and the lower critical region is $X \leqslant 3$

Let $X = c_2$ is the lower boundary of the upper critical region, r
Require $\text{P}(X \geqslant c_2)$ to be as close as possible to 2.5%
From the tables: $\text{P}(X \geqslant 15) = 1 - \text{P}(X \leqslant 14) = 1 - 0.9585 = 0.0415$
and $\text{P}(X \geqslant 16) = 1 - \text{P}(X \leqslant 15) = 1 - 0.9780 = 0.0220$
0.0220 is closer to 0.025, so $c_2 = 16$ and the upper critical region is $X \geqslant 16$

Critical region is $X \leqslant 3$ or $X \geqslant 16$

c Actual significance level $= \text{P}(X \leqslant 3) + \text{P}(X \geqslant 18) = 0.0212 + 0.0220 = 0.0432$ or 4.32%

d Let the random variable Y be the number of incoming calls received in a 10-minute interval,
so $Y \sim \text{Po}(10 \times 0.45)$, i.e. $Y \sim \text{Po}(4.5)$
H_0: $\lambda = 4.5$ H_1: $\lambda < 4.5$
Assume H_0, so that $Y \sim \text{Po}(4.5)$
Significance level 5%, so require $\text{P}(Y \leqslant c) < 0.05$

From the tables: $\text{P}(Y \leqslant 1) = 0.0611$
As $0.061 > 0.05$ there is insufficient evidence to reject H_0
So reject the hypothesis that there are fewer incoming calls during the school holidays.

12 Let X be the number of bacteria.
H_0: $\lambda = 5$, H_1: $\lambda > 5$
$X \sim \text{Po}(2.5)$
Critical region is $X \geq 6$.
Therefore reject H_0.
There is significant evidence that near the factory the river is polluted with bacteria at the 5% level.

13 a $X \sim \text{B}(20, 0.15)$
H_0: $\lambda = 0.15$, H_0: $\lambda \neq 0.15$
The critical region is $X \geq 7$
N.B. The lower tail is empty in this circumstance.

b $(1 – 0.9781) + 0 = 0.0219$

c There is no evidence to reject the null hypothesis.
The probability that a pin chosen at random is not less than 0.15.

14 a H_0: p = 0.3 H_1: p ≠ 0.3 Significance level 2.5%
If H_0 is true $X \sim B(40, 0.3)$
Let c_1 and c_2 be the two critical values, so $P(X \leqslant c_1) = 0.0125$ and $P(X \geqslant c_2) = 0.0125$

For the lower tail, finding values of P(X) from tables or calculator, those either side of 0.0125 are:
$P(X \leqslant 5) = 0.0086$
$P(X \leqslant 6) = 0.0238$
$0.0125 - 0.0086 = 0.0039$ ($X \leqslant 5$) and $0.0238 - 0.0125 = 0.0113$ ($X \leqslant 6$)
So $c_1 = 5$ as this gives the value closest to 0.0125.

For the upper tail:
$P(X \geqslant 19) = 1 - P(X \leqslant 18) = 1 - 0.9852 = 0.0148$
$P(X \geqslant 20) = 1 - P(X \leqslant 19) = 1 - 0.9937 = 0.0063$
$0.0148 - 0.0125 = 0.0023$ ($X \geq 19$) and $0.0125 - 0.0063 = 0.0062$ ($X \geqslant 20$)
So $c_2 = 19$

$P(X \leqslant 5)$ and $P(X \geqslant 19)$, so the critical region is $0 \leqslant X \leqslant 5$ and $19 \leqslant X \leqslant 40$

b The probability of incorrectly rejecting the null hypothesis is the same as the probability that X falls within the critical region.
$P(X \leqslant 5) + P(X \geqslant 19) = 0.0086 + 0.0148 = 0.0234$

15 a $X \sim B(10, 0.75)$ where X is the random variable 'number of patients who recover when treated'.

b Using tables or calculator
$P(X = 6) = 0.146$

Alternative method (tables)
$P(X = 6) = P(X \leqslant 6) - P(X \leqslant 5) = 0.9219 - 0.7759 = 0.146$

Alternative method (calculating)
$$P(X = 6) = \binom{10}{6}(0.75)^6(0.25)^4 = \frac{10!}{6!4!} \times (0.75)^6(0.25)^4 = 0.146$$

c H_0: p = 0.75 H_1: p < 0.75
$X \sim B(20, 0.75)$

$P(X \leqslant 13) = 1 - 0.7858 = 0.2142$
As the probability is greater than 5%, there is insufficient evidence to reject the null hypothesis that 75% of patients will recover. Therefore, there is no evidence to support the doctor's belief that fewer than 75% of patients will recover.

d Using tables/calculator to find values either side of 1 − 0.01 = 0.99:
$P(X \leqslant 9) = 1 - 0.9961 = 0.0039 < 0.01$
i.e. if 9 patients recover, the null hypothesis is accepted.
$P(X \leqslant 10) = 1 - 0.9861 = 0.0139 > 0.01$
i.e. if 10 patients recover, the null hypothesis is rejected.
Therefore, no more than 9 patients should recover for the test to be significant at this level.

16 a H_0: p = 0.3 i.e. the new fertiliser will have no effect.
H_1: p > 0.3 i.e. the new fertiliser will increase the probability of the tomatoes having a diameter greater than 4 cm.

b Significance level 5%
If H_0 is true $X \sim B(40, 0.3)$
Let c be the critical value.

$P(X \geqslant 17) = 1 - P(X \leqslant 16) = 1 - 0.9367 = 0.0633 > 0.05$
$P(X \geqslant 18) = 1 - P(X \leqslant 17) = 1 - 0.9680 = 0.032 < 0.05$
So $c = 18$
$P(X \geqslant 18)$, so the critical region is $18 \leqslant X \leqslant 40$

c Actual significance level = 0.032 = 3.2%

d The observed value of 18 lies in the critical region so it is reasonable to reject the null hypothesis that the fertiliser has no effect. Dhriti's claim that the fertiliser works is supported.

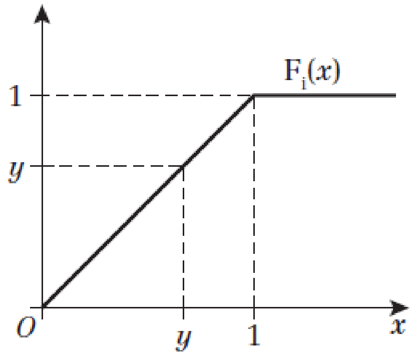
17 a Let the random number X be the number of microchips tested before a faulty one is found
H_0: p = 0.0005 H_1: p > 0.0005
Assume H_0, so that $X \sim \text{Geo}(0.0005)$
Significance level 5%, so require $P(X \leqslant c) < 0.05$
$P(X \leqslant c) = 1 - (1 - 0.0005)^c$
So $1 - (1 - 0.0005)^c < 0.05$
$(0.9995)^c > 0.95$
$$c < \frac{\log 0.95}{\log 0.9995}$$
$c < 102.56$
So the critical region is $X \leqslant 102$

b As 115 is outside the critical region $X \leqslant 102$, at the 5% significance level there is no evidence that the failure rate is greater than claimed.

Challenge

1 $X_i \sim \mathrm{U}[0,1]$ $Y = \max(X_i)$

a



$$\mathrm{f}_i(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leqslant x \leqslant 1 \\ 1 & x > 1 \end{cases}$$

$$\mathrm{F}_i(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leqslant x \leqslant 1 \\ 1 & x > 1 \end{cases}$$

$$\mathrm{F}_Y(y) = \mathrm{P}\big((X_1 \leqslant y) \cap (X_2 \leqslant y) \cap \ldots \cap (X_n \leqslant y)\big) = y^n$$

So $\mathrm{f}_Y(y) = \dfrac{\mathrm{d}}{\mathrm{d}y}(y^n) = ny^{n-1}$

Hence, $\mathrm{E}(Y) = \int_{-\infty}^{\infty} y\mathrm{f}_Y(y)\mathrm{d}y = \int_0^1 ny^n \mathrm{d}y = \dfrac{n}{n+1}\left[y^{n+1}\right]_0^1 = \dfrac{n}{n+1}$

b Median of Y is the value m such that $\mathrm{F}_Y(m) = 0.5$

$\Rightarrow \mathrm{F}_Y(m) = m^n = 0.5 \Rightarrow m = \sqrt[n]{0.5}$

1 c Let $Y = X_1 + X_2$

$\mathrm{f}_Y(z) = \int_{-\infty}^{\infty} \mathrm{f}_1(z-t)\mathrm{f}_2(t)\mathrm{d}t = \int_0^1 \mathrm{f}_1(z-t)\mathrm{d}t$ since $\mathrm{f}_2(t) = 0$ for values of t outside the interval $[0,1]$

$\mathrm{f}_1(z-t)$ is only non-zero for $0 \leqslant z-t \leqslant 1 \Rightarrow z-1 \leqslant t \leqslant z$

Since X_i takes values in [0, 1], Y takes values in [0, 2].

Consider $0 \leqslant z \leqslant 1$ so $z-1 \leqslant 0 \leqslant t \leqslant z \leqslant 1$:

$$\mathrm{f}_Y(z) = \int_0^z \mathrm{f}_1(z-t)\mathrm{d}t = \int_0^z \mathrm{d}t = [t]_0^z = z$$

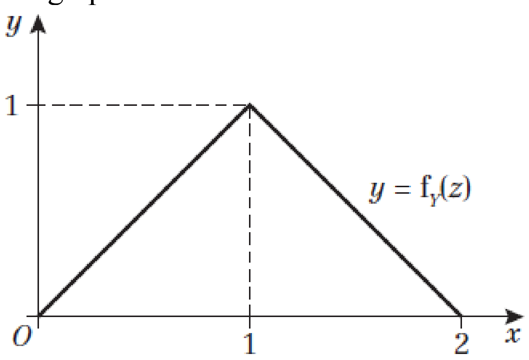
Consider $1 \leqslant z \leqslant 2$ so $0 < z-1 \leqslant t \leqslant 1 \leqslant z$:

$$\mathrm{f}_Y(z) = \int_{z-1}^1 \mathrm{f}_1(z-t)\mathrm{d}t = \int_{z-1}^1 \mathrm{d}t = [t]_{z-1}^1 = 1-(z-1) = 2-z$$

So:

$$\mathrm{f}_Y(z) = \begin{cases} z & 0 \leqslant z \leqslant 1 \\ 2-z & 1 < z \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

The graph is:



1 d Let $Z = Y_1 + X_3$ where Y is defined as in part **c**. So:

$$\mathrm{f}_Y(z) = \begin{cases} z & 0 \leqslant z \leqslant 1 \\ 2-z & 1 < z \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{f}_3(x) = \begin{cases} 1 & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{f}_Z(z) = \int_{-\infty}^{\infty} \mathrm{f}_Y(z-t)\mathrm{f}_3(t)\,\mathrm{d}t = \int_0^1 \mathrm{f}_Y(z-t)\,\mathrm{d}t \qquad \text{since } \mathrm{f}_3(t) = 0 \text{ for values of } t \text{ outside } [0,1]$$

$\mathrm{f}_Y(z-t)$ is only non-zero for $0 \leqslant z-t \leqslant 2 \Rightarrow z-2 \leqslant t \leqslant z$

Since Y takes values in [0, 2] and X_3 takes values in [0, 1], Z takes values in [0, 3].

Consider $0 \leqslant z \leqslant 1$ so $z-2 \leqslant -1 < 0 \leqslant t \leqslant z \leqslant 1$:

$$\mathrm{f}_Z(z) = \int_0^z (z-t)\,\mathrm{d}t = \left[zt - \tfrac{1}{2}t^2\right]_0^z = \frac{1}{2}z^2$$

Consider $1 \leqslant z \leqslant 2$ so $z-2 \leqslant 0 \leqslant t \leqslant 1 < z$:

$$\mathrm{f}_Z(z) = \int_0^{z-1} (2-z+t)\,\mathrm{d}t + \int_{z-1}^{1} (z-t)\,\mathrm{d}t = \left[2t - zt + \frac{1}{2}t^2\right]_0^{z-1} + \left[zt - \frac{1}{2}t^2\right]_{z-1}^{1}$$

$$= 2(z-1) - z(z-1) + \frac{1}{2}(z^2-2z+1) + z - \frac{1}{2} - \left(z(z-1) - \frac{1}{2}(z^2-2z+1)\right)$$

$$= 2z - 2 - z^2 + z + \frac{1}{2}(z^2-2z+1) + z - \frac{1}{2} - z^2 + z + \frac{1}{2}(z^2-2z+1)$$

$$= 5z - \frac{5}{2} - 2z^2 + z^2 - 2z + 1 = -z^2 + 3z - \frac{3}{2}$$

$$= -\left(z - \frac{3}{2}\right)^2 + \frac{9}{4} - \frac{3}{2} = \frac{3}{4} - \left(z - \frac{3}{2}\right)^2$$

Consider $2 \leqslant z \leqslant 3$ so $0 < z-2 \leqslant t \leqslant 1 < 2 < z$:

$$\mathrm{f}_Z(z) = \int_{z-2}^{1} (2-z+t)\,\mathrm{d}t = \left[2t - zt + \frac{1}{2}t^2\right]_{z-2}^{1}$$

$$= 2 - z + \frac{1}{2} - \left(2(z-2) - z(z-2) + \frac{1}{2}(z^2-4z+4)\right)$$

$$= 2 - z + \frac{1}{2} - 2z + 4 + z^2 - 2z - \frac{1}{2}z^2 + 2z - 2$$

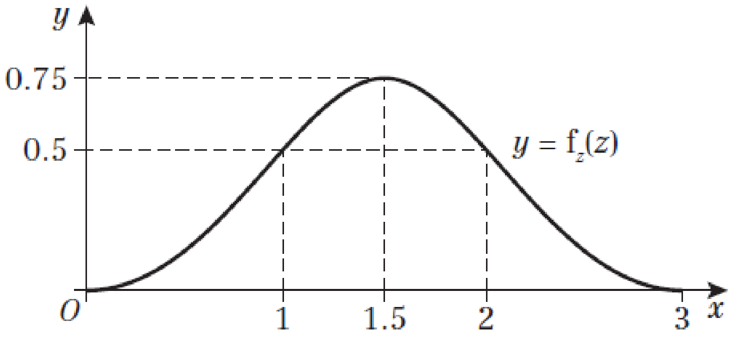
$$= \frac{1}{2}z^2 - 3z + \frac{9}{2} = \frac{1}{2}(z^2 - 6z + 9) = \frac{1}{2}(z-3)^2$$

So:

$$\mathrm{f}_Z(z) = \begin{cases} \frac{1}{2}z^2 & 0 \leqslant z \leqslant 1 \\ \frac{3}{4} - (z - \frac{3}{2})^2 & 1 < z \leqslant 2 \\ \frac{1}{2}(z-3)^2 & 2 < z \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

1 d (continued)

Between (0, 0) and (1, 0.5), $f_Z(z)$ is a positive quadratic; between (1, 0.5) and (2, 0.5), $f_Z(z)$ is a negative quadratic with a maximum at (1.5, 0.75); between (2, 0.5) and (3, 0), $f_Z(z)$ is a positive quadratic; otherwise $f_Z(z)$ is zero.
The graph is:



2 a Significance level 10%
If H_0 is true $X \sim B(30, 0.65)$
Let c be the critical value.

$P(X \leqslant 15) = 0.0652$
$P(X \leqslant 16) = 0.1263$
$0.1 - 0.0652 = 0.0348\ (X \leqslant 15)$ and $0.1263 - 0.1 = 0.0263\ (X \leqslant 16)$
So $c = 16$
$P(X \leqslant 16)$, so the critical region is $0 \leqslant X \leqslant 16$

b $P(X \leqslant 16) = 0.1263$
$P(X \leqslant 16 \text{ and } X \leqslant 16) = 0.1263^2 = 0.0160$