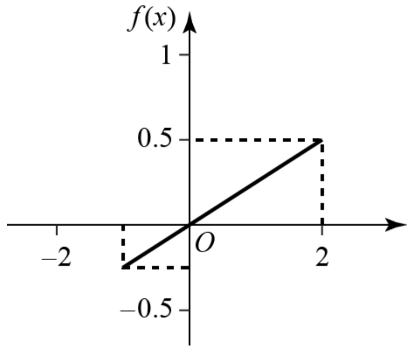
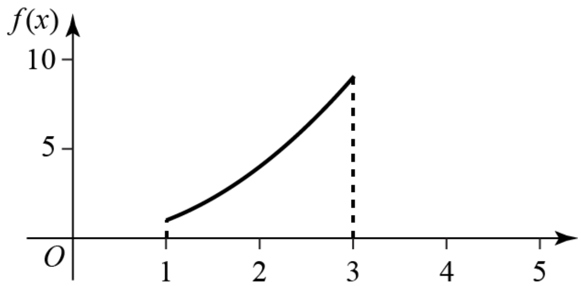
Exercise 4A

1 **a** Sketching the function:



There are negative values for f(x) when $-1 \leqslant x < 0$, so this is not a probability density function.

b Sketching the function:

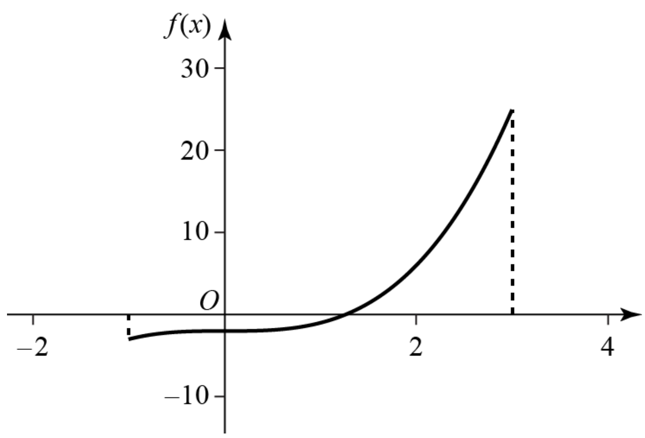


There is no negative value of f(x)

Area under $\mathrm{f}(x) = \int_1^3 x^2 \mathrm{d}x = \left[\frac{x^3}{3}\right]_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$

Area is not equal to 1, therefore this is not a valid probability density function.

c When f(x) = 0, $x = 2^{\frac{1}{3}} = 1.26$ (2 d.p.). So for $-1 \leqslant x < 1.26$, f(x) < 0. As there are negative values for f(x), this is not a probability density function.
Alternatively, reach this result by sketching the function:



2 The area under the curve must equal 1, so:

$$\int_{-4}^{-2} k(x^2-1)\,dx = 1$$

$$k\left[\frac{x^3}{3}-x\right]_{-4}^{-2} = 1$$

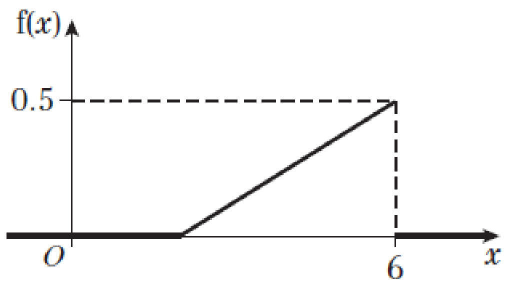
$$k\left(\left(-\frac{8}{3}+2\right)-\left(-\frac{64}{3}+4\right)\right) = 1$$

$$k\left(\frac{56}{3}-2\right) = 1$$

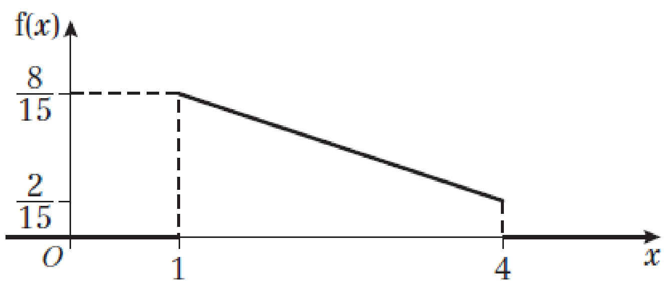
$$\frac{50}{3}k = 1$$

$$k = \frac{3}{50}$$

3 a For the non-zero parts of the function, its graph is a straight line running from (2, 0) to (6, 0.5).



b For the non-zero parts of the function, its graph is a straight line running from $\left(1, \frac{8}{15}\right)$ to $\left(4, \frac{2}{15}\right)$.



4 a The area under the curve must equal 1, so:

$$\int_{1}^{3} kx\,dx = 1$$

$$\left[\frac{kx^2}{2}\right]_{1}^{3} = 1$$

$$\frac{9k}{2}-\frac{k}{2} = 1$$

$$4k = 1$$

$$k = \frac{1}{4}$$

4 b $\int_0^3 kx^2\,dx = 1$

$$\left[\frac{kx^3}{3}\right]_0^3 = 1$$

$$\frac{27k}{3} = 1$$

$$9k = 1$$

$$k = \frac{1}{9}$$

c $\int_{-1}^{2} k(1+x^2)dx = 1$

$$k\left[x+\frac{x^3}{3}\right]_{-1}^{2} = 1$$

$$k\left(\left(2+\frac{8}{3}\right)-\left(-1-\frac{1}{3}\right)\right) = 1$$

$$k\left(3+\frac{9}{3}\right) = 1$$

$$6k = 1$$

$$k = \frac{1}{6}$$

5 a The area under the curve must equal 1, so:

$$\int_0^2 k(4-x)dx = 1$$

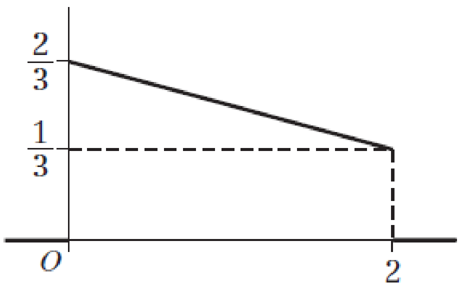
$$k\left[4x-\frac{x^2}{2}\right]_0^2 = 1$$

$$k(8-2) = 1$$

$$6k = 1$$

$$k = \frac{1}{6}$$

b For the non-zero parts of the function, its graph is a straight line running from $\left(0, \frac{2}{3}\right)$ to $\left(2, \frac{1}{3}\right)$.



5 c $\mathrm{P}(X>1)=\int_1^2 \frac{1}{6}(4-x)\,\mathrm{d}x=\left[\frac{2}{3}x-\frac{1}{12}x^2\right]_1^2$

$$=\left(\frac{4}{3}-\frac{1}{3}\right)-\left(\frac{2}{3}-\frac{1}{12}\right)=\frac{1}{3}+\frac{1}{12}=\frac{5}{12}$$

6 a The area under the curve must equal 1, so:

$$\int_0^2 kx^2(2-x)\,\mathrm{d}x=1$$

$$k\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2=1$$

$$k\left(\frac{16}{3}-\frac{16}{4}\right)=1$$

$$\frac{16k}{12}=1$$

$$k=\frac{3}{4}=0.75$$

b $\mathrm{P}(0<X<1)=\int_0^1 \frac{3}{4}x^2(2-x)\,\mathrm{d}x=\left[\frac{1}{2}x^3-\frac{3}{16}x^4\right]_0^1=\frac{5}{16}$

7 a The area under the curve must equal 1, so:

$$\int_1^4 kx^3\,\mathrm{d}x=1$$

$$\left[\frac{kx^4}{4}\right]_1^4=1$$

$$\frac{256k}{4}-\frac{k}{4}=1$$

$$\frac{255k}{4}=1$$

$$k=\frac{4}{255}$$

b $\int_1^2 \frac{4}{255}x^3\,\mathrm{d}x=\left[\frac{1}{255}x^4\right]_1^2=\frac{15}{255}=\frac{1}{17}=0.0588 \text{ (4 d.p.)}$

8 a The area under the curve must equal 1, so:

$$\int_0^2 k\,\mathrm{d}x+\int_2^3 k(2x-3)\,\mathrm{d}x=1$$

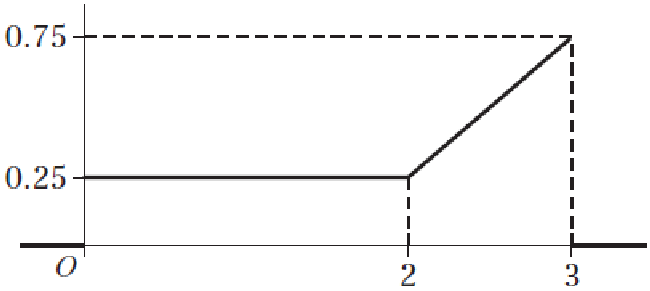
$$[kx]_0^2+\left[kx^2-3kx\right]_2^3=1$$

$$2k+\left[(9k-9k)-(4k-6k)\right]=1$$

$$2k+2k=1$$

$$k=\frac{1}{4}=0.25$$

8 b For the non-zero parts of the function, its graph is a horizontal line running from $(0, 0.25)$ to $(2, 0.25)$ and then a straight line from $(2, 0.25)$ to $(3, 0.75)$.



c $\text{P}(X < 1) = \int_0^1 0.25\,\text{d}x = [0.25x]_0^1 = 0.25$

$\text{P}(Y < 1) = \int_{-2}^{1} \frac{3}{16}y^2\,\text{d}y = \left[\frac{1}{16}y^3\right]_{-2}^{1} = \frac{1}{16} - \left(-\frac{8}{16}\right) = \frac{9}{16}$

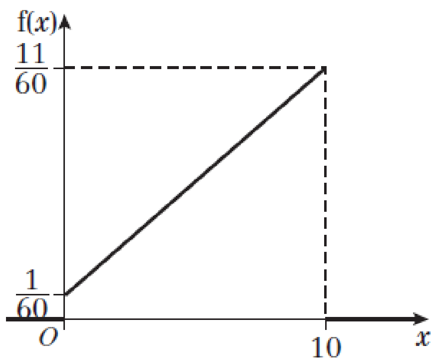
As X and Y are independent:

$\text{P}(X < 1 \cap Y < 1) = \text{P}(X < 1) \times \text{P}(Y < 1) = \frac{1}{4} \times \frac{9}{16} = \frac{9}{64}$

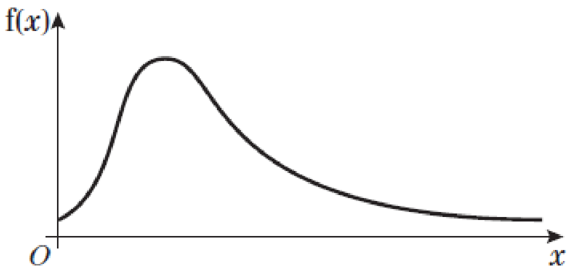
9 a $\text{P}(X < 0.5) = \int_0^{0.5} \frac{1}{60}(x+1)\,\text{d}x = \frac{1}{60}\left[0.5x^2 + x\right]_0^{0.5}$

$= \frac{1}{60}\left(\frac{1}{8} + \frac{1}{2}\right) = \frac{1}{60} \times \frac{5}{8} = \frac{1}{96} = 0.0104 \text{ (4 d.p.)}$

b For the non-zero parts of the function, its graph is a straight line running from $\left(0, \frac{1}{60}\right)$ to $\left(10, \frac{11}{60}\right)$.



9 c By definition, every visitor would spend some time (however short) on the site, but the probability of spending a long time on the site would be very low but not become zero as x gets larger. So in reality the probability density might look like this:



10 a

$$f(x) = \begin{cases} k & 0 \le x < 1 \\ k(x-1)^2 & 1 \le x < 2 \\ k(3-x) & 2 \le x < 3 \\ 0 & \text{otherwise} \end{cases}$$

Since f(x) represents a probability density function

$$\int_0^3 f(x)\,dx = 1$$

$$\int_0^1 k\,dx + \int_1^2 k(x-1)^2\,dx + \int_2^3 k(3-x)\,dx = 1$$

$$\int_0^1 k\,dx + \int_1^2 k(x^2-2x+1)\,dx + \int_2^3 k(3-x)\,dx = 1$$

$$1 + \left[\left(\frac{1}{3}(2)^3 - (2)^2 + (2)\right) - \left(\frac{1}{3}(1)^3 - (1)^2 + (1)\right)\right]_1^2 + \left[\left(3(3) - \frac{1}{2}(3)^2\right) - \left(3(2) - \frac{1}{2}(2)^2\right)\right]_2^3 = \frac{1}{k}$$

$$1 + \frac{1}{3} + \frac{1}{2} = \frac{1}{k}$$

$$k = \frac{6}{11}$$

b $$P(0.5 \le X \le 1.5) = \int_{0.5}^{1} \frac{6}{11}\,dx + \int_1^{1.5} \frac{6}{11}(x^2-2x+1)\,dx$$

$$= \frac{6}{11}[x]_{0.5}^1 + \frac{6}{11}\left[\frac{1}{3}x^3 - x^2 + x\right]_1^{1.5}$$

$$= \frac{6}{11}(1-0.5) + \frac{6}{11}\left[\left(\frac{1}{3}(1.5)^3 - (1.5)^2 + (1.5)\right) - \left(\frac{1}{3}(1)^3 - (1)^2 + (1)\right)\right]$$

$$= \frac{3}{11} + \frac{6}{11}\left(\frac{3}{8} - \frac{1}{3}\right)$$

$$= \frac{13}{44}$$

11 a The area under the curve must equal 1, so:

$$\int_1^5 \frac{k}{x}\,dx = 1$$

$$\left[k\ln x\right]_1^5 = 1$$

$$k\ln 5 = 1$$

$$k = \frac{1}{\ln 5}$$

b $\mathrm{P}(2 < X < 4) = \frac{1}{\ln 5}\int_2^4 \frac{1}{x}\,dx = \frac{1}{\ln 5}\left[\ln x\right]_2^4$

$$= \frac{1}{\ln 5}(\ln 4 - \ln 2) = \frac{\ln 2}{\ln 5}$$

12 a The area under the curve must equal 1, so:

$$\int_{-1}^4 \frac{k}{x+2}\,dx = 1$$

$$\left[k\ln(x+2)\right]_{-1}^4 = 1$$

$$k\ln 6 = 1$$

$$k = \frac{1}{\ln 6}$$

b $\mathrm{P}(1 < X < 3) = \frac{1}{\ln 6}\int_1^3 \frac{1}{x+2}\,dx = \frac{1}{\ln 6}\left[\ln(x+2)\right]_1^3$

$$= \frac{1}{\ln 6}(\ln 5 - \ln 3) = \frac{\ln 1.666\ldots}{\ln 6} = 0.285 \text{ (3 d.p.)}$$

13 a The area under the curve must equal 1, so:

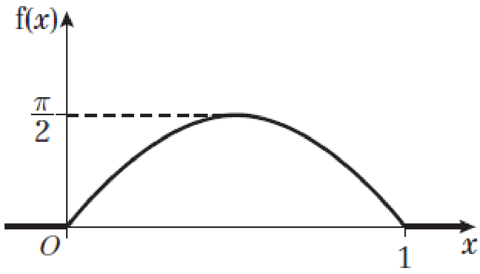
$$\int_0^1 k\sin \pi x\,dx = 1$$

$$\left[-\frac{k}{\pi}\cos \pi x\right]_0^1 = 1$$

$$\frac{k}{\pi}\big(1-(-1)\big) = 1$$

$$k = \frac{\pi}{2}$$

13 b For the non-zero parts of the function, its graph is a sine curve of amplitude $\frac{\pi}{2}$ running from $(0, 0)$ to $(1, 0)$.



c $P(0 < X < \frac{1}{3}) = \frac{\pi}{2}\int_0^{\frac{1}{3}} \sin \pi x \, dx = \frac{\pi}{2}\left[-\frac{1}{\pi}\cos \pi x\right]_0^{\frac{1}{3}} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

Challenge

a The area under the curve must equal 1, so:

$\int_1^{\infty} \frac{k}{t^3} dt = 1 \Rightarrow k\int_1^{\infty} \frac{1}{t^3} dt = 1 \Rightarrow k \times \frac{1}{2} = 1 \Rightarrow k = 2$

b i $P(0 < T < 3) = \int_1^3 \frac{2}{t^3} dt = \left[-t^{-2}\right]_1^3 = 1 - \frac{1}{9} = \frac{8}{9}$

ii $P(T > 20) = \int_{20}^{\infty} \frac{2}{t^3} dt = \left[-t^{-2}\right]_{20}^{\infty} = \frac{1}{20^2} = \frac{1}{400}$

c $P(p < T < 2p) = \int_p^{2p} \frac{2}{t^3} dt = \left[-t^{-2}\right]_p^{2p} = \frac{1}{p^2} - \frac{1}{4p^2}$

So $\frac{1}{p^2} - \frac{1}{4p^2} = 0.12$

$\Rightarrow \frac{3}{4p^2} = 0.12$

$\Rightarrow p^2 = \frac{3}{4 \times 0.12} = \frac{1}{4 \times 0.04} = \frac{1}{0.16} = 6.25$

$\Rightarrow p = 2.5$