

Exercise 4C

1 a The area under the curve must equal 1, so:

$$\int_0^2 kx^2 = 1$$

$$\left[\frac{kx^3}{3} \right]_0^2 = 1$$

$$\frac{8k}{3} = 1$$

$$k = \frac{3}{8}$$

b $E(X) = \int x f(x) dx = \int_0^2 \frac{3x^3}{8} dx = \left[\frac{3x^4}{32} \right]_0^2 = \frac{48}{32} = \frac{3}{2} = 1.5$

c $\text{Var}(X) = \int x^2 f(x) dx - \mu^2 = \int_0^2 \frac{3x^4}{8} dx - 1.5^2$

$$= \left[\frac{3x^5}{40} \right]_0^2 - 1.5^2 = \frac{96}{40} - 2.25 = \frac{48}{20} - \frac{45}{20} = \frac{3}{20} = 0.15$$

2 a $E(Y) = \int_0^3 \frac{y^3}{9} dy = \left[\frac{y^4}{36} \right]_0^3 = \frac{81}{36} = \frac{9}{4} = 2.25$

b Use $\text{Var}(Y) = \int y^2 f(y) dy - \mu^2 = \int y^2 f(y) dx - (E(Y))^2$

$$\text{Var}(Y) = \int_0^3 \frac{y^4}{9} dy - 2.25^2 = \left[\frac{y^5}{45} \right]_0^3 - \frac{81}{16}$$

$$= \frac{243}{45} - \frac{81}{16} = \frac{27}{5} - \frac{81}{16} = \frac{432 - 405}{80} = \frac{27}{80} = 0.3375$$

c $\sigma = \sqrt{0.3375} = 0.581$ (3 s.f.)

3 a $E(Y) = \int_0^4 \frac{y^2}{8} dy = \left[\frac{y^3}{24} \right]_0^4 = \frac{64}{24} = \frac{8}{3}$

b $\text{Var}(Y) = \int_0^4 \frac{y^3}{8} dy - \left(\frac{8}{3} \right)^2 = \left[\frac{y^4}{32} \right]_0^4 - \frac{64}{9}$

$$= \frac{256}{32} - \frac{64}{9} = 8 - \frac{64}{9} = \frac{72 - 64}{9} = \frac{8}{9}$$

c $\sigma = \sqrt{\frac{8}{9}} = 0.943$ (3 s.f.)

3 d From part a, $\mu = \frac{8}{3}$, so

$$\begin{aligned} P(Y > \mu) &= P\left(Y > \frac{8}{3}\right) = \int_{\frac{8}{3}}^4 \frac{y}{8} dy \\ &= \left[\frac{y^2}{16}\right]_{\frac{8}{3}}^4 = 1 - 0.4444\dots = 0.556 \text{ (3 s.f.)} \end{aligned}$$

e $\text{Var}(3Y + 2) = 3^2 \text{Var}(Y) = 9 \times \frac{8}{9} = 8$

f $E(Y + 2) = E(Y) + 2 = \frac{8}{3} + 2 = \frac{14}{3}$

4 a The area under the curve must equal 1, so:

$$\int_0^1 k(1-x) dx = 1$$

$$\left[kx - \frac{kx^2}{2}\right]_0^1 = 1$$

$$k - \frac{1}{2}k = 1$$

$$k = 2$$

b $E(X) = \int_0^1 2(x - x^2) dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

c Use $\text{Var}(X) = \int x^2 f(x) dx - \mu^2 = \int x^2 f(x) dx - (E(X))^2$

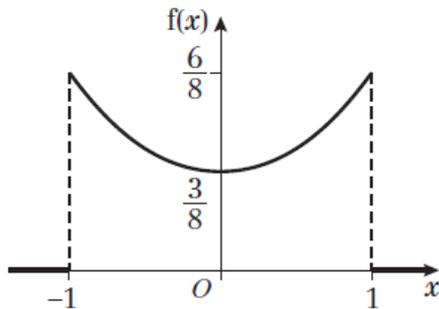
$$\begin{aligned} \text{Var}(X) &= \int_0^1 2(x^2 - x^3) dx - \left(\frac{1}{3}\right)^2 = \left[\frac{2x^3}{3} - \frac{x^4}{2}\right]_0^1 - \frac{1}{9} \\ &= \left(\frac{2}{3} - \frac{1}{2}\right) - \frac{1}{9} = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} = \frac{1}{18} \end{aligned}$$

d $P\left(X > \frac{1}{3}\right) = \int_{\frac{1}{3}}^1 2(1-x) dx = \left[2x - x^2\right]_{\frac{1}{3}}^1 = (2-1) - \left(\frac{2}{3} - \frac{1}{9}\right) = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

5 a $P(X < 0.5) = \int_0^{0.5} 12x^2 - 12x^3 dx = \left[4x^3 - 3x^4\right]_0^{0.5} = \frac{1}{2} - \frac{3}{16} = \frac{5}{16} = 0.3125$

b $E(X) = \int_0^1 x f(x) dx = \int_0^1 12x^3 - 12x^4 dx = \left[3x^4 - \frac{12x^5}{5}\right]_0^1 = 3 - \frac{12}{5} = \frac{3}{5} = 0.6$

- 6 a Between $\left(-1, \frac{3}{4}\right)$ and $\left(1, \frac{3}{4}\right)$, the curve is a positive quadratic with a minimum at $\left(0, \frac{3}{8}\right)$.
Otherwise $f(x)$ is 0.



- b As the probability density function is symmetrical about $x = 0$, $E(X) = 0$

$$\text{Alternatively, } E(X) = \int_{-1}^1 \frac{3}{8}(x + x^3) dx = \left[\frac{3}{16}x^2 + \frac{3}{32}x^4 \right]_{-1}^1 = \frac{3}{16} + \frac{3}{32} - \left(\frac{3}{16} + \frac{3}{32} \right) = 0$$

- c $\sigma^2 = \text{Var}(X) = E(X^2) - (E(X))^2 = \int_{-1}^1 \frac{3x^2}{8} + \frac{3x^4}{8} dx - 0^2$

$$\begin{aligned} &= \left[\frac{3x^3}{24} + \frac{3x^5}{40} \right]_{-1}^1 = \left(\frac{3}{24} + \frac{3}{40} \right) - \left(-\frac{3}{24} - \frac{3}{40} \right) \\ &= \frac{6}{24} + \frac{6}{40} = \frac{10}{40} + \frac{6}{40} = \frac{16}{40} = \frac{2}{5} = 0.4 \end{aligned}$$

- d $P(-\sqrt{0.4} < X < \sqrt{0.4}) = \int_{-\sqrt{0.4}}^{\sqrt{0.4}} \frac{3}{8} + \frac{3x^2}{8} dx = \left[\frac{3x}{8} + \frac{3x^3}{24} \right]_{-\sqrt{0.4}}^{\sqrt{0.4}}$
- $$\begin{aligned} &= \left(\frac{3}{8} \times \sqrt{0.4} + \frac{3}{24} \times (\sqrt{0.4})^3 \right) - \left(\frac{3}{8} \times (-\sqrt{0.4}) + \frac{3}{24} \times (\sqrt{0.4})^3 \right) \\ &= \frac{3}{4} \times \sqrt{0.4} + \frac{1}{4} (\sqrt{0.4})^3 = 0.4743 + 0.0632 = 0.538 \text{ (3 s.f.)} \end{aligned}$$

- 7 a The area under the curve must equal 1, so:

$$\int_0^2 kt^3 dt = 1$$

$$\left[\frac{kt^4}{4} \right]_0^2 = 1$$

$$4k = 1$$

$$k = \frac{1}{4}$$

- b $E(T) = \int_0^2 \frac{t^4}{4} dt = \left[\frac{t^5}{20} \right]_0^2 = \frac{32}{20} = 1.6$

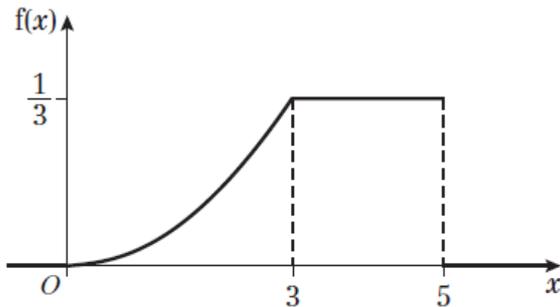
- c $E(2T + 3) = 2E(T) + 3 = 2 \times 1.6 + 3 = 6.2$

$$\begin{aligned}
 7 \text{ d } \text{Var}(T) &= \int_0^2 \frac{t^5}{4} dt - \left(\frac{8}{5}\right)^2 = \left[\frac{t^6}{24}\right]_0^2 - \left(\frac{8}{5}\right)^2 \\
 &= \frac{64}{24} - \frac{64}{25} = \frac{8}{3} - \frac{64}{25} = \frac{200}{75} - \frac{192}{75} = \frac{8}{75}
 \end{aligned}$$

$$e \text{ } \text{Var}(2T + 3) = 2^2 \text{Var}(T) = 4 \times \frac{8}{75} = \frac{32}{75}$$

$$f \text{ } P(T < 1) = \int_0^1 \frac{t^3}{4} dt = \left[\frac{t^4}{16}\right]_0^1 = \frac{1}{16}$$

- 8 a Between $(0, 0)$ and $\left(3, \frac{1}{3}\right)$, the curve is a positive quadratic with a minimum at $(0, 0)$.
Between $\left(3, \frac{1}{3}\right)$ and $\left(5, \frac{1}{3}\right)$, the graph is a horizontal straight line. Otherwise $f(x)$ is 0.

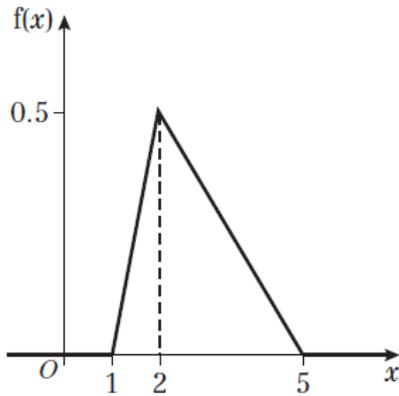


$$\begin{aligned}
 b \text{ } E(X) &= \int_0^3 \frac{x^3}{27} dx + \int_3^5 \frac{x}{3} dx = \left[\frac{x^4}{108}\right]_0^3 + \left[\frac{x^2}{6}\right]_3^5 \\
 &= \frac{81}{108} + \left(\frac{25}{6} - \frac{9}{6}\right) = \frac{9}{12} + \frac{32}{12} = \frac{41}{12} = 3.417 \text{ (3 d.p.)}
 \end{aligned}$$

$$\begin{aligned}
 c \text{ } \text{Var}(X) &= \int_0^3 \frac{x^4}{27} dx + \int_3^5 \frac{x^2}{3} dx - \left(\frac{41}{12}\right)^2 = \left[\frac{x^5}{135}\right]_0^3 + \left[\frac{x^3}{9}\right]_3^5 - \frac{1681}{144} \\
 &= \frac{243}{135} + \left(\frac{125}{9} - \frac{27}{9}\right) - \frac{1681}{144} = \frac{243}{135} + \frac{98}{9} - \frac{1681}{144} = 1.015 \text{ (3 d.p.)}
 \end{aligned}$$

$$d \text{ } \sigma = \sqrt{1.015} = 1.01 \text{ (2 d.p.)}$$

- 9 a Between (1,0) and (2,0.5), the graph is a straight line with a positive gradient, and between (2,0.5) and (5,0), the graph is a straight line with a negative gradient. Otherwise $f(x)$ is 0.



$$\begin{aligned} \text{b } E(X) &= \int_1^2 \left(\frac{x^2}{2} - \frac{x}{2} \right) dx + \int_2^5 \left(\frac{5x}{6} - \frac{x^2}{6} \right) dx = \left[\frac{x^3}{6} - \frac{x^2}{4} \right]_1^2 + \left[\frac{5x^2}{12} - \frac{x^3}{18} \right]_2^5 \\ &= \left(\frac{8}{6} - 1 \right) - \left(\frac{1}{6} - \frac{1}{4} \right) + \left(\frac{125}{12} - \frac{125}{18} \right) - \left(\frac{20}{12} - \frac{8}{18} \right) = \frac{7}{6} - \frac{3}{4} + \frac{105}{12} - \frac{117}{18} \\ &= \frac{7}{6} - \frac{3}{4} + \frac{35}{4} - \frac{13}{2} = \frac{7}{6} + \frac{48}{6} - \frac{39}{6} = \frac{16}{6} = \frac{8}{3} \end{aligned}$$

$$\begin{aligned} \text{c } \text{Var}(X) &= \int_1^2 \left(\frac{x^3}{2} - \frac{x^2}{2} \right) dx + \int_2^5 \left(\frac{5x^2}{6} - \frac{x^3}{6} \right) dx - \left(\frac{8}{3} \right)^2 = \left[\frac{x^4}{8} - \frac{x^3}{6} \right]_1^2 + \left[\frac{5x^3}{18} - \frac{x^4}{24} \right]_2^5 - \frac{64}{9} \\ &= \left(\frac{16}{8} - \frac{8}{6} \right) - \left(\frac{1}{8} - \frac{1}{6} \right) + \left(\frac{625}{18} - \frac{625}{24} \right) - \left(\frac{40}{18} - \frac{16}{24} \right) - \frac{64}{9} \\ &= \frac{15}{8} - \frac{7}{6} + \frac{585}{18} - \frac{609}{24} - \frac{64}{9} = \frac{15}{8} - \frac{21}{18} + \frac{585}{18} - \frac{203}{8} - \frac{128}{18} = \frac{436}{18} - \frac{188}{8} \\ &= \frac{436}{18} - \frac{47}{2} = \frac{436}{18} - \frac{423}{18} = \frac{13}{18} \end{aligned}$$

- 10 a The area under the curve must equal 1, so:

$$\int_0^{10} kt^2 dt = 1$$

$$\left[\frac{kt^3}{3} \right]_0^{10} = 1$$

$$\frac{1000k}{3} = 1$$

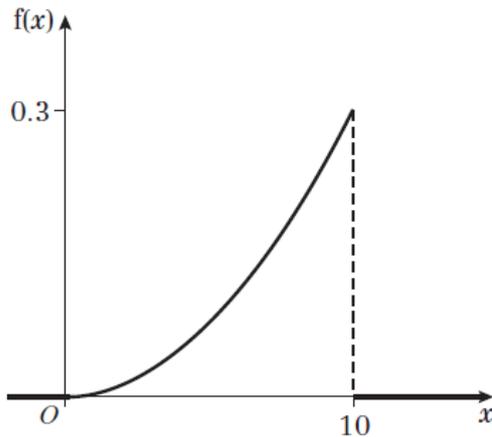
$$k = 0.003$$

$$\text{b } E(T) = \int_0^{10} 0.003t^3 dt = \left[\frac{0.003t^4}{4} \right]_0^{10} = \frac{30}{4} = 7.5$$

$$10 \text{ c } \text{Var}(x) = \int_0^{10} 0.003t^4 dt - 7.5^2 = \left[\frac{0.003t^5}{5} \right]_0^{10} - 7.5^2 = 60 - 56.25 = 3.75$$

$$10 \text{ d } P(7 < T < 9) = \int_7^9 0.003t^2 dt = \left[\frac{0.003t^3}{3} \right]_7^9 = 0.729 - 0.343 = 0.386$$

10 e Between (0, 0) and (10, 0.3), the curve is a positive quadratic with a minimum at (0, 0). Otherwise $f(x)$ is 0.



$$11 \text{ a } E(X) = \int_0^2 \frac{3}{4}x - \frac{3}{16}x^3 dx = \left[\frac{3}{8}x^2 - \frac{3}{64}x^4 \right]_0^2 = \frac{12}{8} - \frac{48}{64} = \frac{6}{4} - \frac{3}{4} = \frac{3}{4}$$

$$11 \text{ b } E(X^2) = \int_0^2 \frac{3}{4}x^2 - \frac{3}{16}x^4 dx = \left[\frac{1}{4}x^3 - \frac{3}{80}x^5 \right]_0^2 = \frac{8}{4} - \frac{96}{80} = 2 - \frac{6}{5} = \frac{4}{5}$$

$$11 \text{ c } \text{Var}(X) = E(X^2) - (E(X))^2 = \frac{4}{5} - \left(\frac{3}{4}\right)^2 = \frac{4}{5} - \frac{9}{16} = \frac{64 - 45}{80} = \frac{19}{80}$$

12 a $f(x) = \frac{d}{dx}F(x) = \frac{x}{50}$ for $0 \leq x \leq 10$; and $f(x) = 0$ otherwise

$$E(X) = \int_0^{10} \frac{x^2}{50} dx = \left[\frac{x^3}{150} \right]_0^{10} = \frac{1000}{150} = \frac{20}{3}$$

$$E(X^2) = \int_0^{10} \frac{x^3}{50} dx = \left[\frac{x^4}{2000} \right]_0^{10} = \frac{10000}{200} = 50$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = 50 - \frac{400}{9} = \frac{50}{9}$$

$$12 \text{ b } E(X^3) = \int x^3 f(x) dx = \int_0^{10} \frac{x^4}{50} dx = \left[\frac{x^5}{250} \right]_0^{10} = \frac{100000}{250} = 400$$

13 a The area under the curve must equal 1, so:

$$\int_1^3 \frac{k}{x} dx = 1$$

$$[k \ln x]_1^3 = 1$$

$$k \ln 3 = 1$$

$$k = \frac{1}{\ln 3}$$

$$\mathbf{b} \quad E(X) = \int_1^3 \frac{1}{\ln 3} dx = \frac{1}{\ln 3} [x]_1^3 = \frac{3-1}{\ln 3} = \frac{2}{\ln 3}$$

$$\mathbf{c} \quad E(X^2) = \int_1^3 \frac{x}{\ln 3} dx = \frac{1}{\ln 3} \left[\frac{x^2}{2} \right]_1^3 = \frac{9-1}{2 \ln 3} = \frac{4}{\ln 3}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{4}{\ln 3} - \left(\frac{2}{\ln 3} \right)^2 = 3.64096 - 3.3141 = 0.3268 \text{ (4 d.p.)}$$

14 a The area under the curve must equal 1, so:

$$\int_1^2 \frac{c}{x(3-x)} dx = \frac{c}{3} \int_1^2 \left(\frac{1}{x} + \frac{1}{3-x} \right) dx = 1$$

$$\frac{c}{3} [\ln x - \ln(3-x)]_1^2 = 1$$

$$\frac{c}{3} (\ln 2 - (-\ln 2)) = 1$$

$$\frac{c}{3} (2 \ln 2) = 1$$

$$\frac{c}{3} \ln 4 = 1$$

$$c = \frac{3}{\ln 4}$$

$$\mathbf{b} \quad \mu = E(X) = \frac{3}{\ln 4} \int_1^2 \frac{1}{3-x} dx = \frac{3}{\ln 4} [-\ln(3-x)]_1^2 = \frac{3 \ln 2}{\ln 4} = \frac{3 \ln 2}{2 \ln 2} = 1.5$$

$$E(X^2) = \frac{3}{\ln 4} \int_1^2 \frac{x}{3-x} dx = \frac{3}{\ln 4} \int_1^2 -1 + \frac{3}{3-x} dx = \frac{3}{\ln 4} [-x - 3 \ln(3-x)]_1^2 = \frac{3}{\ln 4} (3 \ln 2 - 1)$$

$$\sigma^2 = \text{Var}(X) = \frac{3}{\ln 4} (3 \ln 2 - 1) - 1.5^2 = 2.33596 - 2.25 = 0.0860 \text{ (4 d.p.)}$$

Challenge

Using integration by parts

$$E(\ln X) = \int_0^1 2x \ln x dx = [x^2 \ln x]_0^1 - \int_0^1 x dx = [x^2 \ln x]_0^1 - [0.5x^2]_0^1$$

$$\text{Since } \lim_{x \rightarrow 0} (x^2 \ln x) = 0, [x^2 \ln x]_0^1 = 0$$

$$\text{So } E(\ln X) = -[0.5x^2]_0^1 = -0.5$$