

## Exercise 7D

1 a  $f(x) = \sqrt{x} = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$ ,  
where  $a = 1$

$$f(x) = \sqrt{x}$$

$$f(1) = 1$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

$$f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$f''(1) = -\frac{1}{4}$$

$$f'''(x) = \frac{3}{8}x^{-\frac{5}{2}}$$

$$f'''(1) = \frac{3}{8}$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-\frac{7}{2}}$$

$$f^{(4)}(1) = -\frac{15}{16}$$

$$\text{So } \sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{4 \times 2!}(x-1)^2 + \frac{3}{8 \times 3!}(x-1)^3 - \frac{15}{16 \times 4!}(x-1)^4 + \dots$$

$$= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \dots$$

b  $\sqrt{1.2} \approx 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 - \frac{5}{128}(0.2)^4$   
 $\approx 1 + 0.1 - 0.005 + 0.0005 - 0.0000625$   
 $= 1.095 \text{ (3 d.p.)}$

2 All solutions use the Taylor expansion in the form:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \frac{f^{(4)}(a)(x-a)^4}{4!} + \dots$$

a Let  $f(x) = \ln x$  then  $f(a) = f(e) = \ln e = 1$

$$f'(x) = \frac{1}{x}$$

$$f'(a) = f'(e) = \frac{1}{e}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f''(a) = f''(e) = -\frac{1}{e^2}$$

$$\text{So } f(x) = \ln x = 1 + \frac{1}{e}(x-e) + \frac{\left(-\frac{1}{e^2}\right)}{2!}(x-e)^2 + \dots$$

$$= 1 + \frac{(x-e)}{e} - \frac{(x-e)^2}{2e^2} + \dots$$

**2 b** Let  $f(x) = \tan x$  then  $f(a) = f\left(\frac{\pi}{3}\right) = \sqrt{3}$

$f'(x) = \sec^2 x$   $f'(a) = f'\left(\frac{\pi}{3}\right) = 4$

$f''(x) = 2\sec^2 x \tan x$   $f''(a) = f''\left(\frac{\pi}{3}\right) = 2(4)(\sqrt{3}) = 8\sqrt{3}$

$f'''(x) = 2\sec^4 x + 2\tan x(2\sec^2 x \tan x)$   $f'''(a) = f'''\left(\frac{\pi}{3}\right) = 2(16) + 4(4)(3) = 80$

So  $f(x) = \tan x = \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + \frac{8\sqrt{3}}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{80}{3!}\left(x - \frac{\pi}{3}\right)^3 + \dots$

$= \sqrt{3} + 4\left(x - \frac{\pi}{3}\right) + 4\sqrt{3}\left(x - \frac{\pi}{3}\right)^2 + \frac{40}{3}\left(x - \frac{\pi}{3}\right)^3 + \dots$

**c** Let  $f(x) = \cos x$  then  $f(a) = f(1) = \cos 1$

$f'(x) = -\sin x$   $f'(a) = f'(1) = -\sin 1$

$f''(x) = -\cos x$   $f''(a) = f''(1) = -\cos 1$

$f'''(x) = \sin x$   $f'''(a) = f'''(1) = \sin 1$

$f''''(x) = \cos x$   $f''''(a) = f''''(1) = \cos 1$

So  $f(x) = \cos x = \cos 1 - \sin 1(x-1) - \frac{(\cos 1)}{2}(x-1)^2 + \frac{(\sin 1)}{6}(x-1)^3 + \frac{(\cos 1)}{24}(x-1)^4 + \dots$

**3 a i** Let  $f(x) = \cos(x)$ . Then  $\cos\left(x + \frac{\pi}{4}\right) = f\left(x + \frac{\pi}{4}\right)$

$f'(x) = \cos(x) \Rightarrow f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(x) = -\sin(x) \Rightarrow f'\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f''(x) = -\cos(x) \Rightarrow f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

$f'''(x) = \sin(x) \Rightarrow f'''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f''''(x) = \cos(x) \Rightarrow f''''\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

Using  $f(x+a) = f(a) + f'(a)x + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \frac{f''''(a)x^4}{4!} + \dots$

$\cos\left(x + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 - x - \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} - \dots\right)$

3 a ii Let  $f(x) = \ln(x)$ . Then  $\ln(x+5) = f(x+5)$

$$f(x) = \ln(x) \Rightarrow f(5) = \ln 5$$

$$f'(x) = \frac{1}{x} \Rightarrow f'(5) = \frac{1}{5}$$

$$f''(x) = -\frac{1}{x^2} \Rightarrow f''(5) = -\frac{1}{25}$$

$$f'''(x) = \frac{2}{x^3} \Rightarrow f'''(5) = \frac{2}{125}$$

$$f''''(x) = -\frac{6}{x^4} \Rightarrow f''''(5) = -\frac{6}{625}$$

$$\text{Using } f(x+a) = f(a) + f'(a)x + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \frac{f''''(a)x^4}{4!} + \dots$$

$$\ln(x+5) = \ln 5 + \frac{1}{5}x - \frac{1}{50}x^2 + \frac{1}{375}x^3 - \frac{1}{2500}x^4 + \dots$$

iii Let  $f(x) = \sin(x)$ . Then  $\sin\left(x - \frac{\pi}{3}\right) = f\left(x - \frac{\pi}{3}\right)$

$$f(x) = \sin(x) \Rightarrow f\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$f'(x) = \cos(x) \Rightarrow f'\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

$$f''(x) = -\sin(x) \Rightarrow f''\left(-\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$f'''(x) = -\cos(x) \Rightarrow f'''\left(-\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$f''''(x) = \sin(x) \Rightarrow f''''\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\text{Using } f(x+a) = f(a) + f'(a)x + \frac{f''(a)x^2}{2!} + \frac{f'''(a)x^3}{3!} + \frac{f''''(a)x^4}{4!} + \dots$$

$$\begin{aligned} \sin\left(x - \frac{\pi}{3}\right) &= \frac{1}{2} \left( -\sqrt{3} + x + \frac{\sqrt{3}}{2!}x^2 - \frac{1}{3!}x^3 - \frac{\sqrt{3}}{4!}x^4 + \dots \right) \\ &= \frac{1}{2} \left( -\sqrt{3} + x + \frac{\sqrt{3}}{2}x^2 - \frac{1}{6}x^3 - \frac{\sqrt{3}}{24}x^4 + \dots \right) \end{aligned}$$

b Taking  $x = 0.2$ ,

$$\begin{aligned} \ln 5.2 &\approx \ln 5 + \frac{0.2}{5} - \frac{0.2^2}{50} + \frac{0.2^3}{375} - \frac{0.2^4}{2500} \\ &\approx 1.649 \text{ to 4 s.f.} \end{aligned}$$

4 a  $y = xe^x, \frac{dy}{dx} = xe^x + e^x = e^x(x+1)$

Product rule.

$$\frac{d^2y}{dx^2} = xe^x + e^x + e^x = e^x(x+2)$$

$$\frac{d^3y}{dx^3} = xe^x + 2e^x + e^x = e^x(x+3)$$

Each differentiation adds another  $e^x$ , so  $\frac{d^n y}{dx^n} = (n+x)e^x$

So for  $f(x) = xe^x, f^{(n)}(x) = (n+x)e^x$

b Using the Taylor series with  $a = -1, f(-1) = -e^{-1}, f'(-1) = 0, f''(-1) = e^{-1}$   
 $f'''(-1) = 2e^{-1}, f^{(4)}(-1) = 3e^{-1}$

$$\begin{aligned} \text{So } xe^x &= e^{-1} \left\{ -1 + 0(x+1) + \frac{1}{2!}(x+1)^2 + \frac{2}{3!}(x+1)^3 + \frac{3}{4!}(x+1)^4 + \dots \right\} \\ &= e^{-1} \left\{ -1 + \frac{1}{2}(x+1)^2 + \frac{1}{3}(x+1)^3 + \frac{1}{8}(x+1)^4 + \dots \right\} \end{aligned}$$

5 a Let  $f(x) = x^3 \ln x$  then as  $a = 1$   $f(a) = f(1) = 0$

$$f'(x) = 3x^2 \ln x + x^3 \times \frac{1}{x} = x^2(1 + 3 \ln x) \quad f'(a) = f'(1) = 1$$

$$f''(x) = x^2 \times \frac{3}{x} + 2x(1 + 3 \ln x) = x(5 + 6 \ln x) \quad f''(a) = f''(1) = 5$$

$$f'''(x) = x \times \frac{6}{x} + (5 + 6 \ln x) = (11 + 6 \ln x) \quad f'''(a) = f'''(1) = 11$$

$$f^{(4)}(x) = \frac{6}{x} \quad f^{(4)}(a) = f^{(4)}(1) = 6$$

Using Taylor, form ii

$$\begin{aligned} f(x) = x^3 \ln x &= 0 + 1(x-1) + \frac{5}{2!}(x-1)^2 + \frac{11}{3!}(x-1)^3 + \frac{6}{4!}(x-1)^4 + \dots \\ &= (x-1) + \frac{5}{2}(x-1)^2 + \frac{11}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 + \dots \end{aligned}$$

b Substituting  $x = 1.5$  in series in a, gives

$$\begin{aligned} \frac{27}{8} \ln 1.5 &\approx 0.5 + \frac{5}{2}(0.5)^2 + \frac{11}{6}(0.5)^3 + \frac{1}{4}(0.5)^4 + \dots \\ &\approx 0.5 + 0.625 + 0.22916\dots + 0.015625 (= 1.369791\dots) \end{aligned}$$

So this gives an approximation for  $\ln 1.5$  of  $\frac{8}{27}(1.369791\dots) = 0.4059$  (4d.p.)

- 6 We look to find the Taylor series for the function  $\tan(x - \alpha)$  about  $x = 0$  where,  $\alpha = \arctan\left(\frac{3}{4}\right)$

Let  $f(x) = \tan(x - \alpha)$  then;

$$f(0) = \tan(-\alpha) = -\tan(\alpha) = -\frac{3}{4}$$

$$f'(x) = \sec^2(x - \alpha) \Rightarrow f'(0) = \sec^2(-\alpha) = \sec^2 \alpha = \frac{25}{16}$$

Now we use the fact that;

$$f''(x) = 2 \tan(x - \alpha) \sec^2(x - \alpha) \Rightarrow f''(0) = -2 \tan \alpha \sec^2 \alpha$$

$$\Rightarrow f''(0) = -2 \cdot \frac{3}{4} \cdot \frac{25}{16} = -\frac{75}{32}$$

Finally, using  $f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \dots(x^3)$ , we find that

$$f(x) = \tan(x - \alpha) = -\frac{3}{4} + \frac{25}{16} \cdot x + \frac{1}{2!} \cdot \frac{-75}{32} \cdot x^2 + \dots(x^3)$$

$$\Rightarrow \tan(x - \alpha) = -\frac{3}{4} + \frac{25}{16}x - \frac{75}{64}x^2 + \dots(x^3)$$

- 7 Let  $f(x) = \sin 2x$ , then we calculate the required derivatives;

$$f\left(\frac{\pi}{6}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f'(x) = 2 \cos 2x \Rightarrow f'\left(\frac{\pi}{6}\right) = 2 \cos \frac{\pi}{3} = 1$$

$$f''(x) = -4 \sin 2x \Rightarrow f''\left(\frac{\pi}{6}\right) = -4 \sin \frac{\pi}{3} = -\frac{4\sqrt{3}}{2} = -2\sqrt{3}$$

$$f'''(x) = -8 \cos 2x \Rightarrow f'''\left(\frac{\pi}{6}\right) = -8 \cos \frac{\pi}{3} = -4$$

$$f^{(4)}(x) = 16 \sin 2x \Rightarrow f^{(4)}\left(\frac{\pi}{6}\right) = 8\sqrt{3}$$

$$f(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{1}{2!}f''\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)^2$$

Then, using the Taylor expansion:

$$+ \frac{1}{3!}f'''\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{4!}f^{(4)}\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)^4 + \dots\left(\left(x - \frac{\pi}{6}\right)^5\right)$$

We find that

$$\begin{aligned} \sin 2x &= \frac{\sqrt{3}}{2} + 1 \cdot \left(x - \frac{\pi}{6}\right) + \frac{-2\sqrt{3}}{2!} \cdot \left(x - \frac{\pi}{6}\right)^2 \\ &\quad + \frac{-4}{3!} \cdot \left(x - \frac{\pi}{6}\right)^3 + \frac{8\sqrt{3}}{4!} \cdot \left(x - \frac{\pi}{6}\right)^4 + \dots\left(\left(x - \frac{\pi}{6}\right)^5\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \sin 2x &= \frac{\sqrt{3}}{2} + \left(x - \frac{\pi}{6}\right) - \sqrt{3}\left(x - \frac{\pi}{6}\right)^2 \\ &\quad - \frac{2}{3}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sqrt{3}}{3}\left(x - \frac{\pi}{6}\right)^4 + \dots\left(\left(x - \frac{\pi}{6}\right)^5\right) \end{aligned}$$

## Further Pure Maths 2

## Solution Bank

**8 a** Given  $y = \frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$

$y_3 (= \text{value of } y \text{ when } x = 3) = \frac{1}{2}$

$$\frac{dy}{dx} = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$$

$$\left(\frac{dy}{dx}\right)_3 = -\frac{1}{2} \times \frac{1}{8} = -\frac{1}{16}$$

$$\frac{d^2y}{dx^2} = \frac{3}{4}(1+x)^{-\frac{5}{2}}$$

$$\left(\frac{d^2y}{dx^2}\right)_3 = \frac{3}{4} \times \frac{1}{32} = \frac{3}{128}$$

**b** So using

$$f(x) = f(3) + f'(3)(x-3) + \frac{f''(3)}{2!}(x-3)^2 + \dots \quad \text{with } f^{(n)}(3) \equiv \left(\frac{d^n y}{dx^n}\right)_3$$

$$y = \frac{1}{\sqrt{1+x}} = \frac{1}{2} - \frac{1}{16}(x-3) + \frac{3}{256}(x-3)^2 + \dots$$

$$9 \quad f(x) = \ln x$$

$$f'(x) = \frac{1}{x} = \frac{0!}{x}$$

$$f''(x) = -\frac{1}{x^2} = -\frac{1!}{x^2}$$

$$f'''(x) = \frac{2}{x^3} = \frac{2!}{x^3}$$

$$f^{iv}(x) = -\frac{2 \times 3}{x^4} = -\frac{3!}{x^4}$$

$$f^v(x) = \frac{2 \times 3 \times 4}{x^5} = \frac{4!}{x^5}$$

$$f^{vi}(x) = -\frac{2 \times 3 \times 4 \times 5}{x^6} = -\frac{5!}{x^6}$$

Therefore:

$$\begin{aligned} f^q(x) &= \frac{(-1)^{q-1} (q-1)!}{x^q} \\ &= \frac{(-1)^{q-1} q(q-1)!}{qx^q} \end{aligned}$$

$$= \frac{(-1)^{q-1} q!}{qx^q}$$

$$f^q(2) = \frac{(-1)^{q-1} q!}{q2^q}$$

$$f(x) = \ln x$$

$$= f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!} + \dots + f^n(2)\frac{(x-2)^n}{n!}$$

If  $q = 1$  then:

$$f'(2) = \frac{(-1)^0 1!}{1 \times 2^1}$$

If  $q = n$  then:

$$f^n(2) = \frac{(-1)^{n-1} n!}{n \times 2^n}$$

$$f(x) = \ln x$$

$$= f(2) + \frac{(-1)^0}{1 \times 2^1} 1!(x-2) + \frac{(-1)^1}{2 \times 2^2} 2!\frac{(x-2)^2}{2!} + \dots + \frac{(-1)^{n-1}}{n2^n} n!\frac{(x-2)^n}{n!} + \dots$$

$$= \ln 2 + \frac{(-1)^0}{1 \times 2^1} (x-2) + \frac{(-1)^1}{2 \times 2^2} (x-2)^2 + \dots + \frac{(-1)^{n-1}}{n2^n} (x-2)^n + \dots$$

$$= \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n} (x-2)^n \quad \text{as required}$$

**Challenge**

- a** Let  $f(x) = \ln(\cos 2x)$ , then we will use:

$$f(x) = f(\pi) + f'(\pi)(x - \pi) + \frac{1}{2!}f''(\pi)(x - \pi)^2 + \frac{1}{3!}f'''(\pi)(x - \pi)^3 + \dots((x - \pi)^4)$$

Differentiating:

$$f(\pi) = \ln(\cos 2\pi) = 0$$

$$f'(x) = -\frac{2 \sin 2x}{\cos 2x} = -2 \tan 2x \Rightarrow f'(\pi) = 0$$

$$f''(x) = -4 \sec^2 2x \Rightarrow f''(\pi) = -\frac{4}{1^2} = -4$$

$$f'''(x) = -16 \tan 2x \sec^2 2x \Rightarrow f'''(\pi) = 0$$

$$\text{So, } \ln(\cos 2x) = -2(x - \pi)^2 - \frac{4}{3}(x - \pi)^4 - \dots$$

- b** Note that  $\cos\left(2 \cdot \frac{13\pi}{12}\right) = \frac{\sqrt{3}}{2}$ , so we take the leading order term in the expansion above setting  $x = \frac{13\pi}{12}$

$$\ln\left(\cos \frac{13\pi}{6}\right) = \ln \frac{\sqrt{3}}{2} \approx -2\left(\frac{13\pi}{12} - \pi\right)^2 = -0.1433 \text{ (4 d.p.)}$$