

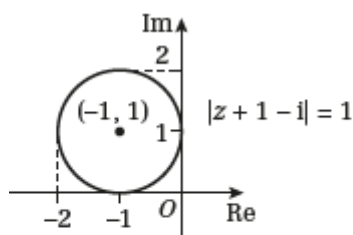
## Chapter review 4

- 1 a  $|z + 1 - i| = 1$  is the circle with centre  $(-1, 1)$  and radius 1

Therefore the Cartesian equation is:

$$(x+1)^2 + (y-1)^2 = 1$$

b



- c The distance  $OC$  where  $C$  is the centre of this circle is  $\sqrt{2}$

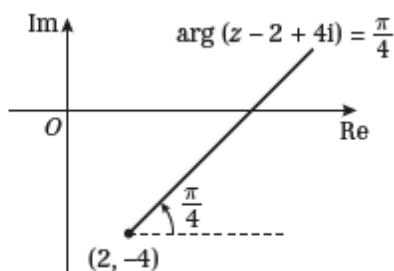
Therefore  $|z|_{\min} = \sqrt{2} - 1$  and  $|z|_{\max} = \sqrt{2} + 1$

- d  $|z - 1|$  is the circle with centre  $(-2, 1)$  and radius 1

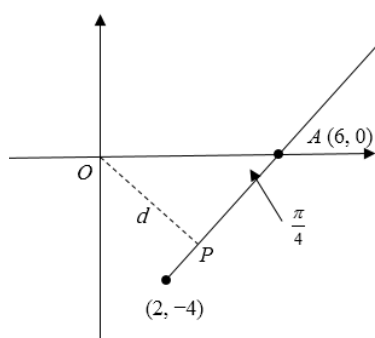
The distance  $OD$  where  $D$  is the centre of this circle is  $\sqrt{5}$

Therefore  $|z - 1|_{\min} = \sqrt{5} - 1$  and  $|z - 1|_{\max} = \sqrt{5} + 1$

- 2 a  $\arg(z - 2 + 4i) = \frac{\pi}{4}$



2 b The half-line cuts the  $x$ -axis at  $(6, 0)$



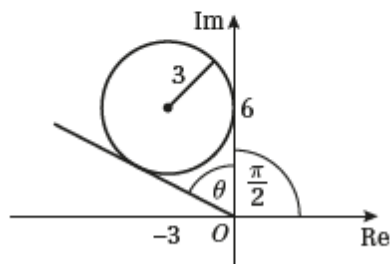
$$\sin\left(\frac{\pi}{4}\right) = \frac{d}{6}$$

$$d = 6 \sin\left(\frac{\pi}{4}\right)$$

$$= 3\sqrt{2}$$

$$\text{Therefore } |z|_{\min} = 3\sqrt{2}$$

3  $|z + 3 - 6i| = 3$  is the circle with centre  $(-3, 6)$  and radius 3



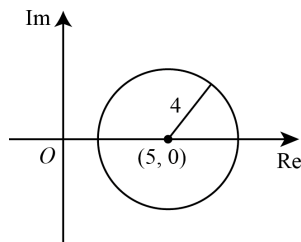
$$\begin{aligned} \sin\left(\frac{\theta}{2}\right) &= \frac{3}{\sqrt{3^2 + 6^2}} \\ &= \frac{1}{\sqrt{5}} \end{aligned}$$

The maximum is at  $\frac{\pi}{2} + \theta$

Therefore:

$$\frac{\pi}{2} + \theta = \frac{\pi}{2} + 2 \sin^{-1}\left(\frac{1}{\sqrt{5}}\right) \text{ as required}$$

- 4 a  $|z - 5| = 4$  is the circle with centre  $(5, 0)$  and radius 4



- b  $\arg(z + 3i) = \frac{\pi}{3}$  is the half-line originating at  $(0, -3)$  at an angle of  $\frac{\pi}{3}$  to the  $x$ -axis.

The half-line has a gradient of  $\sqrt{3}$  and originates at  $(0, -3)$ , therefore is part of the line

$$y = \sqrt{3}x - 3$$

Substituting into  $(x - 5)^2 + y^2 = 16$  gives:

$$(x - 5)^2 + (\sqrt{3}x - 3)^2 = 16$$

$$x^2 - 10x + 25 + 3x^2 - 6\sqrt{3}x + 9 = 16$$

$$4x^2 - (10 + 6\sqrt{3})x + 18 = 0$$

$$x = \frac{10 + 6\sqrt{3} \pm \sqrt{(10 + 6\sqrt{3})^2 - 4(4)(18)}}{2(4)}$$

$$x = \frac{10 + 6\sqrt{3} \pm 11.306...}{8}$$

$$x = 1.135... \text{ or } x = 3.962...$$

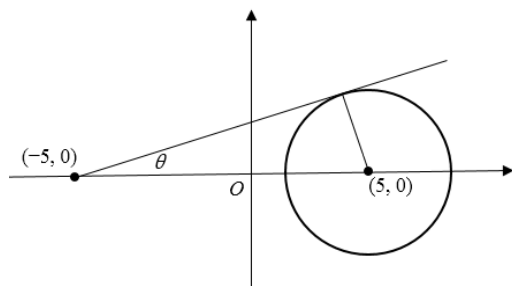
$$\text{when } x = 1.135..., y = -1.032...$$

$$\text{when } x = 3.962..., y = 3.863...$$

Therefore the required solutions are:

$$z = 1.14 - 1.03i \text{ and } z = 3.96 + 3.86i$$

4 c



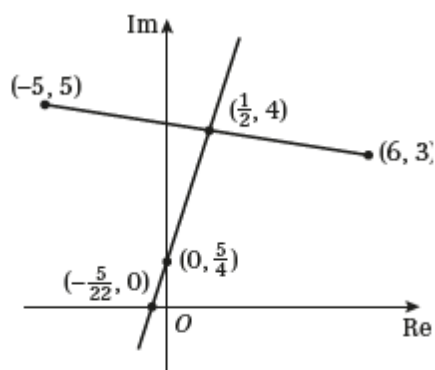
$$\sin \theta = \frac{4}{10}$$

$$\theta = 0.4115\dots$$

Therefore:

$$0.412 < \theta < \pi \text{ and } -\pi < \theta < -0.412$$

5 a



$$5 \text{ b } |z + 5 - 5i| = |z - 6 - 3i|$$

$$|x + yi + 5 - 5i| = |x + yi - 6 - 3i|$$

$$|(x + 5) + i(y - 5)| = |(x - 6) + i(y - 3)|$$

$$|(x + 5) + i(y - 5)|^2 = |(x - 6) + i(y - 3)|^2$$

$$(x + 5)^2 + (y - 5)^2 = (x - 6)^2 + (y - 3)^2$$

$$x^2 + 10x + 25 + y^2 - 10y + 25 = x^2 - 12x + 36 + y^2 - 6y + 9$$

$$22x + 50 = 4y + 45$$

$$y = \frac{11}{2}x + \frac{5}{4}$$

c Since the locus of  $|z + 5 - 5i| = |z - 6 - 3i|$  has gradient  $\frac{11}{2}$ ,

a perpendicular to the locus will have a gradient of  $-\frac{2}{11}$

Using  $y - y_1 = m(x - x_1)$  at  $(0, 0)$  with gradient  $-\frac{2}{11}$  gives:

$$y = -\frac{2}{11}x$$

To find the point of intersection,  $P$ , equate  $y = -\frac{2}{11}x$  and  $y = \frac{11}{2}x + \frac{5}{4}$

$$-\frac{2}{11}x = \frac{11}{2}x + \frac{5}{4}$$

$$-\frac{125}{22}x = \frac{5}{4}$$

$$x = -\frac{11}{50} \text{ and } y = \frac{1}{25}$$

$$|OP| = \sqrt{\left(-\frac{11}{50}\right)^2 + \left(\frac{1}{25}\right)^2}$$

$$= \frac{\sqrt{5}}{10}$$

6 a  $|z - 4| = |z - 8i|$

$$|x + yi - 4| = |x + yi - 8i|$$

$$|(x - 4) + yi| = |x + i(y - 8)|$$

$$|(x - 4) + yi|^2 = |x + i(y - 8)|^2$$

$$(x - 4)^2 + y^2 = x^2 + (y - 8)^2$$

$$x^2 - 8x + 16 + y^2 = x^2 + y^2 - 16y + 64$$

$$-8x + 16 = -16y + 64$$

$$16y = 8x + 48$$

$$y = \frac{1}{2}x + 3$$

- b  $\arg z = \frac{\pi}{4}$  is the half-line originating at the point  $(0, 0)$  at an angle of  $\frac{\pi}{4}$  to positive  $x$ -axis, therefore the half-line is part of the line  $y = x$

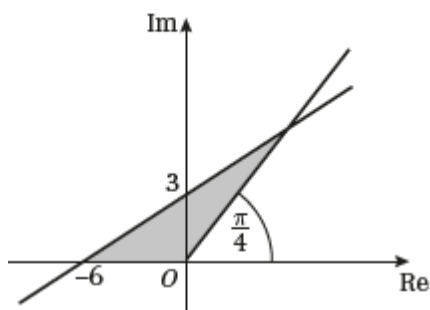
Equating  $y = \frac{1}{2}x + 3$  and  $y = x$  gives:

$$x = \frac{1}{2}x + 3$$

$$x = 6 \text{ and } y = 6$$

Therefore  $z = 6 + 6i$

c  $\{z \in \mathbb{C} : |z - 4| \leq |z - 8i|\} \cap \left\{z \in \mathbb{C} : \frac{\pi}{4} \leq \arg(z) \leq \pi\right\}$



**7 a i**  $|z - 3 + i| = |z - 1 - i|$

$$|x + yi - 3 + i| = |x + yi - 1 - i|$$

$$|(x - 3) + i(y + 1)| = |(x - 1) + i(y - 1)|$$

$$|(x - 3) + i(y + 1)|^2 = |(x - 1) + i(y - 1)|^2$$

$$(x - 3)^2 + (y + 1)^2 = (x - 1)^2 + (y - 1)^2$$

$$x^2 - 6x + 9 + y^2 + 2y + 1 = x^2 - 2x + 1 + y^2 - 2y + 1$$

$$-6x + 2y + 10 = -2x - 2y + 2$$

$$4y = 4x - 8$$

$$y = x - 2$$

**ii**  $|x + yi - 2| = 2\sqrt{2}$

$$|(x - 2) + yi| = 2\sqrt{2}$$

$$|(x - 2) + yi|^2 = 8$$

$$(x - 2)^2 + y^2 = 8$$

**b** Substituting  $y = x - 2$  into  $(x - 2)^2 + y^2 = 8$  gives:

$$(x - 2)^2 = 4$$

$$x - 2 = \pm 2$$

$$x = 2 \pm 2$$

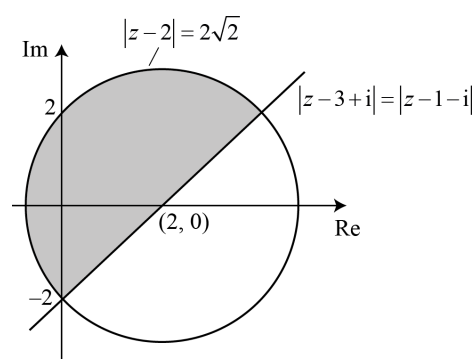
$$x = 0 \text{ or } x = 4$$

$$\text{When } x = 0, y = -2$$

$$\text{When } x = 4, y = 2$$

$$\text{Therefore } z = -2i \text{ and } z = 4 + 2i$$

**c**



**8 a i**  $|z| = |z - 4|$

$$|x + yi| = |x + yi - 4|$$

$$|x + yi| = |(x - 4) + yi|$$

$$|x + yi|^2 = |(x - 4) + yi|^2$$

$$x^2 + y^2 = (x - 4)^2 + y^2$$

$$x^2 + y^2 = x^2 - 8x + 16 + y^2$$

$$0 = -8x + 16$$

$$x = 2$$

**ii** The perpendicular bisector of (0, 0) and (4, 0)

**b i**  $|z| = 2|z - 4|$

$$|x + yi| = 2|(x - 4) + yi|$$

$$|x + yi|^2 = 4|(x - 4) + yi|^2$$

$$x^2 + y^2 = 4(x - 4)^2 + 4y^2$$

$$x^2 + y^2 = 4x^2 - 32x + 64 + 4y^2$$

$$3x^2 - 32x + 3y^2 = -64$$

$$x^2 - \frac{32}{3}x + y^2 = -\frac{64}{3}$$

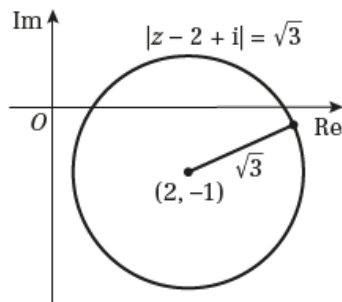
$$\left(x - \frac{32}{6}\right)^2 - \left(\frac{32}{6}\right)^2 + y^2 = -\frac{64}{3}$$

$$\left(x - \frac{16}{3}\right)^2 + y^2 = \frac{64}{9}$$

**ii** Circle with centre  $\left(\frac{16}{3}, 0\right)$  and radius  $\frac{8}{3}$



- 9 a The equation  $|z - 2 + i| = \sqrt{3}$  describes a circle centred at  $(2, -1)$ , radius  $r = \sqrt{3}$ :



- b The half-line  $L$   $y = mx - 1$ ,  $x \geq 0$ ,  $m > 0$  is tangent to the circle from part a. This means that it has to lie on the circle and thus satisfy the equation  $(x - 2)^2 + (y + 1)^2 = 3$ . Substituting the expression for  $y$  into this equation gives:

$$(x - 2)^2 + (mx)^2 = 3$$

$$x^2 - 4x + 4 + m^2x^2 - 3 = 0$$

$$x^2(1 + m^2) - 4x + 1 = 0$$

$$x^2 - \frac{4}{1 + m^2}x + \frac{1}{1 + m^2} = 0$$

This equation must have exactly one solution, as the line and the circle only touch at one point. Therefore, it needs to be of the form:

$$\left(x - \frac{2}{1 + m^2}\right)^2 = 0 \text{ which means that } \left(\frac{2}{1 + m^2}\right)^2 = \frac{1}{1 + m^2}$$

Solving this gives:

$$\frac{4}{(1 + m^2)^2} = \frac{1}{1 + m^2}$$

$$4 = 1 + m^2$$

$$m^2 = 3$$

$$m = \sqrt{3}$$

since we know that  $m > 0$ .

So  $L$  is given by  $y = x\sqrt{3} - 1$ .

- c For  $x = 0$ ,  $y = -1$  so the line goes through  $(0, -1)$ . The gradient is equal to  $\sqrt{3}$ , so  $\tan \theta = \sqrt{3}$ .

Therefore  $\theta = \frac{\pi}{3}$ . So the equation for  $L$  can be written as  $\arg(z + i) = \frac{\pi}{3}$ .

- d Using  $\left(x - \frac{2}{1 + m^2}\right)^2 = 0$  from part b and substituting  $m = \sqrt{3}$  we obtain  $\left(x - \frac{1}{2}\right)^2 = 0$ , so  $x = \frac{1}{2}$ .

Now substituting this value into  $y = x\sqrt{3} - 1$  we see  $y = \frac{\sqrt{3} - 2}{2}$ . So  $a = \frac{1}{2} + \left(\frac{\sqrt{3} - 2}{2}\right)i$ .

**10 a**  $|z + 2| = |2z - 1|$

$$\Rightarrow |x + iy + 2| = |2(x + iy) - 1|$$

$$\Rightarrow |x + iy + 2| = |2x + 2iy - 1|$$

$$\Rightarrow |(x + 2) + iy| = |(2x - 1) + i(2y)|$$

$$\Rightarrow |(x + 2) + iy|^2 = |(2x - 1) + i(2y)|^2$$

$$\Rightarrow (x + 2)^2 + y^2 = (2x - 1)^2 + (2y)^2$$

$$\Rightarrow x^2 + 4x + 4 + y^2 = 4x^2 - 4x + 1 + 4y^2$$

$$\Rightarrow 0 = 3x^2 - 8x + 3y^2 + 1 - 4$$

$$\Rightarrow 3x^2 - 8x + 3y^2 - 3 = 0$$

$$\Rightarrow x^2 - \frac{8}{3}x + y^2 - 1 = 0$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1 = 0$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{16}{9} + 1$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}$$

$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \left(\frac{5}{3}\right)^2$$

This is a circle, centre  $\left(\frac{4}{3}, 0\right)$ , radius  $\frac{5}{3}$ .

The Cartesian equation of the locus of points representing  $|z + 2| = |2z - 1|$  is

$$\left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9}.$$

$$10 \text{ b } |z+2| = |2z-1| \Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 = \frac{25}{9} \quad (1)$$

$$\arg z = \frac{\pi}{4} \Rightarrow \arg(x+iy) = \frac{\pi}{4}$$

$$\Rightarrow \frac{y}{x} = \tan \frac{\pi}{4}$$

$$\Rightarrow \frac{y}{x} = 1$$

$$\Rightarrow y = x \quad \text{where } x > 0, y > 0 \quad (2)$$

Solving simultaneously:

$$\left(x - \frac{4}{3}\right)^2 + x^2 = \frac{25}{9}$$

$$\Rightarrow x^2 - \frac{4}{3}x - \frac{4}{3}x + \frac{16}{9} + x^2 = \frac{25}{9}$$

$$\Rightarrow 2x^2 - \frac{8}{3}x = \frac{25}{9} - \frac{16}{9}$$

$$\Rightarrow 2x^2 - \frac{8}{3}x = \frac{9}{9}$$

$$\Rightarrow 2x^2 - \frac{8}{3}x = 1 \quad (\times 3)$$

$$\Rightarrow 6x^2 - 8x = 3$$

$$\Rightarrow 6x^2 - 8x - 3 = 0$$

$$\Rightarrow x = \frac{8 \pm \sqrt{64 - 4(6)(-3)}}{2(6)}$$

$$\Rightarrow x = \frac{8 \pm \sqrt{136}}{12}$$

$$\Rightarrow x = \frac{8 \pm 2\sqrt{34}}{12}$$

$$\Rightarrow x = \frac{4 \pm \sqrt{34}}{6}$$

As  $x > 0$  then we reject  $x = \frac{4 - \sqrt{34}}{6}$

and accept  $x = \frac{4 + \sqrt{34}}{6}$

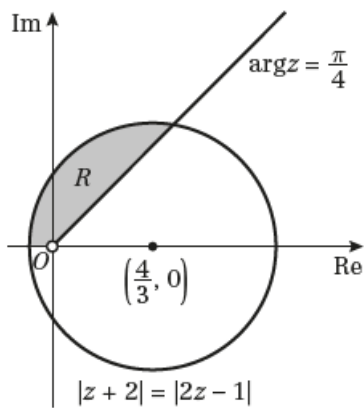
as  $y = x$ , then  $y = \frac{4 + \sqrt{34}}{6}$

$$\text{So } z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)i$$

The value of  $z$  satisfying  $|z+2| = |2z-1|$  and  $\arg z = \frac{\pi}{4}$

$$\text{is } z = \left(\frac{4 + \sqrt{34}}{6}\right) + \left(\frac{4 + \sqrt{34}}{6}\right)i \quad \text{OR } z = 1.64 + 1.64i \text{ (2 d.p.)}$$

10 c



The region R (shaded) satisfies both  $|z + 2| \geq |2z - 1|$  and  $\frac{\pi}{4} \leq \arg z \leq \pi$ .

Note that  $|z + 2| \geq |2z - 1|$

$$\Rightarrow (x + 2)^2 + y^2 \geq (2x - 1)^2 + (2y)^2$$

$$\Rightarrow 0 \geq 3x^2 - 8x + 3y^2 - 3$$

$$\Rightarrow 0 \geq \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1$$

$$\Rightarrow \frac{25}{9} \geq \left(x - \frac{4}{3}\right)^2 + y^2$$

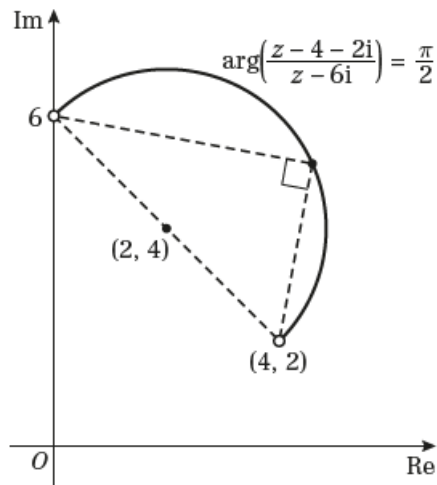
$$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 \leq \frac{25}{9}$$

represents region inside and bounded by the circle, centre  $\left(\frac{4}{3}, 0\right)$ , radius  $\frac{5}{3}$ .

11 a  $\arg\left(\frac{z-4-2i}{z-6i}\right) = \frac{\pi}{2}$

$$\Rightarrow \arg(z-4-2i) - \arg(z-6i) = \frac{\pi}{2}$$

$$\Rightarrow \theta - \phi = \frac{\pi}{2}, \text{ where } \arg(z-4-2i) = \theta \text{ and } \arg(z-6i) = \phi.$$



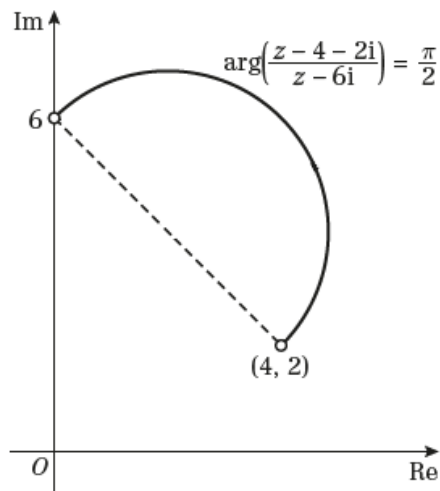
Using geometry,

$$\Rightarrow \hat{APB} = -\phi + \theta$$

$$\Rightarrow \hat{APB} = \theta - \phi$$

$$\Rightarrow \hat{APB} = \frac{\pi}{2}$$

The locus of  $z$  is the arc of a circle (in this case, a semi-circle) cut off at  $(4, 2)$  and  $(0, 6)$  as shown below.



**11 b**  $|z - 2 - 4i|$  is the distance from the point  $(2, 4)$  to the locus of points  $P$ .

Note, as the locus is a semi-circle, its centre is  $\left(\frac{4+0}{2}, \frac{2+6}{2}\right) = (2, 4)$ .

Therefore  $|z - 2 - 4i|$  is the distance from the centre of the semi-circle to points on the locus of points  $P$ .

Hence  $|z - 2 - 4i| = \text{radius of semi-circle}$

$$\begin{aligned}
 &= \sqrt{(0-2)^2 + (6-4)^2} \\
 &= \sqrt{4+4} \\
 &= \sqrt{8} \\
 &= 2\sqrt{2}
 \end{aligned}$$

The exact value of  $|z - 2 - 4i|$  is  $2\sqrt{2}$

**12** We have  $2|z + 3| = |z - 3|$

**a** To show that this describes a circle, write  $z = x + iy$  and square both sides:

$$2|x + 3 + iy| = |x - 3 + iy|$$

$$4|x + 3 + iy|^2 = |x - 3 + iy|^2$$

$$4(x + 3)^2 + 4y^2 = (x - 3)^2 + y^2$$

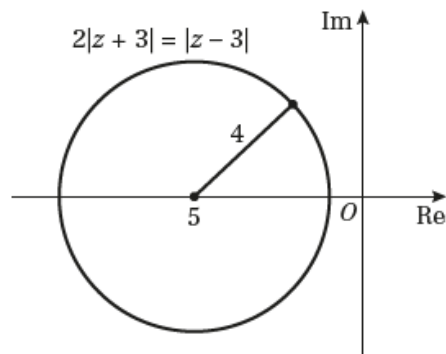
$$4x^2 + 24x + 36 + 4y^2 = x^2 - 6x + 9$$

$$3x^2 + 30x + 27 + 4y^2 = 0$$

$$x^2 + y^2 + 10x + 9 = 0$$

as required.

**b**



**12 c**  $L$  is given by  $bz^* + b^*z = 0$  where  $b, z \in \mathbb{R}$ .

We know that  $L$  is tangent to the circle and that  $\arg b = \theta$ .

We want to find possible values of  $\tan \theta$ .

Write  $z = x + iy$ ,  $b = u + iv$ .

Then the equation for  $L$  becomes:

$$b(x - iy) + b^*(x + iy) = 0$$

$$(u + iv)(x - iy) + (u - iv)(x + iy) = 0$$

$$ux + vy = 0$$

$$y = -\frac{ux}{v}, \quad v \neq 0$$

If  $v = 0$  then  $ux = 0$ , so either  $u = 0$  i.e.  $b = 0$ , which means that the line does not exist, or  $x = 0$  but this is not tangent to the circle.

So we can assume  $v \neq 0$ .

$$\text{Now, since } b = u + iv, \quad \tan \theta = \frac{v}{u}.$$

$$\text{So } L \text{ can be written as } y = -\frac{x}{\tan \theta}$$

We want to find  $\tan \theta$  such that the line is tangent to the circle.

Therefore, it has to satisfy the equation for the circle  $x^2 + y^2 + 10x + 9 = 0$ :

$$x^2 + \frac{x^2}{\tan^2 \theta} + 10x + 9 = 0$$

$$x^2 \tan^2 \theta + x^2 + 10x \tan^2 \theta + 9 \tan^2 \theta = 0$$

$$x^2 (\tan^2 \theta + 1) + 10x \tan^2 \theta + 9 \tan^2 \theta = 0$$

$$\text{Let } a = \tan^2 \theta. \text{ Then } x^2(a + 1) + 10xa + 9a = 0.$$

Since the line is tangent to the circle, this equation can only have one solution.

Therefore we need  $\Delta = 0$ .

Therefore

$$100a^2 - 36a(a + 1) = 0$$

$$100a^2 - 36a^2 - 36a = 0$$

$$a(64a - 36) = 0$$

$$a = 0 \quad \text{or} \quad a = \frac{36}{64} = \frac{9}{16}$$

$$\text{Recall that } a = \tan^2 \theta = \frac{v^2}{u^2}$$

Since  $v \neq 0$ , we have  $a \neq 0$

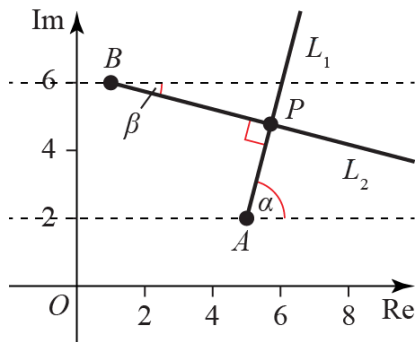
So  $a = \frac{9}{16}$  and we can solve for  $\tan \theta$ :

$$\tan^2 \theta = \frac{9}{16}$$

$$\tan \theta = \pm \frac{3}{4}$$

**13 a** Let  $\arg(z - 5 - 2i) = \alpha$  and  $\arg(z - 1 - 6i) = \beta$ .

Then we have  $\arg\left(\frac{z - 5 - 2i}{z - 1 - 6i}\right) = \arg(z - 5 - 2i) - \arg(z - 1 - 6i) = \alpha - \beta = \frac{\pi}{2}$



As  $\alpha, \beta$  vary,  $P$  i.e. the intersection of  $L_1$  and  $L_2$  creates an arc.

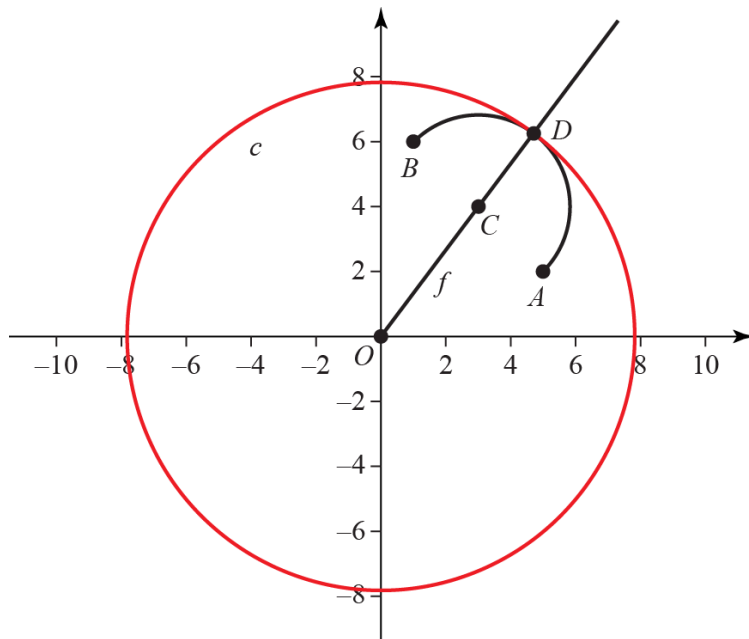
Since  $\angle APB = \frac{\pi}{2}$ , this arc will be a semicircle and the line segment  $AB$  is its diameter.

The centre of the circle lies in the middle of the line segment  $AB$ .  $A = (5, 2)$  and  $B = (1, 6)$  so the midpoint is  $C = (3, 4)$ .

The radius is the distance  $CA$ .  $CA = \sqrt{(5-3)^2 + (2-4)^2} = 2\sqrt{2}$ , so  $r = 2\sqrt{2}$ .



- 13 b** The maximum value will lie on the line connecting the centre of this circle with the origin. This is represented by point  $D$  on the diagram below:



Thus  $D$  satisfies both the equation of the semicircle described in part **a**,  $(x-3)^2 + (y-4)^2 = 8$ , and the equation of the line going through the origin and the centre of that semicircle,  $y = \frac{4x}{3}$ .

Substituting this into the equation for the circle we obtain:

$$(x-3)^2 + \left(\frac{4x}{3} - 4\right)^2 = 8$$

$$x^2 - 6x + 9 + \frac{16x^2}{9} - \frac{32x}{3} + 16 - 8 = 0$$

$$\frac{25x^2}{9} - \frac{50x}{3} + 17 = 0$$

$$x^2 - 6x + \frac{153}{25} = 0$$

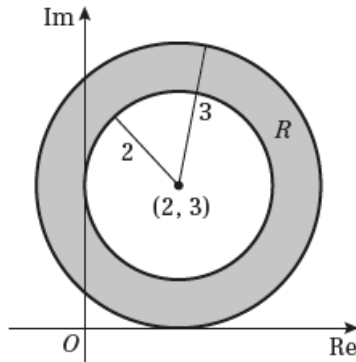
$$x = 3 \pm \frac{6\sqrt{2}}{5}$$

We're looking for the larger value, so  $x = 3 + \frac{6\sqrt{2}}{5}$ ,  $y = 4 + \frac{8\sqrt{2}}{5}$  and so:

$$|z| = \sqrt{x^2 + y^2} = \sqrt{(5 + 2\sqrt{2})^2} = 5 + 2\sqrt{2}.$$

**14 a** First note that both  $2 = |z - 2 - 3i|$  and  $3 = |z - 2 - 3i|$  represent circles centred at  $(2, 3)$  with radius  $r = 2$  and  $r = 3$  respectively.

Thus  $2 \leq |z - 2 - 3i| \leq 3$  represents the region between these two circles, including the circles since the inequalities are not strict.



**b** The area of this region can be found by subtracting the area of the smaller circle from the area of the larger circle  $P_{\text{region}} = P_{\text{large}} - P_{\text{small}} = 9\pi - 4\pi = 5\pi$ .

**c** We want to determine whether  $z = 4 + i$  lies within the region.

We have  $|4 + i - 2 - 3i| = |2 - 2i| = 2|1 - i| = 2\sqrt{2}$ .

Since  $2 \leq 2\sqrt{2} \leq 3$ , the point lies in the region.

$$15 \quad T: w = \frac{1}{z}$$

**a** line  $x = \frac{1}{2}$  in the  $z$ -plane

$$w = \frac{1}{z}$$

$$\Rightarrow wz = 1$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow z = \frac{1}{u + iv}$$

$$\Rightarrow z = \frac{1}{(u + iv)} \times \frac{(u - iv)}{(u - iv)}$$

$$\Rightarrow z = \frac{u - iv}{u^2 + v^2}$$

$$\Rightarrow z = \frac{u}{u^2 + v^2} + i \left( \frac{-v}{u^2 + v^2} \right)$$

$$\text{So, } x + iy = \frac{u}{u^2 + v^2} + i \left( \frac{-v}{u^2 + v^2} \right)$$

$$\Rightarrow x = \frac{u}{u^2 + v^2} \text{ and } y = \frac{-v}{u^2 + v^2}$$

$$\text{As } x = \frac{1}{2}, \text{ then } \frac{1}{2} = \frac{u}{u^2 + v^2}$$

$$\Rightarrow u^2 + v^2 = 2u$$

$$\Rightarrow u^2 - 2u + v^2 = 0$$

$$\Rightarrow (u - 1)^2 - 1 + v^2 = 0$$

$$\Rightarrow (u - 1)^2 + v^2 = 1$$

Therefore the transformation  $T$  maps the line  $x = \frac{1}{2}$  in the  $z$ -plane to a circle  $C$ , with centre  $(1, 0)$ , radius 1. The equation of  $C$  is  $(u - 1)^2 + v^2 = 1$ .

**15 b**  $x \geq \frac{1}{2}$

$$\frac{u}{u^2 + v^2} \geq \frac{1}{2}$$

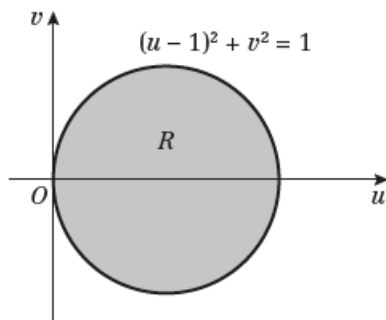
$$\Rightarrow 2u \geq u^2 + v^2$$

$$\Rightarrow 0 \geq u^2 + v^2 - 2u$$

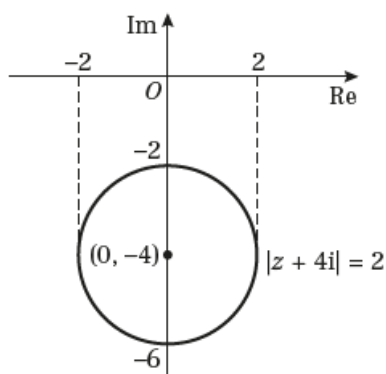
$$\Rightarrow 0 \geq (u-1)^2 + v^2 - 1$$

$$\Rightarrow 1 \geq (u-1)^2 + v^2$$

$$\Rightarrow (u-1)^2 + v^2 \leq 1$$



**16 a**  $|z + 4i| = 2$  is represented by a circle centre  $(0, -4)$ , radius 2.



**b**  $|z|$  represents the distance from  $(0, 0)$  to points on the locus of  $P$ .

Hence  $|z|_{\max}$  occurs when  $z = -6i$

Therefore  $|z|_{\max} = |-6i| = 6$ .

16 c i  $T_1 : w = 2z$

**METHOD (1)**  $z$  lies on circle with equation  $|z + 4i| = 2$

$$\Rightarrow w = 2z$$

$$\Rightarrow \frac{w}{2} = z$$

$$\Rightarrow \frac{w}{2} + 4i = z + 4i$$

$$\Rightarrow \frac{w + 8i}{2} = z + 4i$$

$$\Rightarrow \left| \frac{w + 8i}{2} \right| = |z + 4i|$$

$$\Rightarrow \frac{|w + 8i|}{|2|} = |z + 4i|$$

$$\Rightarrow \frac{|w + 8i|}{2} = 2$$

$$\Rightarrow |w + 8i| = 4$$

So the locus of the image of  $P$  under  $T_1$  is a circle centre  $(0, -8)$ , radius 4, with equation  $u^2 + (v + 8)^2 = 16$ .

**METHOD (2)**  $z$  lies on circle centre  $(0, -4)$ , radius 2



enlargement scale factor 2, centre 0.

$w = 2z$  lies on circle centre  $(0, -8)$ , radius 4.

So the locus of the image of  $P$  under  $T_1$  is a circle centre  $(0, -8)$ , radius 4, with equation  $u^2 + (v + 8)^2 = 16$ .

16 c ii  $T_2 : w = iz$

$z$  lies on a circle with equation  $|z + 4i| = 2$

$$w = iz$$

$$\Rightarrow \frac{w}{i} = z$$

$$\Rightarrow \frac{w}{i} \left( \frac{i}{i} \right) = z$$

$$\Rightarrow \frac{wi}{(-1)} = z$$

$$\Rightarrow -wi = z$$

$$\Rightarrow z = -wi$$

$$\text{Hence } |z + 4i| = 2 \Rightarrow |-wi + 4i| = 2$$

$$\Rightarrow |(-i)(w - 4)| = 2$$

$$\Rightarrow |(-i)| |w - 4| = 2$$

$$\Rightarrow |w - 4| = 2$$

So the locus of the image of  $P$  under  $T_2$  is a circle centre  $(4, 0)$ , radius 2, with equation  $(u - 4)^2 + v^2 = 4$ .

iii  $T_3 : w = -iz$

$z$  lies on a circle with equation  $|z + 4i| = 2$

$$w = -iz$$

$$\Rightarrow iw = i(-iz)$$

$$\Rightarrow iw = z$$

$$\Rightarrow z = iw$$

$$\text{Hence } |z + 4i| = 2 \Rightarrow |iw + 4i| = 2$$

$$\Rightarrow |i(w + 4)| = 2$$

$$\Rightarrow |i| |w + 4| = 2$$

$$\Rightarrow |w + 4| = 2 \quad \leftarrow \boxed{|i| = 1}$$

So the locus of the image of  $P$  under  $T_3$  is a circle centre  $(-4, 0)$ , radius 2, with equation  $(u + 4)^2 + v^2 = 4$ .

16 c iv  $T_4 : w = z^*$

$z$  lies on a circle with equation  $|z + 4i| = 2$

$$w = z^* \Rightarrow u + iv = x - iy$$

So  $u = x$ ,  $v = -y$  and  $x = u$  and  $y = -v$

$$\begin{aligned} z &= x + iy \\ \Rightarrow z^* &= x - iy \end{aligned}$$

$$|z + 4i| = 2 \Rightarrow |x + iy + 4i| = 2$$

$$\Rightarrow |x + i(y + 4)| = 2$$

$$\Rightarrow |u + i(-v + 4)| = 2$$

$$\Rightarrow |u + i(4 - v)| = 2$$

$$\Rightarrow |u + i(4 - v)|^2 = 2^2$$

$$\Rightarrow u^2 + (4 - v)^2 = 4$$

$$\Rightarrow u^2 + (v - 4)^2 = 4$$

So the locus of the image of  $P$  under  $T_4$  is a circle centre  $(0, 4)$ , radius 2, with equation  $u^2 + (v - 4)^2 = 4$ .

$$17 \quad T: w = \frac{z+2}{z+i}, z \neq -i$$

a the imaginary axis in  $z$ -plane  $\Rightarrow x = 0$

$$\begin{aligned} w &= \frac{z+2}{z+i} \\ \Rightarrow w(z+i) &= z+2 \\ \Rightarrow wz + iw &= z+2 \\ \Rightarrow wz - z &= 2 - iw \\ \Rightarrow z(w-1) &= 2 - iw \\ \Rightarrow z &= \frac{2 - iw}{w-1} \\ \Rightarrow z &= \frac{2 - i(u+iv)}{u+iv-1} \\ \Rightarrow z &= \frac{2 - iu + v}{(u-1) + iv} \\ \Rightarrow z &= \left[ \frac{(2+v) - iu}{(u-1) + iv} \right] \times \left[ \frac{(u-1) - iv}{(u-1) - iv} \right] \\ \Rightarrow z &= \frac{(2+v)(u-1) - uv - iv(2+v) - iu(u-1)}{(u-1)^2 + v^2} \\ \Rightarrow z &= \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - i \left( \frac{v(2+v) + u(u-1)}{(u-1)^2 + v^2} \right) \\ \text{So } x + iy &= \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} - i \left( \frac{v(2+v) + u(u-1)}{(u-1)^2 + v^2} \right) \\ \Rightarrow x &= \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} \text{ and } y = \frac{-v(2+v) - u(u-1)}{(u-1)^2 + v^2} \end{aligned}$$

As  $x = 0$ , then

$$\begin{aligned} \frac{(2+v)(u-1) - uv}{(u-1)^2 + v^2} &= 0 \\ \Rightarrow (2+v)(u-1) - uv &= 0 \\ \Rightarrow 2u - 2 + vu - v - uv &= 0 \\ \Rightarrow 2u - 2 - v &= 0 \\ \Rightarrow v &= 2u - 2 \end{aligned}$$

The transformation  $T$  maps the imaginary axis in the  $z$ -plane to the line  $l$  with equation  $v = 2u - 2$  in the  $w$ -plane.



17 b As  $y = x$ , then

$$\frac{-v(2+v)-u(u-1)}{(u-1)^2+v^2} = \frac{(2+v)(u-1)-uv}{(u-1)^2+v^2}$$

$$\Rightarrow -v(2+v)-u(u-1) = (2+v)(u-1)-uv$$

$$\Rightarrow -2v-v^2-u^2+u = 2u-2+vu-v-uv$$

$$\Rightarrow -2v-v^2-u^2+u = 2u-2-v$$

$$\Rightarrow 0 = u^2+v^2+u+v-2$$

$$\Rightarrow \left(u + \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} - 2 = 0$$

$$\Rightarrow \left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{5}{2}$$

$$\Rightarrow \left(u + \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{10}}{2}\right)^2$$

$$\begin{aligned} \sqrt{\frac{5}{2}} &= \frac{\sqrt{5}}{\sqrt{2}} \\ &= \frac{\sqrt{5}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{10}}{2} \end{aligned}$$

The transformation  $T$  maps the line  $y = x$  in the  $z$ -plane to the circle  $C$  with centre

$\left(-\frac{1}{2}, -\frac{1}{2}\right)$ , radius  $\frac{\sqrt{10}}{2}$  in the  $w$ -plane.

18  $T: w = \frac{4-z}{z+i} \quad z \neq -i$

circle with equation  $|z| = 1$  in the  $z$ -plane.

$$w = \frac{4-z}{z+i}$$

$$\Rightarrow w(z+i) = 4-z$$

$$\Rightarrow wz + iw = 4-z$$

$$\Rightarrow wz + z = 4-iw$$

$$\Rightarrow z(w+1) = 4-iw$$

$$\Rightarrow z = \frac{4-iw}{w+1}$$

$$\Rightarrow |z| = \left| \frac{4-iw}{w+1} \right|$$

$$\Rightarrow |z| = \frac{|4-iw|}{|w+1|}$$

Applying  $|z| = 1$  gives  $1 = \frac{|4-iw|}{|w+1|}$

$$\Rightarrow |w+1| = |4-iw|$$

$$\Rightarrow |w+1| = |-i(w+4i)|$$

$$\Rightarrow |w+1| = |-i| |w+4i|$$

$$\Rightarrow |w+1| = |w+4i|$$

$$\Rightarrow |u+iv+1| = |u+iv+4i|$$

$$\Rightarrow |(u+1)+iv| = |u+i(v+4)|$$

$$\Rightarrow |(u+1)+iv|^2 = |u+i(v+4)|^2$$

$$\Rightarrow (u+1)^2 + v^2 = u^2 + (v+4)^2$$

$$\Rightarrow u^2 + 2u + 1 + v^2 = u^2 + v^2 + 8v + 16$$

$$\Rightarrow 2u + 1 = 8v + 16$$

$$\Rightarrow 2u - 8v - 15 = 0$$

The circle  $|z| = 1$  is mapped by  $T$  onto the line  $l: 2u - 8v - 15 = 0$  (i.e.  $a = 2$ ,  $b = -8$ ,  $c = -15$ ).

19  $T: w = \frac{3iz + 6}{1 - z}; z \neq 1$

circle with equation  $|z| = 2$

$$w = \frac{3iz + 6}{1 - z}$$

$$\Rightarrow w(1 - z) = 3iz + 6$$

$$\Rightarrow w - wz = 3iz + 6$$

$$\Rightarrow w - 6 = 3iz + wz$$

$$\Rightarrow w - 6 = z(3i + w)$$

$$\Rightarrow \frac{w - 6}{w + 3i} = z$$

$$\Rightarrow \left| \frac{w - 6}{w + 3i} \right| = |z|$$

$$\Rightarrow \frac{|w - 6|}{|w + 3i|} = |z|$$

Applying  $|z| = 2 \Rightarrow \frac{|w - 6|}{|w + 3i|} = 2$

$$\Rightarrow |w - 6| = 2|w + 3i|$$

$$\Rightarrow |u + iv - 6| = 2|u + iv + 3i|$$

$$\Rightarrow |(u - 6) + iv| = 2|u + i(v + 3)|$$

$$\Rightarrow |(u - 6) + iv|^2 = 2^2 |u + i(v + 3)|^2$$

$$\Rightarrow (u - 6)^2 + v^2 = 4[u^2 + (v + 3)^2]$$

$$\Rightarrow u^2 - 12u + 36 + v^2 = 4[u^2 + v^2 + 6v + 9]$$

$$\Rightarrow u^2 - 12u + 36 + v^2 = 4u^2 + 4v^2 + 24v + 36$$

$$\Rightarrow 0 = 3u^2 + 12u + 3v^2 + 24v$$

$$\Rightarrow 0 = u^2 + 4u + v^2 + 8v$$

$$\Rightarrow 0 = (u + 2)^2 - 4 + (v + 4)^2 - 16$$

$$\Rightarrow 20 = (u + 2)^2 + (v + 4)^2$$

$$\Rightarrow (u + 2)^2 + (v + 4)^2 = (2\sqrt{5})^2$$

$$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$$

Therefore the circle with equation  $|z| = 2$  is mapped onto a circle  $C$ , centre  $(-2, -4)$ , radius  $2\sqrt{5}$ . So  $k = 2$ .

**20 a** We know that the transformation  $T$  given by  $w = \frac{az+b}{z+c}$  maps the origin onto itself, so,

substituting  $w = z = 0$  into  $T$  we get  $0 = \frac{b}{c}$ .

We have to assume  $c \neq 0$  or else it is not possible to map the origin onto itself.

Therefore  $b = 0$ .

We also know that this mapping reflects  $z_1 = 1 + 2i$  in the real axis, i.e.  $w_1 = 1 - 2i$ . Substituting these values into  $T$  we obtain:

$$1 - 2i = \frac{a + 2ai}{1 + 2i + c}$$

$$(1 - 2i)(1 + 2i + c) = a + 2ai$$

$$1 + 2i + c - 2i + 4 - 2ci = a + 2ai$$

$$1 + 4 + c - 2ci = a + 2ai$$

$$5 + c - 2ci = a + 2ai$$

We now equate the real and complex parts:

$$5 + c = a$$

$$-c = a$$

Solving simultaneously gives:

$$2c = -5$$

$$c = -\frac{5}{2}$$

$$a = \frac{5}{2}$$

So  $a = \frac{5}{2}$ ,  $b = 0$  and  $c = -\frac{5}{2}$ .

**b** We know that another complex number,  $\omega$ , is mapped onto itself. i.e. we have:

$$\omega = \frac{\frac{5}{2}\omega}{\omega - \frac{5}{2}}$$

$$\omega^2 - \frac{5}{2}\omega = \frac{5}{2}\omega$$

$$\omega^2 = 5\omega$$

$$\omega^2 - 5\omega = 0$$

$$\omega(\omega - 5) = 0$$

$$\omega = 0 \text{ or } \omega = 5$$

$\omega = 0$  is the origin, so the other number mapped onto itself is  $\omega = 5$ .

**21 a**  $w = \frac{az+b}{z+c}$   $a, b, c \in \mathbb{R}$ .

$$w = 1 \text{ when } z = 0 \quad (1)$$

$$w = 3 - 2i \text{ when } z = 2 + 3i \quad (2)$$

$$(1) \Rightarrow 1 = \frac{a(0)+b}{0+c} \Rightarrow 1 = \frac{b}{c} \Rightarrow c = b \quad (3)$$

$$(3) \Rightarrow w = \frac{az+b}{z+b}$$

$$(2) \Rightarrow 3 - 2i = \frac{a(2+3i)+b}{2+3i+b}$$

$$3 - 2i = \frac{(2a+b)+3ai}{(2+b)+3i}$$

$$(3-2i)[(2+b)+3i] = 2a+b+3ai$$

$$6+3b+9i-4i-2bi+6 = 2a+b+3ai$$

$$(12+3b)+(5-2b)i = (2a+b)+3ai$$

Equate real parts:  $12+3b = 2a+b$

$$\Rightarrow 12 = 2a-2b \quad (4)$$

Equate imaginary parts:  $5-2b = 3a$

$$\Rightarrow 5 = 3a+2b \quad (5)$$

$$(4)+(5): 17 = 5a$$

$$\Rightarrow \frac{17}{5} = a$$

$$(5) \Rightarrow 5 = \frac{51}{5} + 2b$$

$$\Rightarrow -\frac{26}{5} = 2b$$

$$\Rightarrow -\frac{13}{5} = b$$

As  $b = c$  then  $c = -\frac{13}{5}$

The values are  $a = \frac{17}{5}$ ,  $b = -\frac{13}{5}$ ,  $c = -\frac{13}{5}$

$$21 \text{ b } w = \frac{\frac{17}{5}z - \frac{13}{5}}{z - \frac{13}{5}}$$

$$w = \frac{17z - 13}{5z - 13}$$

$$\text{invariant points} \Rightarrow z = \frac{17z - 13}{5z - 13}$$

$$z(5z - 13) = 17z - 13$$

$$5z^2 - 13z = 17z - 13$$

$$5z^2 - 30z + 13 = 0$$

$$z = \frac{30 \pm \sqrt{900 - 4(5)(13)}}{10}$$

$$z = \frac{30 \pm \sqrt{900 - 260}}{10}$$

$$z = \frac{30 \pm \sqrt{640}}{10}$$

$$z = \frac{30 \pm \sqrt{64}\sqrt{10}}{10}$$

$$z = \frac{30 \pm 8\sqrt{10}}{10} = 3 \pm \frac{4\sqrt{10}}{5}$$

The exact values of the two points which remain invariant are

$$z = 3 + \frac{4\sqrt{10}}{5} \text{ and } z = 3 - \frac{4\sqrt{10}}{5}$$

$$22 \text{ T: } w = \frac{z+i}{z}, \quad z \neq 0.$$

**a** the line  $y = x$  in the  $z$ -plane other than  $(0, 0)$

$$w = \frac{z+i}{z}$$

$$\Rightarrow wz = z + i$$

$$\Rightarrow wz - z = i$$

$$\Rightarrow z(w-1) = i$$

$$\Rightarrow z = \frac{i}{w-1}$$

$$\Rightarrow z = \frac{i}{(u+iv)-1} = \frac{i}{(u-1)+iv}$$

$$\Rightarrow z = \left[ \frac{i}{(u-1)+iv} \right] \left[ \frac{(u-1)-iv}{(u-1)-iv} \right]$$

$$\Rightarrow z = \frac{i(u-1)+v}{(u-1)^2+v^2}$$

$$\Rightarrow z = \frac{v}{(u-1)^2+v^2} + i \frac{(u-1)}{(u-1)^2+v^2}$$

$$\text{So } x+iy = \frac{v}{(u-1)^2+v^2} + i \frac{(u-1)}{(u-1)^2+v^2}$$

$$\Rightarrow x = \frac{v}{(u-1)^2+v^2} \text{ and } y = \frac{u-1}{(u-1)^2+v^2}$$

$$\text{Applying } y = x, \text{ gives } \frac{u-1}{(u-1)^2+v^2} = \frac{v}{(u-1)^2+v^2}$$

$$\Rightarrow u-1 = v$$

$$\Rightarrow v = u-1$$

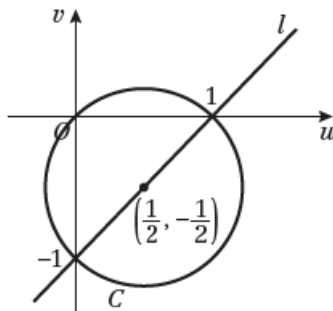
Therefore the line  $l$  has equation  $v = u-1$ .

**22 b** The line with equation  $x + y + 1 = 0$  in the  $z$ -plane

$$\begin{aligned}
 x + y + 1 = 0 &\Rightarrow \frac{v}{(u-1)^2 + v^2} + \frac{u-1}{(u-1)^2 + v^2} + 1 = 0 \quad [\times (u-1)^2 + v^2] \\
 &\Rightarrow v + (u-1) + (u-1)^2 + v^2 = 0 \\
 &\Rightarrow v + u - 1 + u^2 - 2u + 1 + v^2 = 0 \\
 &\Rightarrow u^2 + v^2 - u + v = 0 \\
 &\Rightarrow \left(u - \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0 \\
 &\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2} \\
 &\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2
 \end{aligned}$$

The image of  $x + y + 1 = 0$  under  $T$  is a circle  $C$ , centre  $\left(\frac{1}{2}, -\frac{1}{2}\right)$ , radius  $\frac{\sqrt{2}}{2}$  with equation  $u^2 + v^2 - u + v = 0$ , as required.

**c**





**Challenge**

- 2 We want to find a transformation of the form  $f(z) = az^* + b$ , which reflects the  $z$ -plane in the line  $x + y = 1$ .

First note that points lying on the line will be mapped onto themselves.

This means that for any  $z = x + iy$  such that  $x + y = 1$  we have  $f(z) = z$ .

Choose  $z_1 = 1$ ,  $z_2 = i$ .

Both these points satisfy  $x + y = 1$ , so  $f(z_1) = z_1$  and  $f(z_2) = z_2$ .

Therefore we have:

$$f(1) = a + b = 1$$

$$\text{and } f(i) = -ai + b = i.$$

Solving simultaneously:

$$-(1-b)i + b = i$$

$$-i + bi + b = i$$

$$b(1+i) = 2i$$

$$b = \frac{2i}{1+i} = \frac{2i}{1+i} \left( \frac{1-i}{1-i} \right) = \frac{2i(1-i)}{2} = i(1-i)$$

$$b = 1+i$$

$$\text{So } a = 1 - b = -i \text{ and } f(z) = -iz^* + 1 + i$$