Review exercise 2

1 a $12-7k+d=3k^2 \Rightarrow 3k^2+7k-12=d$

$3k^2+d=k^2-10k \Rightarrow -2k^2-10k=d$

Subtracting the second equation from the first gives

$5k^2+17k-12=(5k-3)(k+4)=0$

So $k=\frac{3}{5}=0.6$ or $k=-4$

b Since the sequence contains only integer terms, $k=-4$.

$u_4=12-7(-4)=40$, $u_5=3(-4)^2=48$

So common difference d is $d=u_5-u_4=48-40=8$

The first term a satisfies

$a+3d=u_4 \Rightarrow a=40-3(8)=16$

So $a=16$, $d=8$

2 a $a=19p-18$

$d=u_2-a=(17p-8)-(19p-18)=10-2p$

So $u_{30}=a+29d=(19p-18)+29(-2p+10)$

$u_{30}=272-39p$

b $S_{31}=\frac{31}{2}(2a+(31-1)d)=0$

$\Rightarrow 2a+30d=0$

So $2(19p-18)+30(10-2p)=0$

$(38-60)p-36+300=0$

$22p=264$

$p=12$

3 a $u_2=ar=256, u_8=ar^7=900$

$\frac{ar^7}{ar}=\frac{900}{256}$

$\Rightarrow r^6=\frac{225}{64}$

$\Rightarrow \ln r^6=\ln\left(\frac{225}{64}\right)$

$\Rightarrow 6\ln r-\ln\left(\frac{225}{64}\right)=0$ (as $\ln x^k=k\ln x$)

$\Rightarrow 6\ln r+\ln\left(\frac{64}{225}\right)=0$ (as $\ln x^{-1}=-\ln x$)

3 b Noting $r > 1$, so r is positive

$$r = \left(\frac{225}{64}\right)^{\frac{1}{6}} = 1.2331060... = 1.23 \text{ (3 s.f.)}$$

4 a $4,\ 4r,\ 4r^2, ...$

$4 + 4r + 4r^2 = 7$

$4r^2 + 4r - 3 = 0$ (as required)

Use ar^{n-1} to write down expressions for the first three terms. Here $a = 4$ and $n = 1, 2, 3$

b $4r^2 + 4r - 3 = 0$

$(2r-1)(2r+3) = 0$

$r = \frac{1}{2},\ r = -\frac{3}{2}$

Factorise $4r^2 + 4r - 3 = -12$

$(-2) + (+6) = +4$, so

$$4r^2 - 2r + 6r - 3 = 2r(2r-1) + 3(2r-1)$$
$$= (2r-1)(2r+3)$$

c $r = \frac{1}{2}$

$$\frac{a}{1-r} = \frac{4}{1-\frac{1}{2}}$$
$$= \frac{4}{\frac{1}{2}}$$
$$= 8$$

Use $S_\infty = \frac{a}{1-r}$

Here $a = 4$ and $r = \frac{1}{2}$

5 a $ar^3 = x,\ ar^4 = 3,\ ar^5 = x + 8$

$$\frac{ar^5}{ar^4} = \frac{ar^4}{ar^3}$$

so $\frac{x+8}{3} = \frac{3}{x}$

$x(x+8) = 9$

$x^2 + 8x - 9 = 0$

$(x+9)(x-1) = 0$

$x = 1,\ x = -9$

$r = \frac{ar^4}{ar^3} = \frac{3}{x}$

When $x = 1,\ r = 3$

When $x = -9,\ r = -\frac{1}{3}$

b $r = -\frac{1}{3}$

$ar^4 = 3$

$$a\left(-\frac{1}{3}\right)^4 = 3$$

$a = 243$

5 c $S_\infty = \frac{a}{1-r} = \frac{243}{1+\frac{1}{3}} = 182.25$

6 a $a_{n+1} = 3a_n + 5$

$n = 1: a_2 = 3a_1 + 5$

$a_2 = 3k + 5$

Use the given formula with $n = 1$

b $n = 2:\ a_3 = 3a_2 + 5$

$= 3(3k+5)+5$

$= 9k + 15 + 5$

$= 9k + 20$

c i $\sum_{r=1}^{4} a_r = a_1 + a_2 + a_3 + a_4$

This is *not* an arithmetic series.

$n = 3: a_4 = 3a_3 + 5$

$= 3(9k+20)+5$

$= 27k + 65$

You cannot use a standard formula, so work out each separate term and then add them together to find the required sum.

$\sum_{r=1}^{4} a_r = k + (3k+5) + (9k+20) + (27k+65)$

$= 40k + 90$

ii $\sum_{r=1}^{4} a_r = 10(4k+9)$

There is a factor of 10, so the sum is divisible by 10.

Give a conclusion.

7 a Common ratio is $r = -4x$

Condition for the convergence of infinite sum is

$|r| < 1 \Rightarrow |-4x| < 1$

$\Rightarrow |x| < \frac{1}{4}$

b $\sum_{r=1}^{\infty} 6 \times (-4x)^{r-1} = S_\infty = \frac{24}{5}$

Another equation for S_∞ is $S_\infty = \frac{a}{1-r} = \frac{6}{1+4x}$

So $\frac{6}{1+4x} = \frac{24}{5}$

$\Rightarrow 30 = 24 + 96x$

$\Rightarrow x = \frac{6}{96} = \frac{1}{16}$

8 a $r = \frac{ar}{a} = \frac{u_2}{u_1}$

So $r = \frac{\frac{50}{6}}{10} = \frac{5}{6}$

$\therefore$ As $|r|<1, S_\infty = \frac{a}{1-r} = \frac{10}{1-\frac{5}{6}} = 60$

8 b $a = 10, r = \frac{5}{6}$

$$S_k = \frac{10\left(1-\left(\frac{5}{6}\right)^k\right)}{1-\frac{5}{6}}$$

As $S_k > 55 \Rightarrow 1-\left(\frac{5}{6}\right)^k > \frac{55}{60}$

$\Rightarrow 1-\left(\frac{5}{6}\right)^k > \frac{11}{12}$

$\Rightarrow \frac{1}{12} > \left(\frac{5}{6}\right)^k \Rightarrow \log\left(\frac{1}{12}\right) > \log\left(\frac{5}{6}\right)^k$

$\Rightarrow \log\left(\frac{1}{12}\right) > k\log\left(\frac{5}{6}\right)$

$\Rightarrow k > \dfrac{\log\left(\frac{1}{12}\right)}{\log\left(\frac{5}{6}\right)}$

(the inequality reverses direction in the final step because $\ln\frac{5}{6} < 0$)

c k must be a positive integer.

$\dfrac{\ln\frac{1}{12}}{\ln\frac{5}{6}} = 13.629$ (3 d.p.)

So the minimum value of k is 14.

9 a $a = 2400,\ r = 1.06$

After 4 years,

$2400(1.06)^3 = 2858.44\ldots = 2860$ to the nearest 10.

b $2400 \times 1.06^{N-1} > 6000 \Rightarrow 1.06^{N-1} > 2.5$

$\Rightarrow \log 1.06^{N-1} > \log 2.5 \Rightarrow (N-1)\log 1.06 > \log 2.5$

c Rearranging the inequality

$$N > \frac{\ln 2.5}{\ln 1.06} + 1 = 16.7 \text{ (1 d.p.)}$$

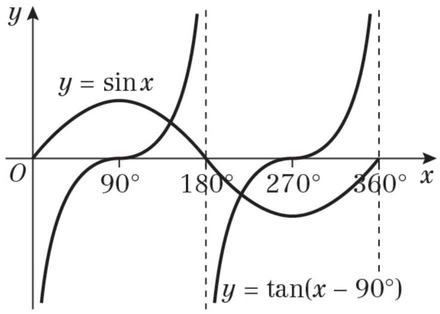
So $N = 17$

d The total amount raised is $5(S_{10})$

$$S_{10} = \frac{a\left(1-r^{10}\right)}{1-r} = \frac{2400\left(1-1.06^{10}\right)}{1.106} = 31633.90 \text{ (2 d.p.)}$$

Therefore the total amount raised is 5 × 31633.9, which to the nearest £1000 is £158 000

10 a



b There are two solutions in the interval $0 \leq x \leq 360°$.

11 a $\sin\theta = \cos\theta$

$\tan\theta = 1$

b $\tan\theta = 1,\ 0 \leq \theta \leq 2\pi$

$\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$

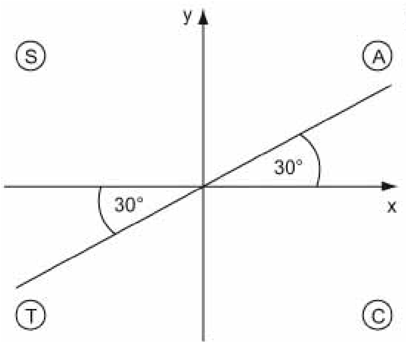
(Note, in earlier versions of the book an erroneous question 12 was removed; all subsequent numbers will therefore by 1 higher than printed here.)

12 $3\tan^2 x = 1$

$\tan x = \pm\frac{1}{\sqrt{3}}$

For $\tan x = \frac{1}{\sqrt{3}}$

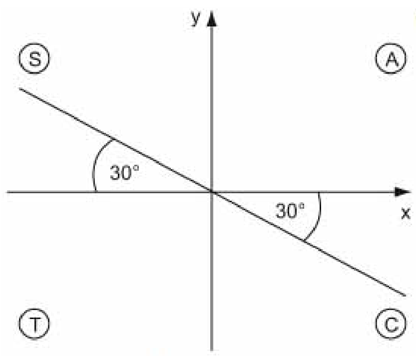
$x = 30°$



So $x = 30°$ or $x = 210°$

For $\tan x = -\frac{1}{\sqrt{3}}$

$x = 330°$ (or $-30°$)



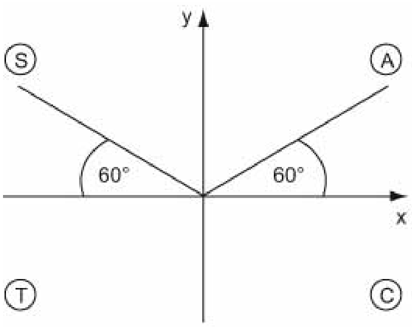
So $x = 330°$ or $x = 150°$

So $x = 30°, 150°, 210°$ or $330°$

13 $2\sin(\theta - 30°) = \sqrt{3}$

$\sin(\theta - 30°) = \frac{\sqrt{3}}{2}$

$\theta - 30° = 60°$



So $\theta - 30° = 60°$ or $\theta - 30° = 120°$
When $\theta - 30° = 60°$
$\theta = 60° + 30°$
$= 90°$
When $\theta - 30° = 120°$
$\theta = 120° + 30°$
$= 150°$
So $\theta = 90°$ or $150°$

14 a $2\cos^2 x = 4 - 5\sin x$
$2(1 - \sin^2 x) = 4 - 5\sin x$
$2 - 2\sin^2 x = 4 - 5\sin x$
$2\sin^2 x - 5\sin x + 2 = 0$ (as required)

b Let $\sin x = y$
$2y^2 - 5y + 2 = 0$
$(2y - 1)(y - 2) = 0$
So $y = \frac{1}{2}$ or $y = 2$
When $\sin x = \frac{1}{2}$, $x = 30°$
or $x = 180° - 30° = 150°$
$\sin x = 2$ is impossible.
$x = 30°$ or $150°$

15 $2\tan^2 x - 4 = 5\tan x$
$2\tan^2 x - 5\tan x - 4 = 0$

Using the quadratic formula:

$\tan x = \frac{5 \pm \sqrt{(-5)^2 - 4(2)(-4)}}{2(2)}$

$= \frac{5 \pm \sqrt{57}}{4}$

When $\tan x = \frac{5 + \sqrt{57}}{4}$, $x = 72.3°$
or $x = 72.3° + 180° = 252.3°$

16 $5\sin^2 x = 6(1 - \cos x)$
$5\sin^2 x + 6\cos x - 6 = 0$
$5(1 - \cos^2 x) + 6\cos x - 6 = 0$
$5 - 5\cos^2 x + 6\cos x - 6 = 0$
$5\cos^2 x - 6\cos x + 1 = 0$
$(5\cos x - 1)(\cos x - 1) = 0$
So $\cos x = \frac{1}{5}$ or $\cos x = 1$
When $\cos x = \frac{1}{5}$, $x = 78.5°$
or $x = 360° - 78.5° = 281.5°$
When $\cos x = 1$, $x = 0°$ or $360°$
$x = 0°, 78.5°, 281.5°$ or $360°$

17 LHS $= \cos^2 x(\tan^2 x + 1)$
$= \cos^2 x\left(\frac{\sin^2 x}{\cos^2 x} + 1\right)$
$= \sin^2 x + \cos^2 x$
= RHS

18 $f(x) = x^3 - 12x^2 + 48x$
$f'(x) = 3x^2 - 24x + 48$
$= 3(x - 4)^2$
$(x - 4)^2 > 0$ for all real values of x
So $3x^2 - 24x + 48 > 0$ for all real values of x.
So $f(x)$ is increasing for all real values of x.

19 a $y = x + \frac{2}{x} - 3$
When $y = 0$, $x + \frac{2}{x} - 3 = 0$
$x^2 + 2 - 3x = 0$
$x^2 - 3x + 2 = 0$
$(x - 1)(x - 2) = 0$
$x = 1$ or $x = 2$
$A(1, 0)$ and $B(2, 0)$

19 b $y = x + 2x^{-1} - 3$

$\frac{dy}{dx} = 1 - 2x^{-2}$

$= 1 - \frac{2}{x^2}$

Let $\frac{dy}{dx} = 0$ to find the minimum

$x^2 = 2$

$x = \pm\sqrt{2}$

x is positive, so $x = \sqrt{2}$.

When $x = \sqrt{2}$,

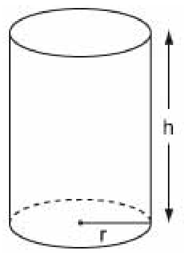
$y = \sqrt{2} + \frac{2}{\sqrt{2}} - 3$

$= \sqrt{2} + \frac{2\sqrt{2}}{2} - 3$

$= 2\sqrt{2} - 3$

C has coordinates $\left(\sqrt{2}, 2\sqrt{2} - 3\right)$

20



Draw a diagram. Let h be the height of the cylinder.

a Surface area, S = $2\pi rh + 2\pi r^2$

Volume = $\pi r^2 h = 128\pi$

$h = \frac{128\pi}{\pi r^2}$

$= \frac{128}{r^2}$

so $S = 2\pi r \times \frac{128}{r^2} + 2\pi r^2$

$= \frac{256\pi}{r} + 2\pi r^2$ (as required)

20 b $\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$

$4\pi r - \frac{256\pi}{r^2} = 0$

$4\pi r = \frac{256\pi}{r^2}$

$r^3 = 64$

$r = 4$ cm

When $r = 4$,

$S = \frac{256\pi}{(4)} + 2\pi(4)^2$

$= 64\pi + 32\pi$

$= 96\pi$ cm^2

21 a $y = 3x^2 + 4\sqrt{x}$

$= 3x^2 + 4x^{\frac{1}{2}}$

$\frac{dy}{dx} = (3 \times 2x^1) + \left(4 \times \frac{1}{2}x^{-\frac{1}{2}}\right)$

$\frac{dy}{dx} = 6x + 2x^{-\frac{1}{2}}$

Or:

$\frac{dy}{dx} = 6x + \frac{2}{x^{\frac{1}{2}}} = 6x + \frac{2}{\sqrt{x}}$

b $\frac{dy}{dx} = 6x + 2x^{\frac{-1}{2}}$

$\frac{d^2y}{dx^2} = 6 + \left(2 \times -\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= 6 - x^{\frac{-3}{2}}$

Or:

$\frac{d^2y}{dx^2} = 6 - \frac{1}{x^{\frac{3}{2}}}$

Or:

$\frac{d^2y}{dx^2} = 6 - \frac{1}{x\sqrt{x}}$

21 c $\int\left(3x^2+4x^{\frac{1}{2}}\right)\mathrm{d}x=\frac{3x^3}{3}+\frac{4x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+C$

$=x^3+4\left(\frac{2}{3}\right)x^{\frac{3}{2}}+C$

$=x^3+\frac{8}{3}x^{\frac{3}{2}}+C$

(Or: $x^3+\frac{8}{3}x\sqrt{x}+C$)

22 $\int_1^8 x^{\frac{1}{3}}-x^{\frac{-1}{3}}\,\mathrm{d}x$

$=\left(\frac{3}{4}(8)^{\frac{4}{3}}-\frac{3}{2}(8)^{\frac{2}{3}}\right)-\left(\frac{3}{4}(1)^{\frac{4}{3}}-\frac{3}{2}(1)^{\frac{2}{3}}\right)$

$=\left(\frac{3}{4}(16)-\frac{3}{2}(4)\right)-\left(\frac{3}{4}(1)-\frac{3}{2}(1)\right)$

$=\frac{27}{4}$

$=6\frac{3}{4}$

23 $\int_0^6\left(x^2-kx\right)\mathrm{d}x$

$=\left[\frac{x^3}{3}-\frac{kx^2}{2}\right]_0^6$

$=\left(\frac{6}{3}-\frac{k(6)^2}{2}\right)-\left(\frac{0^3}{3}-\frac{k(0)^2}{2}\right)$

$=72-18k=0$

$k=4$

24 a $-x^4+3x^2+4=0$
$(-x^2+4)(x^2+1)=0$
$(2-x)(2+x)(x^2+1)=0$
$x^2+1=0$ has no real solutions.
So there are two solutions $x=-2$ or $x=2$.
$A(-2, 0)$and $B(2, 0)$

24 b $R = \int_{-2}^{2}\left(-x^4 + 3x^2 + 4\right)\,dx$

$$= \left[-\frac{x^5}{5} + \frac{3x^3}{3} + 4x\right]_{-2}^{2}$$

$$= \left[-\frac{x^5}{5} + x^3 + 4x\right]_{-2}^{2}$$

$$= \left(-\frac{2^5}{5} + 2^3 + 4(2)\right) - \left(-\frac{(-2)^5}{5} + (-2)^3 + 4(-2)\right)$$

$$= \left(-\frac{32}{5} + 8 + 8\right) - \left(\frac{32}{5} - 8 - 8\right)$$

$$= 19.2 \text{ units}^2$$

25 Area $= \int_{1}^{4}(x-1)(x-4)\,dx$

$$= \int_{1}^{4} x^2 - 5x + 4\,dx$$

$$= \left(\frac{x^3}{3} - \frac{5x^2}{2} + 4x\right)_{1}^{4}$$

$$= \left(\frac{(4)^3}{3} - \frac{5(4)^2}{2} + 4(4)\right)$$

$$= -\left(\frac{(1)^3}{3} - \frac{5(1)^2}{2} + 4(1)\right)$$

$$= -4\frac{1}{2}$$

$\therefore$ Area $= 4\frac{1}{2}$ units^2 (area cannot be a negative value)

26 a Solving simultaneously

$5 - x^2 = 3 - x$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or $x = -1$

when $x = 2$, $y = 1$

when $x = -1$, $y = 4$

$P(-1, 4)$ and $Q(2, 1)$

26 b Shaded area =
area under the curve between P and Q and the x-axis − area of trapezium

$$\text{Area} = \int_{-1}^{2}\left(5-x^2\right)\,dx - \frac{1}{2}\times 3(1+4)$$

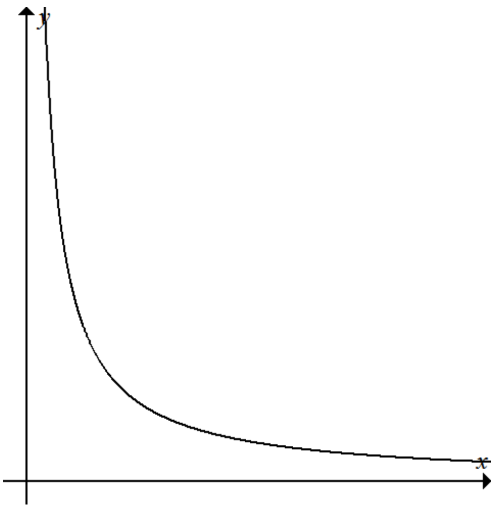
$$= \left[5x - \frac{x^3}{3}\right]_{-1}^{2} - \frac{15}{2}$$

$$= \left(5(2) - \frac{2^3}{3}\right) - \left(5(-1) - \frac{(-1)^3}{3}\right) - \frac{15}{2}$$

$$= \left(10 - \frac{8}{3}\right) - \left(-5 + \frac{1}{3}\right) - \frac{15}{2}$$

$$= 4.5 \text{ units}^2$$

27 a



b

x	1	1.2	1.4	1.6	1.8	2
$\frac{1}{x}$	1	0.833	0.714	0.625	0.556	0.5

c $A = \int_1^2 \left(\frac{1}{x}\right) dx$

$$\int_a^b y\,dx = \frac{1}{2}h\left(y_0 + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right) + y_n\right)$$

$$\int_1^2 \left(\frac{1}{x}\right) dx = \frac{1}{2}(0.2)\left(1 + 2(0.833 + 0.714 + 0.625 + 0.556) + 0.5\right)$$

$$= 0.696 \text{ (3 s.f.)}$$

d Overestimate as the curve is concave.

28 a $y=\sqrt{1-x^2}$

$\sqrt{1-x^2}=0$

$1-x^2=0$

$x=\pm 1$

so $x=1$

x	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\sqrt{1-x^2}$	1	0.9950	0.9798	0.9539	0.9165	0.866	0.8	0.7141	0.6	0.4359	0

$A=\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$

$\int_a^b y\,\mathrm{d}x=\frac{1}{2}h\left(y_0+2\left(y_1+y_2+\ldots+y_{n-1}\right)+y_n\right)$

$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x=\frac{1}{2}(0.1)\left(1+2\left(0.9950+0.9798+0.9539+0.9165+0.866+0.8+0.7141+0.6+0.4359\right)+0\right)$$

$=0.776$ (3 s.f.)

b Error $\dfrac{\frac{\pi}{4}-0.776}{\frac{\pi}{4}}\times 100=1.2\%$

29 $y=\sqrt{x}$ and $y=x^2$

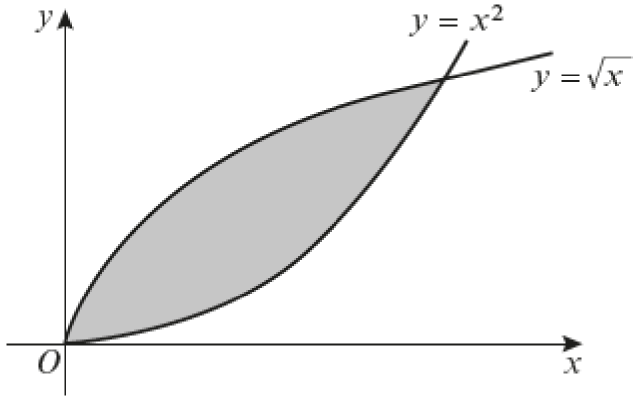
$\sqrt{x}=x^2$

$x=x^4$

$x^4-x=0$

$x\left(x^3-1\right)=0$

$x=0$ or $x=1$



The shaded area is given by

$$A=\int_0^1 \sqrt{x}\ \mathrm{d}x-\int_0^1 x^2\ \mathrm{d}x$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1-\left[\frac{1}{3}x^3\right]_0^1$$

$$=\left[\left(\frac{2}{3}(1)^{\frac{3}{2}}\right)-\left(\frac{2}{3}(0)^{\frac{3}{2}}\right)\right]-\left[\left(\frac{1}{3}(1)^3\right)-\left(\frac{1}{3}(0)^3\right)\right]$$

$$=\frac{2}{3}-\frac{1}{3}$$

$$=\frac{1}{3}$$

Challenge

1 Many possible solutions are available, such as

$a_{n+1} = a_n + k$, $a_1 = m$, $\sum_{i=6}^{11} a_i = \sum_{i=12}^{15} a_i$

$a_1 = m$, $a_2 = m + k$, $a_3 = m + 2k$, ...

$$\sum_{i=6}^{11} a_i = m + 5k + m + 6k + ... + m + 10k$$

$$= 6m + 45k$$

$$\sum_{i=12}^{15} a_i = m + 11k + m + 12k + m + 13k + m + 14k$$

$$= 4m + 50k$$

$6m + 45k = 4m + 50k$

$5k = 2m$

$m = \frac{5}{2}k$ as required

2 $2\sin^3 x - \sin x + 1 = \cos^2 x$

$2\sin^3 x - \sin x + 1 = 1 - \sin^2 x$

$2\sin^3 x + \sin^2 x - \sin x = 0$

$\sin x\left(2\sin^2 x + \sin x - 1\right) = 0$

$\sin x\left(2\sin x - 1\right)\left(\sin x + 1\right) = 0$

$\sin x = 0$, $\sin x = \frac{1}{2}$ and $\sin x = -1$

When $\sin x = 0$, $x = 0$, $x = 180°$ and $x = 360°$

When $\sin x = \frac{1}{2}$, $x = 30°$ and $x = 150°$

When $\sin x = -1$, $x = 270°$