Exercise 2C

1 a $(x_1, y_1) = (3, 2), r = 4$

So $(x-3)^2 + (y-2)^2 = 4^2$

$(x-3)^2 + (y-2)^2 = 16$

b $(x_1, y_1) = (-4, 5), r = 6$

So $(x-(-4))^2 + (y-5)^2 = 6^2$

$(x+4)^2 + (y-5)^2 = 36$

c $(x_1, y_1) = (5, -6), r = 2\sqrt{3}$

So $(x-5)^2 + (y-(-6))^2 = (2\sqrt{3})^2$

$(x-5)^2 + (y+6)^2 = 2^2(\sqrt{3})^2$

$(x-5)^2 + (y+6)^2 = 4 \times 3$

$(x-5)^2 + (y+6)^2 = 12$

d $(x_1, y_1) = (2a, 7a), r = 5a$

So $(x-2a)^2 + (y-7a)^2 = (5a)^2$

$(x-2a)^2 + (y-7a)^2 = 25a^2$

e $(x_1, y_1) = (-2\sqrt{2}, -3\sqrt{2}), r = 1$

So $\left(x-(-2\sqrt{2})\right)^2 + \left(y-(-3\sqrt{2})\right)^2 = 1^2$

$(x+2\sqrt{2})^2 + (y+3\sqrt{2})^2 = 1$

2 a $(x+5)^2 + (y-4)^2 = 9^2$

$(x-(-5))^2 + (y-4)^2 = 9^2$

The centre of the circle is (–5, 4) and the radius is 9.

b $(x-7)^2 + (y-1)^2 = 16$

$(x-7)^2 + (y-1)^2 = 4^2$

The centre of the circle is (7, 1) and the radius is 4.

c $(x+4)^2 + y^2 = 25$

$(x-(-4))^2 + (y-0)^2 = 5^2$

The centre of the circle is (–4, 0) and the radius is 5.

2 d $(x+4a)^2+(y+a)^2=144a^2$

$(x-(-4a))^2+(y-(-a))^2=(12a)^2$

The centre of the circle is ($-4a$, $-a$) and the radius is $12a$.

e $\left(x-3\sqrt{5}\right)^2+\left(y+\sqrt{5}\right)^2=27$

$\left(x-3\sqrt{5}\right)^2+\left(y-\left(-\sqrt{5}\right)\right)^2=\left(\sqrt{27}\right)^2$

Now $\sqrt{27}=\sqrt{9\times 3}=\sqrt{9}\times\sqrt{3}=3\sqrt{3}$

The centre of the circle is $\left(3\sqrt{5},-\sqrt{5}\right)$ and the radius is $3\sqrt{3}$.

3 a Substitute $x = 4$, $y = 8$ into $(x-2)^2+(y-5)^2=13$

$(x-2)^2+(y-5)^2=(4-2)^2+(8-5)^2=2^2+3^2=4+9=13$ ✓

So the circle passes through (4, 8).

b Substitute $x = 0$, $y = -2$ into $(x+7)^2+(y-2)^2=65$

$(x+7)^2+(y-2)^2=(0+7)^2+(-2-2)^2=7^2+(-4)^2=49+16=65$ ✓

So the circle passes through (0, –2).

c Substitute $x = 7$, $y = -24$ into $x^2+y^2=25^2$

$x^2+y^2=7^2+(-24)^2=49+576=625=25^2$ ✓

So the circle passes through (7, –24).

d Substitute $x = 6a$, $y = -3a$ into $(x-2a)^2+(y+5a)^2=20a^2$

$(x-2a)^2+(y+5a)^2=(6a-2a)^2+(-3a+5a)^2=(4a)^2+(2a)^2=16a^2+4a^2=20a^2$ ✓

So the circle passes through ($6a$, $-3a$).

e Substitute $x=\sqrt{5}, y=-\sqrt{5}$ into $\left(x-3\sqrt{5}\right)^2+\left(y-\sqrt{5}\right)^2=\left(2\sqrt{10}\right)^2$

$$\left(x-3\sqrt{5}\right)^2+\left(y-\sqrt{5}\right)^2=\left(\sqrt{5}-3\sqrt{5}\right)^2+\left(-\sqrt{5}-\sqrt{5}\right)^2=\left(-2\sqrt{5}\right)^2+\left(-2\sqrt{5}\right)^2$$

$$=4\times 5+4\times 5=20+20=40=\left(\sqrt{40}\right)^2$$

Now $\sqrt{40}=\sqrt{4\times 10}=\sqrt{4}\times\sqrt{10}=2\sqrt{10}$ ✓

So the circle passes through $\left(\sqrt{5},-\sqrt{5}\right)$.

4 The radius of the circle is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-8)^2+((-2)-1)^2}$$
$$=\sqrt{4^2+3^2}$$
$$=\sqrt{16+9}$$
$$=\sqrt{25}$$
$$=5$$

The centre of the circle is (8, 1) and the radius is 5.

So $(x-8)^2+(y-1)^2=5^2$

or $(x-8)^2+(y-1)^2=25$

5 P (5, 6) and Q (−2, 2)

The centre of the circle is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{5+(-2)}{2}, \frac{6+2}{2}\right) = \left(\frac{3}{2}, \frac{8}{2}\right) = \left(\frac{3}{2}, 4\right)$$

The radius of the circle is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{\left(5-\frac{3}{2}\right)^2+(6-4)^2}$$

$$= \sqrt{\left(\frac{7}{2}\right)^2+(2)^2}$$

$$= \sqrt{\frac{49}{4}+4}$$

$$= \sqrt{\frac{49}{4}+\frac{16}{4}}$$

$$= \sqrt{\frac{65}{4}}$$

So the equation of the circle is

$$\left(x-\frac{3}{2}\right)^2+(y-4)^2 = \left(\sqrt{\frac{65}{4}}\right)^2 \text{ or } \left(x-\frac{3}{2}\right)^2+(y-4)^2 = \frac{65}{4}$$

6 Substitute $x=1,\ y=-3$ into $(x-3)^2+(y+4)^2=r^2$

$$(1-3)^2+(-3+4)^2=r^2$$

$$(-2)^2+(1)^2=r^2$$

$$5=r^2$$

So $r=\sqrt{5}$

7 a Substitute $(2,2)$ into $(x-2)^2+(y-4)^2=r^2$

$$(2-2)^2+(2-4)^2=r^2$$

$$0^2+(-2)^2=r^2$$

$$r^2=4$$

$$r=2$$

7 b The distance between (2, 2) and $\left(2+\sqrt{3},5\right)$ is

$$\begin{aligned}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2} &= \sqrt{\left(2+\sqrt{3}-2\right)^2+\left(5-2\right)^2} \\ &= \sqrt{\left(\sqrt{3}\right)^2+3^2} \\ &= \sqrt{3+9} \\ &= \sqrt{12}\end{aligned}$$

The distance between (2, 2) and $\left(2-\sqrt{3},5\right)$ is

$$\begin{aligned}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2} &= \sqrt{\left(2-\sqrt{3}-2\right)^2+\left(5-2\right)^2} \\ &= \sqrt{\left(-\sqrt{3}\right)^2+\left(3\right)^2} \\ &= \sqrt{3+9} \\ &= \sqrt{12}\end{aligned}$$

The distance between $\left(2+\sqrt{3},5\right)$ and $\left(2-\sqrt{3},5\right)$ is

$$\begin{aligned}\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2} &= \sqrt{\left(\left(2-\sqrt{3}\right)-\left(2+\sqrt{3}\right)\right)^2+\left(5-5\right)^2} \\ &= \sqrt{\left(2-\sqrt{3}-2-\sqrt{3}\right)^2+0^2} \\ &= \sqrt{\left(-2\sqrt{3}\right)^2} \\ &= \sqrt{4\times 3} \\ &= \sqrt{12}\end{aligned}$$

PQ, QR and PR all equal $\sqrt{12}$.

So ΔPQR is equilateral.

8 a Rearrange $x^2+y^2-4x-11=0$ into the form $\left(x-a\right)^2+y^2=r^2$

$x^2-4x-11+y^2=0$

Completing the square gives

$\left(x-2\right)^2-4-11+y^2=0$

$\left(x-2\right)^2+y^2=15$

b Centre of the circle = (2, 0), radius = $\sqrt{15}$

9 a $x^2 + y^2 - 10x + 4y - 20 = 0$

$x^2 - 10x + y^2 + 4y - 20 = 0$

Completing the square gives

$(x - 5)^2 - 25 + (y + 2)^2 - 4 - 20 = 0$

$(x - 5)^2 + (y + 2)^2 = 49$

9 b Centre of the circle = $(5, -2)$, radius = 7

10 a $x^2 + y^2 - 2x + 8y - 8 = 0$
$x^2 - 2x + y^2 + 8y - 8 = 0$
Completing the square gives
$(x-1)^2 - 1 + (y+4)^2 - 16 - 8 = 0$
$(x-1)^2 + (y+4)^2 = 25$
Centre of the circle = $(1, -4)$, radius = 5

b $x^2 + y^2 + 12x - 4y = 9$
$x^2 + 12x + y^2 - 4y = 9$
Completing the square gives
$(x+6)^2 - 36 + (y-2)^2 - 4 = 9$
$(x+6)^2 + (y-2)^2 = 49$
Centre of the circle = $(-6, 2)$, radius = 7

c $x^2 + y^2 - 6y = 22x - 40$
$x^2 - 22x + y^2 - 6y = -40$
Completing the square gives
$(x-11)^2 - 121 + (y-3)^2 - 9 = -40$
$(x-11)^2 + (y-3)^2 = 90$
Centre of the circle = $(11, 3)$, radius = $\sqrt{90} = 3\sqrt{10}$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$
$x^2 + 5x + y^2 - 3y = 4$
Completing the square gives
$$\left(x+\frac{5}{2}\right)^2 - \frac{25}{4} + \left(y-\frac{3}{2}\right)^2 - \frac{9}{4} = 4$$
$$\left(x+\frac{5}{2}\right)^2 + \left(y-\frac{3}{2}\right)^2 = \frac{25}{2}$$
Centre of the circle = $\left(-\frac{5}{2}, \frac{3}{2}\right)$, radius = $\frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$

e $2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$
$2x^2 - 8x + 2y^2 + 8y = -3$
Completing the square gives
$2((x-2)^2 - 4) + 2((y+2)^2 - 4) = -3$
$2(x-2)^2 - 8 + 2(y+2)^2 - 8 = -3$
$2(x-2)^2 + 2(y+2)^2 = 13$
$(x-2)^2 + (y+2)^2 = \frac{13}{2}$
Centre of the circle = $(2, -2)$, radius = $\sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}$

11 a $x^2 + y^2 + 12x + 2y = k$
$x^2 + 12x + y^2 + 2y = k$
$(x+6)^2 - 36 + (y+1)^2 - 1 = k$
$(x+6)^2 + (y+1)^2 = k + 37$
Centre of the circle = $(-6, -1)$

11 b A circle must have a positive radius, so $k + 37 > 0$
So $k > -37$

13 $(x - k)^2 + y^2 = 41$, (3, 4)
Substitute $x = 3$ and $y = 4$ into the equation $(x - k)^2 + y^2 = 41$
$(3 - k)^2 + 4^2 = 41$
$k^2 - 6k + 9 + 16 = 41$
$k^2 - 6k - 16 = 0$
$(k + 2)(k - 8) = 0$
$k = -2$ or $k = 8$

Challenge

1 $(x - k)^2 + (y - 2)^2 = 50$, (4, −5)
Substitute $x = 4$ and $y = -5$ into the equation $(x - k)^2 + (y - 2)^2 = 50$
$(4 - k)^2 + (-5 - 2)^2 = 50$
$k^2 - 8k + 16 + 49 = 50$
$k^2 - 8k + 15 = 0$
$(k - 3)(k - 5) = 0$
$k = 3$ or $k = 5$
$(x - 3)^2 + (y - 2)^2 = 50$ or $(x - 5)^2 + (y - 2)^2 = 50$

2 $x^2 + y^2 + 2fx + 2gy + c = 0$
Rearranging the equation:
$x^2 + 2fx + y^2 + 2gy + c = 0$
Completing the square gives
$(x + f)^2 - f^2 + (y + g)^2 - g^2 + c = 0$
$(x + f)^2 + (y + g)^2 = f^2 + g^2 - c$
The centre of the circle is $(-f, -g)$ and the radius is $\sqrt{f^2 + g^2 - c}$.