Exam-style practice: A level

1 We apply the Euclidean algorithm to 17 and 75 in order to find the greatest common divisor.

$75 = 4 \times 17 + 7$

$17 = 2 \times 7 + 3$

$7 = 2 \times 3 + 1$

$3 = 3 \times 1 + 0.$

So the greatest common divisor is 1.
We now work backwards in order to find the multiplicative inverse of 17 modulo 75.

$1 = 7 - 2 \times 3$

$= 7 - 2(17 - 2(7)) = 5(7) - 2(17)$

$= 5(75 - 4(17)) - 2(17)$

$= 5(75) - 22(17)$

$\Rightarrow 22(17) = -1 + 5(75)$

So $22 \times 17 \equiv -1$ (mod 75), so $-22 \times 17 \equiv 1$ (mod 75) and since $-22 \equiv 53$ (mod 75) and so we have that 53 is the multiplicative inverse of 17 modulo 75.
However we require solutions congruent to 2 and so multiplying through by a factor of 2 gives us 106 which is congruent to 31 (mod 75).
Hence the solutions to $17x \equiv 2$ (mod 75) are given by the set of all x satisfying $x \equiv 31$ (mod 75).

2 a List the prime numbers less than 20 in order to make the problem easier.
2, 3, 5, 7, 11, 13, 17, 19.
We then calculate "8 choose 3" which is $\frac{8!}{(8-3)!3!} = 56.$

2 b The Cayley table is

$\times_{12}$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

All entries in the Cayley table are in S_A so we have closure. The row and column corresponding to 1 are the same as the column and row headings, so 1 is the identity.
All elements are self-inverse and so we have inverses for all elements.
Thus S_A forms a group under $\times_{12}$.
Since all elements have order $\leqslant 2$, there are no elements that can act as a generator for the group (we would require order equal to the size of the group) so S_A is a non-cyclic group.

c S_B has element 3 with order 4, so S_B is a cyclic group of order 4.
S_C has $1^2 = 3^2 = 5^2 = 7^2 = 1$, so has no elements of order 4, so $S_C \ncong S_B$. Since there are only two possible groups of order 4, S_A must be isomorphic to either S_B or S_C.

d Assume that $n \geqslant 6$. Then $2^2 = 4$, which is not in the set, so the set is not closed under $\times_n$, so cannot be a group.
Assume that $n \leqslant 4$, when n is either 2 or 4, $2^2 = 4 \equiv 0$, but 0 is not in the set either.
This means the set is not closed under $\times_n$.
Therefore the set cannot form a group under $\times_n$, for any even n.

3 a A Cartesian equation for the locus of P is found by converting $z = x + iy$.

$\sqrt{2}|z-i| = |z-4|$

$\Rightarrow \sqrt{2}|x+iy-i| = |x+iy-4|$

$\Rightarrow \sqrt{2}|x+i(y-1)| = |(x-4)+iy|$

$\Rightarrow \sqrt{2}\sqrt{x^2+(y-1)^2} = \sqrt{(x-4)^2+y^2}$

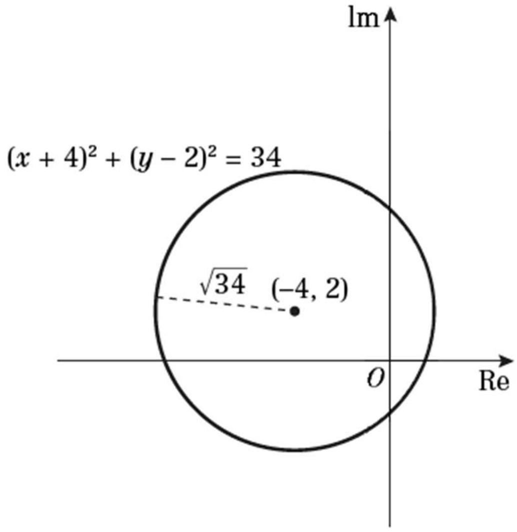
$\Rightarrow 2x^2+2(y-1)^2 = (x-4)^2+y^2$

$\Rightarrow 2x^2+2y^2-4y+2 = x^2-8x+16+y^2$

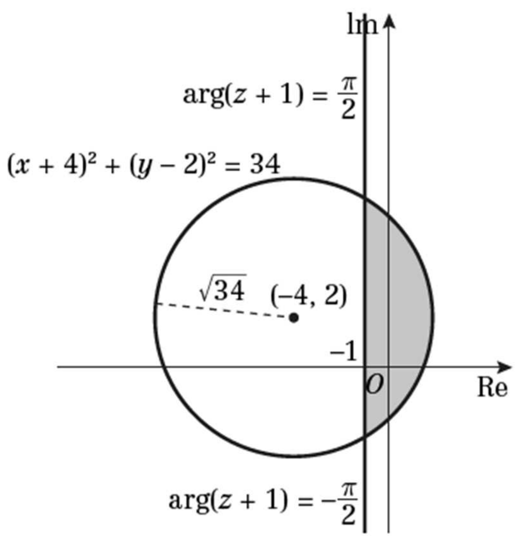
$\Rightarrow x^2+y^2-4y+8x = 14$

$\Rightarrow (x+4)^2+(y-2)^2 = 34.$

b The equation we found above is a circle centred at (−4, 2) with radius $\sqrt{34}$.



c The expression $|\arg(z+1)| < \frac{\pi}{2}$ is the right hand side of the line $z = -1$.



3 d The complex numbers which satisfy both equations are the intersection points of the line and the circle on the graph above.

We can solve this by setting $z+1 = re^{\frac{i\pi}{2}}$ so that we have $|\arg(z+1)| = \frac{\pi}{2}$.

Note that r is real and we have chosen $\frac{\pi}{2}$ instead of $-\frac{\pi}{2}$ for a slightly lower chance of dropping a minus sign, but $\arg(z+1) = \frac{\pi}{2}$ and $\arg(z+1) = -\frac{\pi}{2}$ produce the same line just in opposite directions (if we allow r to be negative), so solving the problem with either is fine.

$z+1 = re^{\frac{i\pi}{2}}$ is equivalent to

$z+1 = r\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right) = ir$

So, $z = -1+ir$.

Since the intersections must lay on the circle we defined in part a, we can substitute $z = x+iy = -1+ir$ into $(x+4)^2+(y-2)^2 = 34$ in order to get

$(-1+4)^2+(r-2)^2 = 34$

$\Rightarrow 3^2+r^2-4r+2^2 = 34$

$\Rightarrow r^2-4r-21 = 0$

$\Rightarrow r = \frac{4\pm\sqrt{4^2-4\times1\times-21}}{2\times1}$

$\Rightarrow r = 2\pm5$

Thus we have $z = -1+7i$ and $z = -1-3i$ as our points of intersection.

If we had forced r to be positive, we would have neglected the second solution and then went ahead with the $-\frac{\pi}{2}$ solution which would come out to $r = -2\pm5$.

Then we would take the $r = 3$ solution and neglect $r = -7$, substitute into

$z+1 = r\left(\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right)$

and get

$z+1 = 3(0-i)$

$\Rightarrow z = -1-3i.$

4 a i We find the determinant of

$$M-\lambda I=\begin{pmatrix}1&0&a\\0&2&0\\a&0&0\end{pmatrix}-\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1-\lambda&0&a\\0&2-\lambda&0\\a&0&-\lambda\end{pmatrix}$$

to be

$$\det(M-\lambda I)=\begin{vmatrix}1-\lambda&0&a\\0&2-\lambda&0\\a&0&-\lambda\end{vmatrix}$$

$$=(1-\lambda)\begin{vmatrix}2-\lambda&0\\0&-\lambda\end{vmatrix}$$

$$-0\begin{vmatrix}0&0\\a&-\lambda\end{vmatrix}$$

$$+a\begin{vmatrix}0&2-\lambda\\a&0\end{vmatrix}$$

$$=(1-\lambda)(\lambda-2)\lambda+a^2(\lambda-2)$$

$$=(\lambda-2)(a^2-\lambda^2+\lambda).$$

Now if we substitute in $\lambda=-1$ and solve this equal to zero, we can deduce conditions for a.

$$(\lambda-2)(a^2-\lambda^2+\lambda)=0$$

$$\Rightarrow(-1-2)(a^2-1-1)=0$$

$$\Rightarrow a^2=2$$

$$\Rightarrow a=\sqrt{2}$$

ii Since we have that $\det(M-\lambda I)=(\lambda-2)(2-\lambda^2+\lambda)$, we can clearly see that this will be zero when $\lambda=2$ and so our other eigenvalue is 2.

We factorise fully in order to find which eigenvalue is repeated.

$$(\lambda-2)(2-\lambda^2+\lambda)=-(\lambda-2)(\lambda-2)(\lambda+1)$$

$$=-(\lambda-2)^2(\lambda+1)$$

So, our repeated eigenvalue is 2.

4 b We find the eigenvectors using the equation

$$\begin{pmatrix}1&0&\sqrt{2}\\0&2&0\\\sqrt{2}&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\lambda\begin{pmatrix}x\\y\\z\end{pmatrix}$$

For $\lambda=2$,

$$\begin{pmatrix}x+\sqrt{2}z\\2y\\\sqrt{2}x\end{pmatrix}=\begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

Equate the top elements to get

$$x+\sqrt{2}z=2x$$

$$\Rightarrow x=\sqrt{2}z$$

If $x\neq0$, then we have a corresponding eigenvector $\begin{pmatrix}\sqrt{2}\\0\\1\end{pmatrix}$ with normalised form

$$\begin{pmatrix}\frac{\sqrt{2}}{\sqrt{3}}\\0\\\frac{1}{\sqrt{3}}\end{pmatrix}$$

If $x=0$, then we have a corresponding normalised eigenvector $\begin{pmatrix}0\\1\\0\end{pmatrix}$

c We already have the components for P as we have 3 normalised eigenvectors.

$$P=\begin{pmatrix}\frac{2}{\sqrt{3}}&0&-\frac{1}{\sqrt{3}}\\0&1&0\\\frac{1}{\sqrt{3}}&0&\sqrt{\frac{2}{3}}\end{pmatrix}$$

We fortunately do not need to calculate D as we know that P is a matrix with the normalised eigenvectors of M as its columns and so $D=P^TMP$ is a matrix with the corresponding eigenvalues along the diagonal.

$$D=\begin{pmatrix}2&0&0\\0&2&0\\0&0&-1\end{pmatrix}$$

5 a Since we have $I_{n+1} = S_n + M_n + D_n$ we can substitute expressions into this until we are only left with I_n terms.

$$I_{n+2} = S_{n+1} + M_{n+1} + D_{n+1}$$
$$= S_{n+1} + (4S_{n+1} - S_n) + d$$
$$= 5S_{n+1} - S_n + d$$
$$= \frac{5}{6}I_{n+1} - \frac{1}{6}I_n + d$$

b First we solve for a complementary function from the equation

$$I_{n+2} = \frac{5}{6}I_{n+1} - \frac{1}{6}I_n$$

That is, $I_n - \frac{5}{6}I_{n-1} + \frac{1}{6}I_{n-2} = 0,$ which gives us an auxiliary equation of

$$r^2 - \frac{5}{6}r + \frac{1}{6} = 0$$
$$\Rightarrow r = \frac{1}{2} \text{ or } r = \frac{1}{3}$$

So the complimentary function is of the form $I_n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n$

Now we try a particular solution of $I_n = \lambda n + \mu,$

$$I_n = \frac{5}{6}I_{n-1} - \frac{1}{6}I_{n-2} + d$$
$$\lambda n + \mu = \frac{5}{6}(\lambda(n-1) + \mu) - \frac{1}{6}(\lambda(n-2) + \mu) + d$$
$$\lambda n + \mu = \left(\frac{2}{3}\lambda n - \frac{1}{2}\lambda + \frac{2}{3}\mu\right) + d$$
$$\frac{1}{3}\lambda n + \frac{1}{3}\mu + \frac{1}{2}\lambda - d = 0$$

Thus, $\lambda = 0$ since there are no terms with n as a factor on the right hand side.
$\mu = 3d$ by equating the remaining terms.
So now we have the general solution

$$I_n = A\left(\frac{1}{2}\right)^n + B\left(\frac{1}{3}\right)^n + 3d$$

5 b (continued)
Finally, we use the initial conditions in order to find the values for A and B.

$$I_0 = A\left(\frac{1}{2}\right)^0 + B\left(\frac{1}{3}\right)^0 + 3d$$
$$= A + B + 3d = d$$
$$I_1 = A\left(\frac{1}{2}\right)^1 + B\left(\frac{1}{3}\right)^1 + 3d$$
$$= \frac{A}{2} + \frac{B}{3} + 3d = \frac{7}{6}d$$
$$\Rightarrow A + \frac{2B}{3} + 6d = \frac{7}{3}d$$

We combine these results by subtracting the first from the second and getting

$$-\frac{B}{3} + 3d = \frac{4}{3}d$$
$$\Rightarrow B = 5d$$
$$\Rightarrow A = -7d$$

So we finally have the solution

$$I_n = -7d\left(\frac{1}{2}\right)^n + 5d\left(\frac{1}{3}\right)^n + 3d$$

c Since the solution has three components, we look at their behaviour as n tends to infinity.
For the first two terms, as $n \to \infty$, they tend to 0. The third term is constant, so as $n \to \infty$, $I_n \to 3d$.

6 a We use the parametric version of the arc length equation which uses the parametric derivatives $\frac{dx}{dt} = 2(t-1)$ and $\frac{dy}{dt} = 4t^{\frac{1}{2}}$.
Substituting these values into the equation for arc length gives

$$s = \int_0^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_0^a \sqrt{4(t-1)^2 + 16t}\, dt$$
$$= \int_0^a \sqrt{4t^2 - 8t + 4 + 16t}\, dt$$
$$= \int_0^a \sqrt{4(t+1)^2}\, dt$$
$$= \int_0^a (2t+2)\, dt$$
$$= \left[t^2 + 2t\right]_0^a$$
$$= a^2 + 2a$$
$$= 8$$

So now we solve

$$a^2 + 2a - 8 = 0$$
$$a = \frac{-2 \pm \sqrt{4+32}}{2} = -1 \pm 3$$

$a > 0$, so $a = 2$

b In order to calculate the area of the generated surface we want to use the equation $S = 2\pi\int_{t_A}^{t_B} x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.
We substitute into the surface area equation to find

$$S = 2\pi\int_0^2 2(t-1)^2(t+1)\, dt$$
$$= 4\pi\int_0^2 (t^3 - t^2 - t + 1)\, dt$$
$$= 4\pi\left[\frac{t^4}{4} - \frac{t^3}{3} - \frac{t^2}{2} + t\right]_0^2$$
$$= \frac{16}{3}\pi$$

7 a Using integration by parts we get

$$I_{n+1} = \int_0^{\pi} \sin^{2(n+1)} x dx$$
$$= \int_0^{\pi} \sin^{2n+1} x \sin x dx$$
$$= \left[-\cos x \sin^{2n+1} x\right]_0^{\pi} + \int_0^{\pi} (2n+1)\cos^2 x \sin^{2n} x dx$$
$$= \int_0^{\pi} (2n+1)(1 - \sin^2 x)\sin^{2n} x dx$$
$$= (2n+1)\int_0^{\pi} \left(\sin^{2n} x - \sin^{2(n+1)} x\right) dx$$
$$= (2n+1)(I_n - I_{n+1})$$
$$\Rightarrow (1 + 2n + 1)I_{n+1} = (2n+1)I_n$$
$$\Rightarrow I_{n+1} = \frac{2n+1}{2n+2} I_n$$

7 b First we prove the base case of $n = 0$;
$\int_0^{\pi} \sin^0 x dx = [x]_0^{\pi} = \pi$ and $\frac{(2\times 0)!\times\pi}{(0!)^2 \times 2^{(2\times 0)}} = \pi$
and so the base case holds true. We assume that $\int_0^{\pi} \sin^{2k} x dx = \frac{(2k)!\pi}{(k!)^2 2^{2k}}$ and now we wish to prove this true for $k+1$.

$$\int_0^{\pi} \sin^{2(k+1)} x dx = I_{k+1}$$
$$= \frac{2k+1}{2k+2} I_k$$
$$= \frac{2k+1}{2k+2}\int_0^{\pi} \sin^{2k} x dx$$
$$= \frac{2k+1}{2k+2} \times \frac{(2k)!\pi}{(k!)^2 2^{2k}}$$
$$= \frac{(2k+1)!\pi}{(k!)(k+1)!2^{2k+1}}$$
$$= \frac{(2k+1)!\pi(2k+2)}{(k!)(k+1)!2^{2k+1}(2k+2)}$$
$$= \frac{(2k+2)!\pi}{((k+1)!)^2 2^{2k+2}}$$
$$= \frac{(2(k+1))!\pi}{((k+1)!)^2 2^{2(k+1)}}$$

So, if the solution is valid for $n = k$, it is valid for $n = k+1$.
Thus, the solution is valid for all $n \in \mathbb{Z}$ and $n \geqslant 0$.

8 **a** 0 does not contain 7 in any position.
We look at all positive integers that are less than or equal to 9999 and we will include the case of 0 for the sake of ease in calculations (this is why we specified that 0 does not contain 7, because then it doesn't matter if it is included or not).
If we want to have a single 7 in the number, the 7 may be in one of four positions, 7____, _7__, __7_, ___7.
The three remaining positions may take any value between 0 and 9 that is not 7, so there are 9 choices. This means there are $4 \times 9 \times 9 \times 9 = 2916$ positive numbers that contain the digit 7 exactly once.

b In order to calculate how many positive integers less than 10 000 contain the digit 7 at least once, we calculate how many don't contain 7 at all.
Similar to above, each position can take any of the 9 values between 0 and 9, not including 7 (note that this is 0 to 9999 not 10 000).
Thus, there are $9^4 = 6561$ non-negative integers that are less than 10 000 that do not contain the digit 7.
That is, 6560 positive integers less than 10 000 that do not contain the digit 7.
So, since there are 9999 positive integers less than 10 000, there are
9999 – 6560 = 3439 positive integers less than 10 000 that contain the digit 7 at least once.