Review exercise 2

1 a We have $u_n = 4u_{n-1} + 1,\ n \geqslant 1$. First, find the complementary function by solving $u_n = 4u_{n-1}$.

Let $u_n = ca^n$. Then the equation becomes $a^n = 4a^{n-1}$, so $a = 4$ and $u_n = c \cdot 4^n$. To find the general solution, we need to determine the particular solution. Since the initial equation is of the form $u_n = au_{n-1} + g(n)$ with $g(n)$ being a constant, we try the particular solution $u_n = \lambda$.
Substituting back to the original equation we get:

$\lambda = 4\lambda + 1$

$3\lambda = -1$

$\lambda = -\frac{1}{3}$

Thus the general solution is of the form $u_n = c \cdot 4^n - \frac{1}{3}$

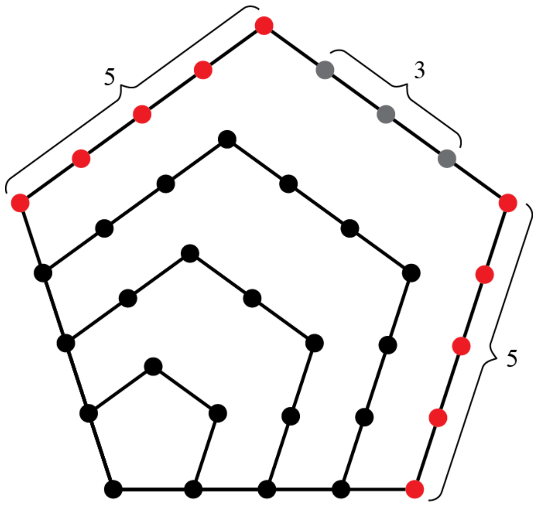
b We know that $u_0 = 7$. Substituting this into the general solution derived in part **a** we get

$7 = c \cdot 4^0 - \frac{1}{3}$

$\frac{22}{3} = c$

Thus the particular solution is $u_n = \dfrac{22(4^n) - 1}{3}$

2 a To find p_5 we need to add two edges of length 5 and connect them with an edge by adding extra 3 dots:



Thus $p_5 = p_4 + 5 + 5 + 3 = 35$. Similarly, to find p_6, we add two edges of length 6 and a connecting edge of extra 4 dots. Thus $p_6 = p_5 + 6 + 6 + 4 = 51$.

b Using the same reasoning as in part **a** we see that each next pentagonal number is created by adding two edges of length n and a connecting edge of length $n - 2$. So we can write:

$p_n = p_{n-1} + 2n + n - 2$

$p_n = p_{n-1} + 3n - 2$

2 **c** To solve this recurrence, write $p_n = p_0 + \sum_{r=1}^{n} g(r)$ where $g(r) = 3r - 2$.

Then:

$$p_n = p_1 + \sum_{r=2}^{n}(3r-2)$$
$$= p_1 + 3\sum_{r=2}^{n} r - 2\sum_{r=2}^{n} 1$$
$$= p_1 + \frac{3(n-1)(n+2)}{2} - 2(n-1)$$
$$= 1 + \frac{3n^2 + 3n - 6 - 4n + 4}{2}$$
$$= \frac{3n^2 - n}{2}$$
$$= \frac{n}{2}(3n-1)$$

So $p_n = \frac{n}{2}(3n-1)$

d Using our answer to part **c** we have $p_{100} = \frac{100}{2}(300-1) = 14\,950$

3 We want to solve $u_n = 2u_{n-1} + 3n + 1$ with $u_0 = 11$. Begin by solving the homogeneous version $u_n = 2u_{n-1}$. Write $u_n = ca^n$, then we have:

$ca^n = 2ca^{n-1}$

$a = 2$

So $u_n = c \cdot 2^n$. Now, for the particular solution, try $u_n = \lambda n + \mu$. Substituting into the original recurrence we get:

$\lambda n + \mu = 2\left(\lambda\left(n-1\right) + \mu\right) + 3n + 1$

$\lambda n + \mu = 2\lambda n + 2\mu + 3n - 2\lambda + 1$

$\lambda n + \mu - 2\lambda = -3n - 1$

Comparing the n terms and the constant terms gives:

$\lambda = -3$

$\mu - 2\lambda = -1$

$\mu + 6 = -1$

$\mu = -7$

So the particular solution is $u_n = -3n - 7$. Thus we can now write $u_n = c \cdot 2^n - 3n - 7$.

Knowing that $u_0 = 11$ gives

$11 = c \cdot 2^0 - 3 \cdot 0 - 7$

$11 = c - 7$

$c = 18$

So

$u_n = 18 \cdot 2^n - 3n - 7$

$u_n = 9 \cdot 2^{n+1} - 3n - 7$

4 **a** We have

$u_{n+1} - 3u_n = 10$

with

$u_1 = 7$

So we have

$u_2 - 3 \cdot 7 = 10$

$u_2 = 31$

And

$u_3 - 3 \cdot 31 = 10$

$u_3 = 103$

4 b i To solve

$u_{n+1} = 3u_n + 10$

first find the complementary function by solving

$u_{n+1} = 3u_n$

Write

$u_{n+1} = ca^{n+1}$

Substituting into the homogeneous recurrence gives:

$ca^{n+1} = 3ca^n$

$a = 3$

And

$u_n = c \cdot 3^n$

To find the particular solution, try

$u_n = \lambda$

Substituting into the original equation gives:

$\lambda = 3\lambda + 10$

$2\lambda = -10$

$\lambda = -5$

So the general solution is

$u_n = c \cdot 3^n - 5$

Using

$u_1 = 7$

we get

$7 = c \cdot 3 - 5$

$3c = 12$

$c = 4$

Thus the particular solution in this case is

$u_n = 4 \cdot 3^n - 5$

ii We want to find the smallest value of n for which

$1\,000\,000 \leqslant 4 \cdot 3^n - 5$

Rearranging this equation gives

$$\frac{1\,000\,005}{4} \leqslant 3^n$$

$$\log_3\left(\frac{1\,000\,005}{4}\right) \leqslant n$$

$n \leqslant 11.32$

Since n has to be a whole number, we see that the smallest such value is $n = 12$.

5 a The drug is administered every 4 hours, so this is the time difference between u_n and u_{n-1}.

Every hour, the amount of the drug in the patient's body diminishes by 20% Assume the last dose was the $n-1^{\text{st}}$ dose. Then, after 1 hour, the patient has $0.8u_{n-1}$ of the drug in her system. After another hour, this decreases by another 20% and becomes

$$0.8\cdot(0.8u_{n-1})=(0.8)^2u_{n-1}$$

So after 4 hours, the amount left in the patient's body is

$$(0.8)^4u_{n-1}$$

This is also when the patient receives another 100 mg dose of the drug. Thus

$$u_n=(0.8)^4u_{n-1}+100$$

b Initially, the drug in not present in the patient's body, so $u_0=0$. To solve the recurrence, begin by finding the complementary function of $u_n=(0.8)^4u_{n-1}$. Write $u_n=c\cdot a^n$. Then

$$c\cdot a^n=(0.8)^4\cdot c\cdot a^{n-1}$$

$$a=(0.8)^4$$

$$u_n=c(0.8)^{4n}$$

Now, to find the particular solution try $u_n=\lambda$. Substitute into $u_n=(0.8)^4u_{n-1}+100$:

$$\lambda=(0.8)^4\lambda+100$$

$$\lambda\left(1-(0.8)^4\right)=100$$

$$\lambda\approx 169.38$$

So

$$u_n=c\cdot(0.8)^{4n}+169.38$$

To find c, note that $u_o=0$

$$0=c+169.38$$

$$c=-169.38$$

Thus the solution to our recurrence is

$$u_n=169.38\left(1-(0.8)^{4n}\right).$$

5 c We need to find the largest n for which

$$169.38\left(1-(0.8)^{4n}\right)\leqslant 160$$

Rearranging we obtain:

$$1-(0.8)^{4n}\leqslant\frac{160}{169.38}$$

$$(0.8)^{4n}\geqslant 1-\frac{160}{169.38}$$

$$4n\geqslant\log_{0.8}\left(1-\frac{160}{169.38}\right)$$

$$4n\leqslant 12.97$$

$\log_{0.8}$ is a decreasing function, so remember to switch the inequality!

$$n\leqslant 3.24$$

Since n has to be a while number, we see that maximum number of doses which can be administered this way is 3.

6 a At the beginning, Heather owes £3000, so $a_0 = 3000$. At the end of the first month, the company adds on the interest of 1.8%, so Heather now owes $1.018\times a_0$. But she then repays £300, so after the first repayment, Heather owes $a_1 = 1.018\times a_0 - 300$. This pattern repeats every month, so $a_n = 1.018\cdot a_{n-1} - 300$.

b To find the value of Heather's last repayment, we need to find the largest n for which $a_n\geqslant 0$ and then find the value of a_n. First, however, we need to solve the recurrence. Begin by finding the complementary function for $a_n = 1.018\cdot a_{n-1}$. Write $a_n = c\cdot r^n$. Then

$$c\cdot r^n = 1.018c\cdot r^{n-1}$$

$$r = 1.018$$

$$a_n = c\cdot(1.018)^n$$

To find the particular solution, try $a_n=\lambda$:

$$\lambda = 1.018\lambda - 300$$

$$0.018\lambda = 300$$

$$\lambda\approx 16666.67$$

So we now have

$$a_n = c\times(1.018)^n + 16\,666.67$$

Finally, to find c, use $a_0 = 3000$:

$$3000 = c + 16\,666.67 \Rightarrow c = -13\,666.67$$

Thus $a_n = -13\,666.67(1.018)^n + 16\,666.67$

Now we need to find the largest n for which $a_n\geqslant 0$:

$$0\leqslant -13\,666.67(1.018)^n + 16\,666.67$$

$$\Rightarrow 1.2195\geqslant(1.018)^n$$

$$\Rightarrow\frac{\log 1.2195}{\log 1.018}\geqslant n$$

$$\Rightarrow 11.12\geqslant n$$

Since n is a whole number, the largest n for which $a_n\geqslant 0$ is 11. For that n we have

$$a_{11} = -13\,666.67(1.018)^{11} + 16\,666.67 = £36.82$$

So the last repayment will be £36.82

7 a We have $u_n = k - 4n$ and $u_{n-1} = k - 4(n-1) = k + 4 - 4n$. Thus:

$u_{n-1} - 4 = k + 4 - 4n - 4$

$u_{n-1} - 4 = k - 4n = u_n$

So the given sequence does indeed satisfy this recurrence relationship.

b Now we want to show that $v_n = c(1.2^{n-1})$ satisfies $v_n = 1.2v_{n-1}$. We have $v_{n-1} = c(1.2^{n-2})$, so $1.2v_{n-1} = 1.2c(1.2^{n-2}) = c(1.2^{n-1}) = v_n$ as required.

c We know that $u_0 = v_0$. Also, by the definition of the two sequences, we have $u_0 = k$ and $v_0 = \dfrac{c}{1.2}$

Thus it must be

$k = \dfrac{c}{1.2}$

$1.2k = c$

8 a We have $u_n - 0.7u_{n-1} = k$, $u_1 = 4$. For the relation to be homogeneous, we need the constant to be 0, so $k = 0$.

b To solve $u_n - 0.7u_{n-1} = 0$

so

$$\begin{aligned}u_n &= 0.7u_{n-1}\\ &= 0.7 \times 0.7u_{n-2} = 0.7^2 u_{n-2}\\ &= 0.7^2 \times 0.7u_{n-3} = 0.7^3 u_{n-3}\\ &= \ldots \qquad\qquad = 0.7^{n-1} u_1\\ &= 4(0.7^{n-1})\end{aligned}$$

9 a The first tower is formed of just 2 blocks, so to build three copies of Tower 1 we need 6 blocks. Thus $b_1 = 6$. To create $(n+1)$st tower, we add $n \cdot n$ blocks. Thus to build three copies of Tower $n+1$, we need additional $3n^2$ blocks. Hence we can write: $b_n = b_{n-1} + 3(n-1)^2$, $b_1 = 6$.

b To solve the recurrence relation write

$$\begin{aligned}b_n &= b_{n-1} + 3(n-1)^2\\ &= b_{n-2} + 3(n-1)^2 + 3(n-2)^2\\ &= \ldots = b_1 + 3\sum_{r=1}^{n-1} r^2\\ &= 6 + 3 \cdot \tfrac{1}{6} n(n-1)(2n-1)\\ &= 6 + \tfrac{1}{2}(2n^3 - 3n^2 + n)\\ &= \tfrac{1}{2}(2n^3 - 3n^2 + n + 12)\end{aligned}$$

Which is the required form.

10 We wish to solve $u_n = 4u_{n-1} + 2n + 1$ with $u_1 = 7$. Begin by finding the complementary function to $u_n = 4u_{n-1}$. Write $u_n = c \cdot a^n$:

$c \cdot a^n = 4c \cdot a^{n-1}$

$a = 4$

$u_n = c \cdot 4^n$

For particular solution try $u_n = \lambda + \mu n$:

$\lambda + \mu n = 4\lambda + 4\mu(n-1) + 2n + 1$

$3\lambda + 3\mu n = 4\mu - 1 - 2n$

By comparing the *n*-terms and the constant terms we get:

$3\mu n = -2n$

$\mu = -\frac{2}{3}$

$3\lambda = -4 \cdot \frac{2}{3} - 1$

$3\lambda = -\frac{11}{3}$

$\lambda = -\frac{11}{9}$

So we can write $u_n = c \cdot 4^n - \frac{2}{3}n - \frac{11}{9}$. To find *c*, use $u_1 = 7$:

$7 = c \cdot 4 - \frac{2}{3} - \frac{11}{9}$

$7 = 4c - \frac{17}{9}$

$4c = \frac{80}{9}$

$c = \frac{20}{9}$

Thus $u_n = \frac{20}{9} \cdot 4^n - \frac{2}{3}n - \frac{11}{9}$

11 a For a number that is one digit long, we can use any of the digits 1–9 but we cannot use 0 (since that would mean an odd dumber of zeros). Thus $a_1 = 9$. Now, we want to count the number of valid strings of length *n*. For any of the a_{n-1} valid string of length *n – 1*, we can add numbers 1–9 at the end and still maintain a valid string. This gives us $9a_{n-1}$ possibilities. Any invalid string of length *n* – 1 can be made valid by adding a 0 at the end. The number of invalid strings of length *n* – 1 can be calculated by subtracting the valid strings from all possible strings. There are 10^{n-1} possible strings of length *n* – 1, so the number of invalid strings can be expressed as $10^{n-1} - a_{n-1}$. Thus:

$a_n = 9a_{n-1} + 10^{n-1} - a_{n-1}$

$a_n = 8a_{n-1} + 10^{n-1}$

As required.

11 b To solve this recurrence begin by finding the complementary function to $a_n = 8a_{n-1}$.

Write $a_n = c \cdot r^n$, then:

$c \cdot r^n = 8c \cdot r^{n-1}$

$r = 8$

$a_n = c \cdot 8^n$

To find the particular solution, try $a_n = \lambda \cdot 10^{n-1}$:

$\lambda \cdot 10^{n-1} = 8\lambda \cdot 10^{n-2} + 10^{n-1}$

$2\lambda \cdot 10^{n-2} = 10^{n-1}$

$2\lambda = 10$

$\lambda = 5$

So we can write $a_n = c \cdot 8^n + 5 \cdot 10^{n-1}$. To find c, use $a_1 = 9$:

$9 = c \cdot 8 + 5$

$8c = 4$

$c = \frac{1}{2}$

Thus $a_n = \frac{1}{2} \cdot 8^n + 5 \cdot 10^{n-1} = \frac{1}{2}\left(8^n + 10^n\right)$

12 We are given a sequence which satisfies $u_n = (n+3)u_{n-1}$, $u_1 = 1$. We want to show that the closed form of this sequence is $u_n = \frac{(n+3)!}{24}$. Begin by checking the induction basis:

$n = 1$: $u_1 = \frac{4!}{24} = 1$ as given in the question. Now, assume that for all $n=k$ we have $u_k = \frac{(k+3)!}{24}$.

We want to show that u_{k+1} also follows this formula:

$$u_{k+1} = (k+4)\frac{(k+3)!}{24} = \frac{(k+4)!}{24}$$

Thus, if the formula is valid for $n = k$, then it's also valid for $n = k + 1$. Hence, by induction, the formula is valid for all positive integers n.

13 a Assume $u_n = \mathrm{F}(n)$ and $u_n = \mathrm{G}(n)$ are both particular solutions to $u_n = au_{n-1}$. We want to show that $u_n = b\mathrm{F}(n) + c\mathrm{G}(n)$ is also a particular solution. Let $u_n = b\mathrm{F}(n) + c\mathrm{G}(n)$. Then

$u_{n-1} = b\mathrm{F}(n-1) + c\mathrm{G}(n-1)$ and

$au_{n-1} = ab\mathrm{F}(n-1) + ac\mathrm{G}(n-1)$

$= b(a\mathrm{F}(n-1)) + c(a\mathrm{G}(n-1))$

Now, since both $\mathrm{F}(n)$ and $\mathrm{G}(n)$ are particular solutions, they satisfy $\mathrm{F}(n) = a\mathrm{F}(n-1)$ and $\mathrm{G}(n) = a\mathrm{G}(n-1)$. So the above can be rewritten as:

$u_{n-1} = b\mathrm{F}(n) + c\mathrm{G}(n) = u_n$.

This proves that if $u_n = \mathrm{F}(n)$ and $u_n = \mathrm{G}(n)$ are both particular solutions to $u_n = au_{n-1}$, then $u_n = b\mathrm{F}(n) + c\mathrm{G}(n)$ is also a particular solution.

b We have $a_n = 3a_{n-1} - 4n + 3\cdot 2^n$, $a_1 = 8$. To solve this, begin by finding the complementary function to $a_n = 3a_{n-1}$. Write $a_n = c\cdot r^n$:

$c\cdot r^n = 3c\cdot r^{n-1}$

$r = 3$

$a_n = c\cdot 3^n$

Next, for the particular solution try $a_n = \lambda\cdot 2^n + pn + q$:

$\lambda\cdot 2^n + pn + q = 3\lambda\cdot 2^{n-1} + 3p(n-1) + 3q - 4n + 3\cdot 2^n$

$\lambda\cdot 2^{n-1} + 2pn - 3p + 2q = 4n - 6\cdot 2^{n-1}$

Now by comparing appropriate terms we get:

$\lambda = -6$

$2p = 4$

$p = 2$

$-3p + 2q = 0$

$2q = 6$

$q = 3$

So we can write $a_n = c\cdot 3^n - 6\cdot 2^n + 2n + 3$. To find c, use $a_1 = 8$:

$8 = c\cdot 3 - 6\cdot 2 + 2 + 3$

$3c = 15$

$c = 5$

So $a_n = 5\cdot 3^n - 6\cdot 2^n + 2n + 3$ as required.

13 c Begin by checking the base: $a_1 = 5\cdot 3 - 6\cdot 2 + 2 + 3 = 8$ as required. Now, assume that for $n = k$, $a_k = 5\cdot 3^k - 6\cdot 2^k + 2k + 3$. We want to check if it also works for $n = k + 1$:

$$a_{k+1} = 3a_k - 4(k+1) + 3\cdot 2^{k+1}$$
$$= 3(5\cdot 3^k - 6\cdot 2^k + 2k + 3) - 4k - 4 + 3\cdot 2^{k+1}$$
$$= 5\cdot 3^{k+1} - 9\cdot 2^{k+1} + 6k + 9 - 4k - 4 + 3\cdot 2^{k+1}$$
$$= 5\cdot 3^{k+1} - 6\cdot 2^{k+1} + 2k + 5$$
$$= 5\cdot 3^{k+1} - 6\cdot 2^{k+1} + 2(k+1) + 3$$

Which is of the required form. Thus if the formula works for $n = k$ then it also works for $n = k + 1$. Hence, by induction, $a_n = 5\cdot 3^n - 6\cdot 2^n + 2n + 3$ for all positive integers n.

14 a We have the size of a population of bacteria modelled by $2p_n = 7p_{n-1} - 5p_{n-2}$, $p_0 = 400$, $p_1 = 448$. This is a second order recurrence relation, since the difference between the highest and the lowest subscript is 2.

b To solve this recurrence relation, write $p_n = c\cdot a^n$:

$$2c\cdot a^n = 7ca^{n-1} - 5ca^{n-2}$$
$$2a^2 = 7a - 5$$
$$a_1 = 1,\ \ a_2 = \tfrac{5}{2}$$

So $p_n = c_1 + c_2\left(\tfrac{5}{2}\right)^n$. To find c_1 and c_2 use $p_0 = 400$ and $p_1 = 448$:

$$400 = c_1 + c_2 \quad\Rightarrow\quad c_1 = 400 - c_2$$
$$448 = c_1 + \tfrac{5}{2}c_2$$
$$= 400 + \tfrac{3}{2}c_2$$
$$\tfrac{3}{2}c_2 = 48$$
$$c_2 = 32 \quad\Rightarrow\quad c_1 = 368$$

Hence $p_n = 368 + 32\cdot\left(\tfrac{5}{2}\right)^n$

c After 12 hours the population is $p_{12} = 368 + 32\cdot\left(\tfrac{5}{2}\right)^{12} \approx 1907\,717$.

d This model assumes that the population grows exponentially to infinity, which is not realistic. Taking into account food and space constraints (for example) would limit the colony's growth.

15 a To solve $u_{n+2} = 4u_{n+1} + 5u_n$ write $u_n = c\cdot a^n$:

$$c\cdot a^{n+2} = 4ca^{n+1} + 5ca^n$$
$$a^2 - 4a - 5 = 0$$
$$(a+1)(a-5) = 0$$
$$a_1 = -1,\ \ a_2 = 5$$

So the general solution is $u_n = A(-1)^n + B\cdot 5^n$

15 b We are given $u_0 = 8,\ \ u_1 = -20$. So:

$8 = A + B \ \ \Rightarrow \ \ A = 8 - B$

$-20 = -A + 5B$

$-20 = B - 8 + 5B = 6B - 8$

$6B = -12$

$B = -2 \ \ \Rightarrow \ \ A = 10$

Thus the particular solution is $u_n = 10 \cdot (-1)^n - 2 \cdot 5^n$

16 a We want to find a constant k such that $u_n = k$ is a particular solution to $3u_{n+2} + 10u_{n+1} - 8u_n = 20$. Substituting k into this equation gives:

$3k + 10k - 8k = 20$

$5k = 20$

$k = 4$

b Now, to solve the homogeneous version of this recurrence, write $u_n = c \cdot a^n$:

$3c \cdot a^{n+2} + 10c \cdot a^{n+1} - 8c \cdot a^n = 0$

$3a^2 + 10a - 8 = 0$

$a_1 = -4,\ \ a_2 = \frac{2}{3}$

So, combining this with part **a**, we can write $u_n = A(-4)^n + B\left(\frac{2}{3}\right)^n + 4$. To find A and B use $u_0 = 0$ and $u_1 = 1$:

$0 = A + B + 4 \ \ \Rightarrow \ \ A = -4 - B$

$1 = -4A + \frac{2}{3}B + 4$

$1 = 16 + 4B + \frac{2}{3}B + 4$

$-19 = \frac{14}{3}B$

$B = -\frac{57}{14} \ \ \Rightarrow \ \ A = \frac{1}{14}$

Thus $u_n = \frac{1}{14}(-4)^n - \frac{57}{14}\left(\frac{2}{3}\right)^n + 4$

17 a To get a row of length $n + 2$ we can either add a vertical domino to a row of length $n - 1$ (and there are x_{n+1} of them), or add a horizontal domino to a row of length n (there are x_n of those). Thus $x_{n+2} = x_{n+1} + x_n$. There is only one way to build a row of length 1 (one vertical domino) so $x_1 = 1$. There are two ways of getting a row of length 2 (two vertical dominoes or one horizontal domino), so $x_2 = 2$.

b Based on part **a** we have $x_3 = 2 + 1 = 3$, $x_4 = 3 + 2 = 5$, $x_5 = 5 + 3 = 8$, $x_6 = 8 + 5 = 13$, $x_7 = 13 + 8 = 21$ and finally $x_8 = 21 + 13 = 34$.

17 c **i** To solve this recurrence relation, write $x_n = c \cdot a^n$:

$$c \cdot a^{n+2} = c \cdot a^{n+1} + c \cdot a^n$$

$$a^2 - a - 1 = 0$$

$$a_1 = \frac{1-\sqrt{5}}{2},\ \ a_2 = \frac{1+\sqrt{5}}{2}$$

So we can write $x_n = A\left(\frac{1-\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n$. To find A and B use the initial conditions:

$$1 = A\left(\frac{1-\sqrt{5}}{2}\right) + B\left(\frac{1+\sqrt{5}}{2}\right) \quad \Rightarrow \quad A = \frac{2 - B\left(1+\sqrt{5}\right)}{1-\sqrt{5}}$$

$$2 = A\frac{6-2\sqrt{5}}{4} + B\frac{6+2\sqrt{5}}{4}$$

$$4 = A\left(3-\sqrt{5}\right) + B\left(3+\sqrt{5}\right)$$

$$4 = \frac{\left(3-\sqrt{5}\right)\left(2-B-B\sqrt{5}\right)}{1-\sqrt{5}} + \frac{B\left(3+\sqrt{5}\right)\left(1-\sqrt{5}\right)}{1-\sqrt{5}}$$

$$4 = \frac{6-2\sqrt{5} - B\left(3-\sqrt{5}+3\sqrt{5}-5\right) - B\left(2+2\sqrt{5}\right)}{1-\sqrt{5}}$$

$$4 = \frac{6-2\sqrt{5}-4\sqrt{5}B}{1-\sqrt{5}}$$

$$4-4\sqrt{5} = 6-2\sqrt{5}-4\sqrt{5}B$$

$$2\sqrt{5}B = 1+\sqrt{5}$$

$$B = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{5+\sqrt{5}}{10}$$

$$A = \frac{2 - \frac{5+\sqrt{5}}{10}\left(1+\sqrt{5}\right)}{1-\sqrt{5}}$$

$$A = \frac{20-10-6\sqrt{5}}{10-10\sqrt{5}}$$

$$A = \frac{5-3\sqrt{5}}{5-5\sqrt{5}} = -\frac{\left(5-3\sqrt{5}\right)\left(5+5\sqrt{5}\right)}{100}$$

$$A = -\frac{-50+10\sqrt{5}}{100} = \frac{5-\sqrt{5}}{10}$$

So the closed form is $x_n = \frac{5-\sqrt{5}}{10} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n$

17 c ii 2 feet is equal to 24 inches, so we're looking for x_{24}:

$$x_{24} = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{24} + \frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{24} = 75\,025$$

18 a We know that $u_n = A\cdot 3^n + B\cdot 5^n$ is the general solution to $u_n = ru_{n-1} - su_{n-2}$. Substituting we get: $A\cdot 3^n + B\cdot 5^n = rA\cdot 3^{n-1} + rB\cdot 5^{n-1} - sA\cdot 3^{n-2} - sB\cdot 5^{n-2}$. Comparing the appropriate terms we obtain:

$9A = 3rA - sA$

$25B = 5rB - sB$

$9 = 3r - s \quad\Rightarrow\quad s = 3r - 9$

$25 = 5r - s$

$25 = 5r - 3r + 9$

$16 = 2r$

$r = 8 \quad\Rightarrow\quad s = 15$

b We know that $u_0 = 1$ and $B = 3A$, we wish to find u_1. Substituting u_0 into $u_n = A\cdot 3^n + B\cdot 5^n$ gives

$1 = A + B = 4A$

$A = \frac{1}{4},\ B = \frac{3}{4}$

Thus $u_1 = \frac{1}{4}\cdot 3 + \frac{3}{4}\cdot 5 = \frac{18}{4} = \frac{9}{2}$

19 a We have $u_n = pu_{n-2} - 4u_{n-1}$ and we know that the auxiliary equation has complex roots. Write $u_n = c\cdot a^n$:

$c\cdot a^n = pc\cdot a^{n-2} - 4ca^{n-1}$

$a^2 + 4a - p = 0$

For this equation to have complex roots, we need $\Delta = 16 + 4p < 0$, so $p < -4$.

b Let $p = -5$, i.e. the equation becomes $u_n + 4u_{n-1} + 5u_{n-2} = 0$. Using the auxiliary equation from part **a** we have:

$a^2 + 4a + 5 = 0$

$a = -2 \pm \mathrm{i}$

So we can write $u_n = A(-2-\mathrm{i})^n + B(-2+\mathrm{i})^n$

c Given $u_0 = 1$ and $u_1 = 2$ we want to find the particular solution:

$1 = A + B \quad\Rightarrow\quad A = 1 - B$

$2 = -2A - A\mathrm{i} - 2B + B\mathrm{i}$

$2 = -2 + 2B - \mathrm{i} + B\mathrm{i} - 2B + B\mathrm{i}$

$4 + \mathrm{i} = 2B\mathrm{i}$

$$B = \frac{4+\mathrm{i}}{2\mathrm{i}} = -\frac{4\mathrm{i}-1}{2} = \frac{1-4\mathrm{i}}{2}$$

$$B = \frac{1-4\mathrm{i}}{2} \quad\Rightarrow\quad \mathrm{A} = \frac{1+4\mathrm{i}}{2}$$

So the particular solution is $u_n = \frac{1+4\mathrm{i}}{2}(-2-\mathrm{i})^n + \frac{1-4\mathrm{i}}{2}(-2+\mathrm{i})^n$

20 a We have $u_n = 10u_{n-1} - 25u_{n-2}$ with $u_0 = 1$ and $u_1 = 3$. To solve, write $u_n = c \cdot a^n$:

$c \cdot a^n = 10c \cdot a^{n-1} - 25c \cdot a^{n-2}$

$a^2 - 10a + 25 = 0$

$(a-5)^2 = 0$

$a = 5$

Since the auxiliary equation has a repeated root, we write the general solution as $u_n = (A + Bn) \cdot 5^n$.

Now, to find A and B we use the initial conditions:

$1 = A$

$3 = 5(A + 3B)$

$3 = 5 + 15B$

$15B = -2$

$B = -\frac{2}{15}$

So the closed form is $u_n = \left(1 - \frac{2}{5}n\right) \cdot 5^n$

b Begin by checking the basis: $u_0 = \left(1 - \frac{2}{5} \cdot 0\right) = 1$ which is the same as the initial condition. Now assume that $u_n = \left(1 - \frac{2}{5}n\right) \cdot 5^n$ holds for all $n = k$. We want to show that then it also holds for $n = k + 1$. We have:

$u_{k+1} = 10u_k - 25u_{k-1}$

$= 10 \cdot 5^k \left(1 - \frac{2}{5}k\right) - 25 \cdot 5^{k-1}\left(1 - \frac{2}{5}(k-1)\right)$

$= 2 \cdot 5^{k+1}\left(1 - \frac{2}{5}k\right) - 5^{k+1}\left(1 - \frac{2}{5}k + \frac{2}{5}\right)$

$= 2 \cdot 5^{k+1}\left(1 - \frac{2}{5}k\right) - 5^{k+1}\left(1 - \frac{2}{5}k\right) - \frac{2}{5} \cdot 5^{k+1}$

$= 5^{k+1}\left(1 - \frac{2}{5}k\right) - \frac{2}{5} \cdot 5^{k+1}$

$= 5^{k+1}\left(1 - \frac{2}{5}k - \frac{2}{5}\right)$

$= 5^{k+1}\left(1 - \frac{2}{5}(k+1)\right)$

Which is of the required form. So, if this formula works for $n = k - 1$ and $n = k$, then it also works for $n = k + 1$. Thus, by mathematical induction, $u_n = \left(1 - \frac{2}{5}n\right) \cdot 5^n$ for all positive integers *n*.

21 a i We wish to solve $u_n = 4u_{n-1} + 5u_{n-2} + 2n^2$. To find the complementary function, consider the homogenous version $u_n = 4u_{n-1} + 5u_{n-2}$ and write $u_n = c \cdot a^n$:

$c \cdot a^n = 4c \cdot a^{n-1} + 5c \cdot a^{n-2}$

$a^2 - 4a - 5 = 0$

$(a-5)(a+1) = 0$

$a_1 = 5, \ a_2 = -1$

So the complementary function is of the form $u_n = A \cdot 5^n + B(-1)^n$

21 a ii For the particular solution, try $u_n = \lambda n^2 + \mu n + \nu$:

$\lambda n^2 + \mu n + \nu = 4\lambda(n-1)^2 + 4\mu(n-1) + 4\nu + 5\lambda(n-2)^2 + 5\mu(n-2) + 5\nu + 2n^2$

$\lambda n^2 + \mu n + \nu = n^2(9\lambda + 2) - n(28\lambda - 9\mu) + 24\lambda - 14\mu + 9\nu$

$n^2(8\lambda + 2) - n(28\lambda - 8\mu) + 24\lambda - 14\mu + 8\nu = 0$

Now comparing the appropriate terms to zero, we get:

$8\lambda = -2$

$\lambda = -\frac{1}{4}$

$28\lambda = 8\mu$

$-7 = 8\mu$

$\mu = -\frac{7}{8}$

$24\lambda - 14\mu + 8\nu = 0$

$-6 + \frac{49}{4} + 8\nu = 0$

$8\nu = -\frac{25}{4}$

$\nu = -\frac{25}{32}$

So the general solution is of the form $u_n = A\cdot 5^n + B(-1)^n - \frac{1}{4}n^2 - \frac{7}{8}n - \frac{25}{32}$

b We know that $u_0 = u_1 = 0$. Substituting into $u_n = A\cdot 5^n + B(-1)^n - \frac{1}{4}n^2 - \frac{7}{8}n - \frac{25}{32}$ we get:

$0 = A + B - \frac{25}{32} \quad \Rightarrow \quad A = \frac{25}{32} - B$

$0 = 5A - B - \frac{1}{4} - \frac{7}{8} - \frac{25}{32}$

$0 = \frac{125}{32} - 5B - B - \frac{61}{32}$

$6B = 2$

$B = \frac{1}{3} \quad \Rightarrow \quad A = \frac{75}{96} - \frac{32}{96} = \frac{43}{96}$

So the particular solution is $u_n = \frac{43}{96}\cdot 5^n + \frac{1}{3}\cdot(-1)^n - \frac{1}{4}n^2 - \frac{7}{8}n - \frac{25}{32}$

22 We want use induction to show that $r_n = r_{n-1} + 12r_{n-2}$ with $r_0 = 1$ and $r_1 = 11$ has closed for $r_n = 2\cdot 4^n - (-3)^n$. Begin by proving the base:

$r_0 = 2\cdot 4^0 - (-3)^0 = 2 - 1 = 1$ which agrees with the initial condition.

$r_1 = 2\cdot 4^1 - (-3)^1 = 8 + 3 = 11$ which also agrees with the initial condition.

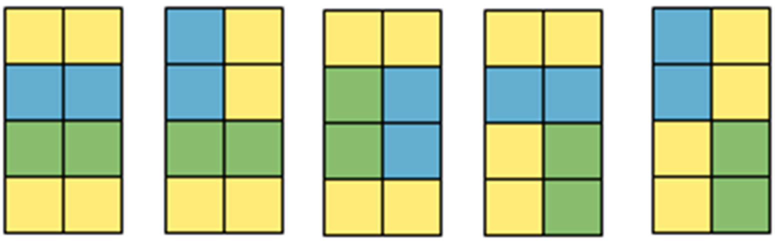
Now assume that for $n = k$ $r_k = 2\cdot 4^k - (-3)^k$ and $r_{k-1} = 2\cdot 4^{k-1} - (-3)^{k-1}$. We want to show that $r_{k+1} = 2\cdot 4^{k+1} - (-3)^{k+1}$. By the definition of this sequence, we have that:

$$\begin{aligned} r_{k+1} &= r_k + 12r_{k-1} \\ &= 2\cdot 4^k - (-3)^k + 12\cdot 2\cdot 4^{k-1} - 12\cdot(-3)^{k-1} \\ &= 2\cdot 4^k - (-3)^k + 6\cdot 4^k + 4\cdot(-3)^k \\ &= 8\cdot 4^k + 3\cdot(-3)^k \\ &= 2\cdot 4^{k+1} - (-3)^{k+1} \end{aligned}$$

Thus, if $r_k = 2\cdot 4^k - (-3)^k$ and $r_{k-1} = 2\cdot 4^{k-1} - (-3)^{k-1}$, then $r_{n+1} = 2\cdot 4^{n+1} - (-3)^{n+1}$. So, by mathematical induction, $r_n = 2\cdot 4^n - (-3)^n$ for all positive integers *n*.

23 a For $n = 3$ we have the following possibilities:

So $a_3 = 3$. For $n = 4$ we have the following possibilities:



So $a_4 = 5$.

b Assume we know a_{n-1} and a_{n-2}. To determine the number of arrangements for *n*, we can lay the first tile horizontally and then think about the floor as if it were a $2\times(n-1)$ area, so this gives us a_{n-1} arrangements. Alternatively, we could lay the first two tiles vertically to cover a 2×2 area and then think about the floor as a $2\times(n-2)$ area, which gives us another a_{n-2} possibilities. Thus $a_{n+1} = a_n + a_{n-1}$. From part **a**, we know that $a_1 = 1$ and $a_2 = 2$, which gives us the initial conditions.

23 c To solve this recurrence, write $a_n = k^n$, then we have:

$$k^n = k^{n-1} + k^{n-2}$$

$$k^2 - k - 1 = 0$$

$$k = \frac{1 \pm \sqrt{5}}{2}$$

So we can write $a_n = A\left(\frac{1-\sqrt{5}}{2}\right)^n + B\left(\frac{1+\sqrt{5}}{2}\right)^n$. To find A and B use the initial conditions:

$$1 = A\left(\frac{1-\sqrt{5}}{2}\right) + B\left(\frac{1+\sqrt{5}}{2}\right) \quad \Rightarrow \quad A = \frac{2 - B\left(1+\sqrt{5}\right)}{1-\sqrt{5}}$$

$$2 = A\frac{6-2\sqrt{5}}{4} + B\frac{6+2\sqrt{5}}{4}$$

$$4 = A\left(3-\sqrt{5}\right) + B\left(3+\sqrt{5}\right)$$

$$4 = \frac{\left(3-\sqrt{5}\right)\left(2-B-B\sqrt{5}\right)}{1-\sqrt{5}} + \frac{B\left(3+\sqrt{5}\right)\left(1-\sqrt{5}\right)}{1-\sqrt{5}}$$

$$4 = \frac{6-2\sqrt{5} - B\left(3-\sqrt{5}+3\sqrt{5}-5\right) - B\left(2+2\sqrt{5}\right)}{1-\sqrt{5}}$$

$$4 = \frac{6-2\sqrt{5}-4\sqrt{5}B}{1-\sqrt{5}}$$

$$4-4\sqrt{5} = 6-2\sqrt{5}-4\sqrt{5}B$$

$$2\sqrt{5}B = 1+\sqrt{5}$$

$$B = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{5+\sqrt{5}}{10}$$

$$A = \frac{2 - \frac{5+\sqrt{5}}{10}\left(1+\sqrt{5}\right)}{1-\sqrt{5}}$$

$$A = \frac{20-10-6\sqrt{5}}{10-10\sqrt{5}}$$

$$A = \frac{5-3\sqrt{5}}{5-5\sqrt{5}} = -\frac{\left(5-3\sqrt{5}\right)\left(5+5\sqrt{5}\right)}{100}$$

$$A = -\frac{-50+10\sqrt{5}}{100} = \frac{5-\sqrt{5}}{10}$$

So the closed form is $a_n = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n$

23 d We want to use mathematical induction to prove that the closed form found in part **c** is correct. Begin by checking the basis:

$$a_1 = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)$$

$$=\frac{10-6\sqrt{5}}{20}+\frac{10+6\sqrt{5}}{20}$$

$$=1$$

Which agrees with the initial condition.

$$a_2 = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^2+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$=\frac{5-\sqrt{5}}{10}\cdot\frac{6-2\sqrt{5}}{4}+\frac{5+\sqrt{5}}{10}\cdot\frac{6+2\sqrt{5}}{4}$$

$$=\frac{40-16\sqrt{5}}{40}+\frac{40+16\sqrt{5}}{40}$$

$$=2$$

Which also agrees with the initial condition. Now, assume that the statement is true for $n = k$ and inductively assume that $a_k = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^k+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^k$ and

$$a_{k-1} = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$$

We want to show that then $a_{k+1} = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$. Using the recurrence relation we can write:

$$a_{k+1} = a_k + a_{k-1}$$

$$=\frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^k+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^k+\frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$$

$$=\frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^k+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^k-\frac{\sqrt{5}}{5}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^k+\frac{\sqrt{5}}{5}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^k$$

$$=\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{5-\sqrt{5}}{10}-\frac{2\sqrt{5}}{10}\right)+\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{5+\sqrt{5}}{10}+\frac{2\sqrt{5}}{10}\right)$$

$$=\frac{5-3\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^k+\frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^k$$

$$=\frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}+\frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$$

23 d (continued)

Which is the required form. So if $a_k = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^k + \frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^k$ and

$a_{k-1} = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k-1} + \frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$ then

$a_{k+1} = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$. So, by mathematical induction,

$a_n = \frac{5-\sqrt{5}}{10}\cdot\left(\frac{1-\sqrt{5}}{2}\right)^n + \frac{5+\sqrt{5}}{10}\cdot\left(\frac{1+\sqrt{5}}{2}\right)^n$ for all positive integers n.

24 a $\mathbf{M} = \begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}$. To find the eigenvalues, we look for the roots of the determinant of

$\mathbf{M} - \lambda\mathbf{I} = \begin{pmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{pmatrix}$:

$$\begin{aligned} \det(\mathbf{M} - \lambda\mathbf{I}) &= \begin{vmatrix} 4-\lambda & -5 \\ 6 & -9-\lambda \end{vmatrix} \\ &= (4-\lambda)(-9-\lambda) + 30 \\ &= \lambda^2 + 5\lambda - 36 + 30 \\ &= \lambda^2 + 5\lambda - 6 \\ &= (\lambda+6)(\lambda-1) \end{aligned}$$

So the eigenvalues of $\mathbf{M}$ are 1 and −6.

b To determine the lines which are invariant under the transformation T represented by $\mathbf{M}$, we need to find the eigenvectors (there are two eigenvalues, so we'll have two such lines. The question only asks for one, so you can determine whichever):

$$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$4x - 5y = x \quad \Rightarrow \quad x = \frac{5}{3}y$

So if we let $y = 3$, we see that the eigenvector associated with the eigenvalue 1 is $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and so the line which is invariant under this transformation lies in the direction of this vector. Its equation is $y = \frac{3}{5}x$.

For the other eigenvalue we have:

$$\begin{pmatrix} 4 & -5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -6\begin{pmatrix} x \\ y \end{pmatrix}$$

$4x - 5y = -6x \quad \Rightarrow \quad x = \frac{1}{2}y$

So if we let $y = 2$, we see that the eigenvector associated with the eigenvalue −6 is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and so the line which is invariant under this transformation lies in the direction of this vector. Its equation is $y = 2x$.

25 a We have $\mathbf{A}=\begin{pmatrix}-4 & 2\\ 6 & -1\end{pmatrix}$ and we want to find the image of $y=2x+1$ under the transformation T represented by this matrix. This line lies in the direction of $\mathbf{v}=\begin{pmatrix}x\\ 2x+1\end{pmatrix}$ for $x\in\mathbb{R}$. To determine the image, we calculate $\mathbf{Av}$:

$$\mathbf{Av}=\begin{pmatrix}-4 & 2\\ 6 & -1\end{pmatrix}\begin{pmatrix}x\\ 2x+1\end{pmatrix}$$
$$=\begin{pmatrix}-4x+4x+2\\ 6x-2x-1\end{pmatrix}$$
$$=\begin{pmatrix}2\\ 4x-1\end{pmatrix}$$

Thus we see that for any x, T transforms it to the value 2, while y will take all possible real values. So the image of $y=2x+1$ under T is the line $x=2$.

b Now let $\mathbf{A}=\begin{pmatrix}-2 & 2\\ 6 & -1\end{pmatrix}$. To find the eigenvalues of **A**, we need to find the roots of the determinant of $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}-2-\lambda & 2\\ 6 & -9-\lambda\end{pmatrix}$:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}-2-\lambda & 2\\ 6 & -1-\lambda\end{vmatrix}$$
$$=(-2-\lambda)(-1-\lambda)-12$$
$$=\lambda^2+3\lambda-10$$
$$=(\lambda+5)(\lambda-2)$$

So the eigenvalues of **A** are −5 and 2.

c To find the Cartesian equations of the lines which are invariant under T, we need to find the eigenvector associated with each of the eigenvalues:

$$\begin{pmatrix}-2 & 2\\ 6 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-5x\\ -5y\end{pmatrix}$$
$$-2x+2y=-5x \quad\Rightarrow\quad x=-\tfrac{2}{3}y$$

So if we let $y=3$, we get the eigenvector $\begin{pmatrix}2\\ 3\end{pmatrix}$. So the line lying in the direction of this vector is invariant under T. Its equation is $y=-\tfrac{3}{2}x$. For the other eigenvalue we write:

$$\begin{pmatrix}-2 & 2\\ 6 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2x\\ 2y\end{pmatrix}$$
$$-2x+2y=2x \quad\Rightarrow\quad x=\tfrac{1}{2}y$$

So if we let $y=2$, we get the eigenvector $\begin{pmatrix}1\\ 2\end{pmatrix}$. We know that the line lying in the direction of this vector is invariant under T. Its equation is $y=2x$.

26 a We want to find the eigenvalues of $\mathbf{M}=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}$. To that end, we need to find the roots of the determinant of $\mathbf{M}-\lambda\mathbf{I}=\begin{pmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{pmatrix}$:

$$\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}4-\lambda & -2\\ 1 & 1-\lambda\end{vmatrix}$$
$$=(4-\lambda)(1-\lambda)+2$$
$$=\lambda^2-5\lambda+6$$
$$=(\lambda-3)(\lambda-2)$$

We know that $\lambda_1<\lambda_2$, so the eigenvalues are $\lambda_1=2$ and $\lambda_2=3$.

b We have $\det\mathbf{M}=4+2=6$, so $\mathbf{M}^{-1}=\frac{1}{6}\begin{pmatrix}1 & 2\\ -1 & 4\end{pmatrix}$.

c To find the eigenvalues of $\mathbf{M}^{-1}$, we need to find the roots of the determinant of $\mathbf{M}^{-1}-\lambda\mathbf{I}=\begin{pmatrix}\frac{1}{6}-\lambda & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\lambda\end{pmatrix}$:

$$\det(\mathbf{M}^{-1}-\lambda\mathbf{I})=\begin{vmatrix}\frac{1}{6}-\lambda & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}-\lambda\end{vmatrix}$$
$$=\left(\tfrac{1}{6}-\lambda\right)\left(\tfrac{2}{3}-\lambda\right)+\tfrac{1}{18}$$
$$=\lambda^2-\tfrac{5}{6}\lambda+\tfrac{1}{6}$$
$$=\left(\lambda-\tfrac{1}{3}\right)\left(\lambda-\tfrac{1}{2}\right)$$

So the eigenvalues of $\mathbf{M}^{-1}$ are $\frac{1}{2}=\lambda_1^{-1}$ and $\frac{1}{3}=\lambda_2^{-1}$ as required.

d To find the Cartesian equations of the lines which are invariant under T, we need to find the eigenvector associated with each of the eigenvalues of **M**:

$$\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}2x\\ 2y\end{pmatrix}$$
$$4x-2y=2x \quad\Rightarrow\quad x=y$$

So letting $y=1$ we get the eigenvector $\begin{pmatrix}1\\ 1\end{pmatrix}$. Thus we know that the line lying in the direction of this vector is invariant under T. Its equation is $y=x$. For the other eigenvalue we have:

$$\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}3x\\ 3y\end{pmatrix}$$
$$4x-2y=3x \quad\Rightarrow\quad x=2y$$

So letting $y=1$ we get the eigenvector $\begin{pmatrix}2\\ 1\end{pmatrix}$. Thus we know that the line lying in the direction of this vector is invariant under T. Its equation is $y=\frac{1}{2}x$.

27 a We have $\mathbf{P} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 4 & 2 \\ -6 & -3 \end{pmatrix}$. To find their eigenvalues, we need to look for the roots of their characteristic equations:

$$\det(\mathbf{P} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 1 \\ 6 & 2-\lambda \end{vmatrix}$$
$$= (3-\lambda)(2-\lambda) - 6$$
$$= \lambda^2 - 5\lambda$$
$$= \lambda(\lambda - 5)$$

So the eigenvalues of **P** are 0 and 5. Solving $\mathbf{P}u = \lambda u$ for the two eigenvalues we obtain the respective eigenvectors $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\det(\mathbf{Q} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 2 \\ -6 & -3-\lambda \end{vmatrix}$$
$$= (4-\lambda)(-3-\lambda) + 12$$
$$= \lambda^2 - \lambda$$
$$= \lambda(\lambda - 1)$$

So the eigenvalues of **Q** are 0 and 1. Solving $\mathbf{Q}u = \lambda u$ for the two eigenvalues we obtain the respective eigenvectors $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

b The eigenvalue which was common to both **P** and **Q** was 0. So we know that **R** also has an eigenvalue 0. We now want to show that **R** is singular, i.e. $\det \mathbf{R} = 0$. **R** has an eigenvalue 0, which means that $\det(\mathbf{R} - 0 \cdot \mathbf{I}) = 0$ but simplifying the bracket gives $\det(\mathbf{R}) = 0$ which means that **R** is singular as required.

28 $\mathbf{C} = \begin{pmatrix} 2 & 3 \\ k & -2 \end{pmatrix}$ has complex eigenvalues. We want to find k. Begin by finding the characteristic polynomial of **C**:

$$\det(\mathbf{C} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 3 \\ k & -2-\lambda \end{vmatrix}$$
$$= (2-\lambda)(-2-\lambda) - 3k$$
$$= \lambda^2 - 4 - 3k$$

For **C** to have complex eigenvalues we need

$-4 - 3k > 0$

$3k < -4$

$k < -\frac{4}{3}$

29 a $\mathbf{M}=\begin{pmatrix} p & 3 \\ -3 & 1 \end{pmatrix},\ p\in\mathbb{Z}^+$

We know that **M** has repeated eigenvalues and thus we want to find p. Begin by finding the characteristic equation of **M**:

$$\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix} p-\lambda & 3 \\ -3 & 1-\lambda \end{vmatrix}$$
$$=(p-\lambda)(1-\lambda)+9$$
$$=\lambda^2-\lambda(p+1)+p+9$$

For **M** to have repeated eigenvalues, we need $\lambda^2-\lambda(p+1)+p+9=(\lambda-a)^2$ for some a. Thus:

$\lambda^2-\lambda(p+1)+p+9=\lambda^2-2a\lambda+a^2$. Comparing the appropriate coefficients gives:

$$p+1=2a \quad\Rightarrow\quad a=\frac{p+1}{2}$$
$$p+9=a^2$$
$$p+9=\left(\frac{p+1}{2}\right)^2$$
$$4(p+9)=p^2+2p+1$$
$$p^2-2p-35=0$$
$$(p-7)(p+5)=0$$

Since $p\in\mathbb{Z}^+$, $p=7$.

b To find the line which is invariant under T, we need to first find the eigenvectors of **M**. The eigenvalues of **M** can be found using part **a**, where we defined them using the constant a:

$$a=\frac{p+1}{2}=4$$

So we can write:

$$\begin{pmatrix} 7 & 3 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=4\begin{pmatrix} x \\ y \end{pmatrix}$$
$$7x+3y=4x \quad\Rightarrow\quad x=-y$$

So letting $y=1$ gives the eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thus the line lying in the direction of this vector is invariant under T. Its equation is $y=-x$.

30 a **A** has a real eigenvalue λ with the corresponding eigenvector **v**. We want to show that $\mathbf{A}^3$ has an eigenvalue λ^3 with the corresponding eigenvector **v**. We have $\mathbf{Av}=\lambda\mathbf{v}$. So:

$$\mathbf{A}^2\mathbf{v}=\mathbf{A}(\mathbf{Av})=\mathbf{A}\cdot\lambda\mathbf{v}=\lambda(\mathbf{Av})=\lambda^2\mathbf{v} \text{ and}$$
$$\mathbf{A}^3\mathbf{v}=\mathbf{A}(\mathbf{A}^2\mathbf{v})=\mathbf{A}(\lambda^2\mathbf{v})=\lambda^2(\mathbf{Av})=\lambda^3\mathbf{v}$$

Which shows that λ^3 is an eigenvalue of $\mathbf{A}^3$ with the corresponding eigenvector **v**.

30 b We can repeat the above reasoning for any of the eigenvalues of **A**. Which means that all of the eigenvalues of $\mathbf{A}^4$ are of the form λ^4, where λ is an eigenvalue of **A**. Since λ^4 is assumed to be real, we can write $\lambda^4 = \left(\lambda^2\right)^2 \geqslant 0$.

31 We know that $\begin{pmatrix}1\\3\end{pmatrix}$ is mapped onto $\begin{pmatrix}-3\\-9\end{pmatrix} = -3\begin{pmatrix}1\\3\end{pmatrix}$. So -3 is an eigenvalue of **M** and $\begin{pmatrix}1\\3\end{pmatrix}$ is the corresponding eigenvector. We also know that $\begin{pmatrix}6\\4\end{pmatrix}$ is mapped onto $\begin{pmatrix}3\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}6\\4\end{pmatrix}$, so $\frac{1}{2}$ is the other eigenvalue of **M** with the corresponding eigenvector $\begin{pmatrix}6\\4\end{pmatrix}$.

32 a We want to show that 5 is an eigenvalue of **A** and find the corresponding eigenvector. Considering $\begin{pmatrix}6 & -1\\2 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 5\begin{pmatrix}x\\y\end{pmatrix}$ gives:

$6x - y = 5x$

$x = y$

Letting $y = 1$ gives the eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$. So 5 is indeed an eigenvalue with corresponding eigenvector $\begin{pmatrix}1\\1\end{pmatrix}$.

b We know that the other eigenvalue of **A** is 4. Find the associated eigenvector:

$\begin{pmatrix}6 & -1\\2 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 4\begin{pmatrix}x\\y\end{pmatrix}$

$6x - y = 4x \quad\Rightarrow\quad 2x = y$

Letting $y = 2$ gives the eigenvector $\begin{pmatrix}1\\2\end{pmatrix}$. To find **P** we need to normalise the eigenvectors, so:

Thus $\mathbf{P} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix}5 & 0\\0 & 4\end{pmatrix}$.

33 a i To find the two lines which are invariant under T, we first need to find the eigenvalues and eigenvectors of **M**. Begin by finding the roots of the characteristic equation:

$$\det(\mathbf{M}-\lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & 6 \\ 6 & -1-\lambda \end{vmatrix}$$
$$= (4-\lambda)(-1-\lambda) - 36$$
$$= \lambda^2 - 3\lambda - 40$$
$$= (\lambda-8)(\lambda+5)$$

So the eigenvalues of **M** are 8 and −5. Now find the associated eigenvectors:

$$\begin{pmatrix} 4 & 6 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -5\begin{pmatrix} x \\ y \end{pmatrix}$$
$$4x + 6y = -5x$$
$$x = -\tfrac{2}{3}y$$

Letting $y = 3$, we get the eigenvector $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$. For the other eigenvalue we have:

$$\begin{pmatrix} 4 & 6 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 8\begin{pmatrix} x \\ y \end{pmatrix}$$
$$4x + 6y = 8x$$
$$x = \tfrac{3}{2}y$$

Letting $y = 2$, we get the eigenvector $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$. Now, we know that the invariant lines lie in the directions of the eigenvectors respectively. So to show that they are normal to each other, it is enough to show that the eigenvectors are normal to each other. Check the dot product:

$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} = -6 + 6 = 0$. So the lines are normal to each other as required.

ii **M** is symmetric.

b Since we already found the eigenvectors and the corresponding eigenvalues in part **a**, we can readily find **D**:

$$\mathbf{D} = \begin{pmatrix} 8 & 0 \\ 0 & -5 \end{pmatrix}$$

To find **P**, we need to normalise the eigenvectors:

$\left|\begin{pmatrix} -2 \\ 3 \end{pmatrix}\right| = \sqrt{13}$, so the normalised eigenvector is $\begin{pmatrix} \frac{-2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}$

$\left|\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right| = \sqrt{13}$, so the normalised eigenvector is $\begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix}$

Thus $\mathbf{P} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$

34 a $\mathbf{A}=\begin{pmatrix}3 & 2\\ -1 & 4\end{pmatrix}$, we want to find the characteristic equation of **A**:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & 2\\ -1 & 4-\lambda\end{vmatrix}$$
$$=(3-\lambda)(4-\lambda)+2$$
$$=\lambda^2-7\lambda+14=0$$

b We know that matrices satisfy their own characteristic equations, hence we know that

$\mathbf{A}^2-7\mathbf{A}+14\mathbf{I}=0$

$\mathbf{A}^2=7\mathbf{A}-14\mathbf{I} \quad\Rightarrow\quad \mathbf{A}^3=7\mathbf{A}^2-14\mathbf{A}$

In the last equation, we substitute the expression for $\mathbf{A}^2$:

$$\mathbf{A}^3=7(7\mathbf{A}-14\mathbf{I})-14\mathbf{A}$$
$$=49\mathbf{A}-98\mathbf{I}-14\mathbf{A}$$
$$=35\mathbf{A}-98\mathbf{I}$$

As required.

35 We want to find the eigenvalues and eigenvectors of $\begin{pmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{pmatrix}$. Begin by finding the characteristic equation:

$$\det\begin{pmatrix}2-\lambda & -3 & 1\\ 3 & 1-\lambda & 3\\ -5 & 2 & -4-\lambda\end{pmatrix}=(2-\lambda)(1-\lambda)(-4-\lambda)+45+6-(-5(1-\lambda)-9(-4-\lambda)+6(2-\lambda))$$
$$=-(\lambda^2-3\lambda+2)(4+\lambda)+51+5-5\lambda-36-9\lambda-12+6\lambda$$
$$=-\lambda^3-\lambda^2+10\lambda-8-8\lambda+8$$
$$=-\lambda^3-\lambda^2+2\lambda$$
$$=-\lambda(\lambda^2+\lambda-2)$$
$$=-\lambda(\lambda+2)(\lambda-1)$$

So the eigenvalues are 0, −2 and 1. The associated eigenvectors:

$$\begin{pmatrix}2 & -3 & 1\\ 3 & 1 & 3\\ -5 & 2 & -4\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0$$

$2x-3y+z=0 \quad\Rightarrow\quad z=3y-2x$

$3x+y+3z=0$

$-5x+2y-4z=0$

Substitute the value for z:

$3x+y+3(3y-2x)=0$

$-5x+2y-4(3y-2x)=0$

$-3x+10y=0$

$3x-10y=0$

35 (continued)

These two equations are equivalent, so we just have $x = \frac{10}{3}y$. Letting $y = 3$ gives eigenvector $\begin{pmatrix} 10 \\ 3 \\ -11 \end{pmatrix}$

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$2x - 3y + z = -2x \quad \Rightarrow \quad z = 3y - 4x$

$3x + y + 3z = -2y$

$-5x + 2y - 4z = -2z$

Substitute the value for z:

$3x + y + 3(3y - 4x) = -2y$

$-5x + 2y - 4(3y - 4x) = -2(3y - 4x)$

$-9x + 12y = 0$

$3x - 4y = 0$

Again, these two equations are equivalent, so we have $x = \frac{4}{3}y$. Letting $y = 3$ gives $\begin{pmatrix} 4 \\ 3 \\ -7 \end{pmatrix}$.

Finally:

$$\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$2x - 3y + z = x \quad \Rightarrow \quad z = 3y - x$

$3x + y + 3z = y$

$-5x + 2y - 4z = z$

Substitute the value for z:

$3x + y + 3(3y - x) = y$

$-5x + 2y - 4(3y - x) = 3y - x$

$9y = 0$

$-13y = 0$

So $y = 0$ and $z = -x$, so letting $x = 1$ we get $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

36 a We want to show that the only real eigenvalue of $\mathbf{X}=\begin{pmatrix}4&3&0\\0&1&4\\2&1&5\end{pmatrix}$ is 7. Begin by factorising the characteristic polynomial:

$$\det(\mathbf{X}-\lambda)=\begin{vmatrix}4-\lambda&3&0\\0&1-\lambda&4\\2&1&5-\lambda\end{vmatrix}$$
$$=(4-\lambda)(1-\lambda)(5-\lambda)+24-4(4-\lambda)$$
$$=-\lambda^3+10\lambda^2-25\lambda+28$$
$$=-(\lambda-7)(\lambda^2-3\lambda+4)$$

Now, consider the quadratic above. We have $\Delta=9-16=-7<0$, so there are no more real roots. Thus we've shown the **X** has only one real eigenvalue: 7.

b Consider $\begin{pmatrix}4&3&0\\0&1&4\\2&1&5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=7\begin{pmatrix}x\\y\\z\end{pmatrix}$. Equating the corresponding coefficients we get:

$4x+3y=7x \quad\Rightarrow\quad x=y$

$y+4z=7y \quad\Rightarrow\quad y=\tfrac{2}{3}z$

$2x+y+5z=7z \quad\Rightarrow\quad y=\tfrac{2}{3}z$

The second and third equations are equivalent, so let $z=3$.

Then $y=x=2$ and the eigenvector is $\begin{pmatrix}2\\2\\3\end{pmatrix}$

c We know that complex eigenvalues come in conjugate pairs. A 3×3 matrix has 3 eigenvalues. So if two of them are complex conjugates, the third one has to be real.

37 a i We know that $\begin{pmatrix}0\\2\\-1\end{pmatrix}$ is an eigenvector of $\mathbf{A}=\begin{pmatrix}3&2&4\\2&0&2\\4&2&k\end{pmatrix}$. To find the corresponding eigenvalue, consider $\begin{pmatrix}3&2&4\\2&0&2\\4&2&k\end{pmatrix}\begin{pmatrix}0\\2\\-1\end{pmatrix}=\lambda\begin{pmatrix}0\\2\\-1\end{pmatrix}$. Equating the appropriate y-coefficients we get $-2=2\lambda$, so the corresponding eigenvalue is $\lambda=-1$.

ii Comparing the z-coefficients in the above equation, we get

$4-k=-\lambda=1$

$k=3$

37 b i To find the remaining eigenvalue, consider the characteristic equation:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}$$

$$= -\lambda(3-\lambda)^2 + 32 - (-16\lambda + 8(3-\lambda))$$

$$= -\lambda^3 + 6\lambda^2 + 15\lambda + 8$$

$$= -(\lambda-8)(\lambda+1)^2 = 0$$

So the repeated eigenvalue is –1.

ii Consider $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Equating the corresponding coefficients we get:

$3x + 2y + 4z = -x$

$2x + 2z = -y \quad \Rightarrow \quad y = -2x - 2z$

$4x + 2y + 3z = -z$

Substituting the expression for y into any of the remaining equations gives the identity 0 = 0.

We're looking for an eigenvector, which is linearly independent of $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$, so let (for example)

$y = 0$ and $x = -1$, then we get $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For eigenvalue $\lambda = 8$, consider $\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 8\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Equating the appropriate coefficients we get:

$3x + 2y + 4z = 8x$

$2x + 2z = 8y \quad \Rightarrow \quad y = \frac{1}{4}x + \frac{1}{4}z$

$4x + 2y + 3z = 8z$

Substituting the value for y into the top equation, we get:

$3x + \frac{1}{2}x + \frac{1}{2}z + 4z = 8x$

$9x = 9z$

$x = z$

Letting $x = 2$ we have the eigenvector $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, which is linearly independent of $\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

38 a We want to find the eigenvalues of $\mathbf{M}=\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}$. Consider the characteristic equation:

$$\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda&0&1\\0&2-\lambda&0\\4&3&1-\lambda\end{vmatrix}$$
$$=(1-\lambda)^2(2-\lambda)-4(2-\lambda)$$
$$=-\lambda^3+4\lambda^2-\lambda-6$$
$$=-(\lambda+1)(\lambda-2)(\lambda-3)$$

So the eigenvalues are –1, 2 and 3. To find the corresponding eigenvectors, consider:

$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-\begin{pmatrix}x\\y\\z\end{pmatrix}$. Comparing the appropriate coefficients we get:

$x+z=-x \quad\Rightarrow\quad z=-2x$

$2y=-y \quad\Rightarrow\quad y=0$

$4x+3y+z=-z$

Letting $x=1$ we find the eigenvector $\begin{pmatrix}1\\0\\-2\end{pmatrix}$. For $\lambda=2$ consider:

$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2\begin{pmatrix}x\\y\\z\end{pmatrix}$. Comparing the appropriate coefficients we get:

$x+z=2x \quad\Rightarrow\quad z=x$

$2y=2y$

$4x+3y+z=2z$

Substituting the value for z into the last equation we have:

$4x+3y+x=2x$

$3x=-3y \quad\Rightarrow\quad x=-y$

So letting $x=1$ we find the eigenvector $\begin{pmatrix}1\\-1\\1\end{pmatrix}$. For $\lambda=3$ consider:

$\begin{pmatrix}1&0&1\\0&2&0\\4&3&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}$. Comparing the appropriate coefficients we get:

$x+z=3x \quad\Rightarrow\quad z=2x$

$2y=3y \quad\Rightarrow\quad y=0$

$4x+3y+z=3z$

Letting $x=1$ we find the eigenvector $\begin{pmatrix}1\\0\\2\end{pmatrix}$.

38 b We want to find the image of $\frac{x}{2} = y = -z$ under T. This line lies along $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, so we begin by finding the image of this vector under T:

$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 4 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 10 \end{pmatrix}$ which gives the line $x = \frac{1}{2}y = \frac{1}{10}z$

39 a We want to find a *symmetric* matrix **S** whose eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = 8$ and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_3 = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$ respectively. From here, we have $\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix}$. Now we need to normalise the eigenvectors to get the **P** matrix:

$|\mathbf{v}_1| = \sqrt{1+4+9} = \sqrt{14}$, so $\hat{\mathbf{v}}_1 = \begin{pmatrix} \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \end{pmatrix}$. $|\mathbf{v}_2| = \sqrt{1+4+1} = \sqrt{6}$, so $\hat{\mathbf{v}}_2 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

Lastly, $|\mathbf{v}_3| = \sqrt{16+1+4} = \sqrt{21}$, so $\hat{\mathbf{v}}_3 = \begin{pmatrix} \frac{4}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ \frac{-2}{\sqrt{21}} \end{pmatrix}$. Thus $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{21}} \end{pmatrix}$.

39 b Using $\mathbf{P}^{\mathrm{T}}\mathbf{SP} = \mathbf{D}$ and $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$ we have $\mathbf{S} = \mathbf{PDP}^{\mathrm{T}}$. Thus:

$$\mathbf{S} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{4}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{6}} & \frac{32}{\sqrt{21}} \\ \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{8}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & \frac{-1}{\sqrt{6}} & \frac{-16}{\sqrt{21}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{21}} & \frac{1}{\sqrt{21}} & \frac{-2}{\sqrt{21}} \end{pmatrix}$$

$$\mathbf{S} = \begin{pmatrix} \frac{1}{14}-\frac{1}{6}+\frac{128}{21} & \frac{2}{14}+\frac{2}{6}+\frac{32}{21} & \frac{3}{14}-\frac{1}{6}-\frac{64}{21} \\ \frac{2}{14}+\frac{2}{6}+\frac{32}{21} & \frac{4}{14}-\frac{4}{6}+\frac{8}{21} & \frac{6}{14}+\frac{2}{6}-\frac{16}{21} \\ \frac{3}{14}-\frac{1}{6}-\frac{64}{21} & \frac{6}{14}+\frac{2}{6}-\frac{16}{21} & \frac{9}{14}-\frac{1}{6}+\frac{32}{21} \end{pmatrix}$$

(Remember that **S** is symmetric, so you don't have to calculate all the elements)

$$\mathbf{S} = \begin{pmatrix} 6 & 2 & -3 \\ 2 & 0 & 0 \\ -3 & 0 & 2 \end{pmatrix}$$

40 a We know that a non-symmetric matrix **A** can be written as $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, where $\mathbf{P}^{-1} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ -2 & 1 & -2 \end{pmatrix}$

and $\mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{pmatrix}$. From here we see that the eigenvalues are 3, 0 and –4. Next, using your

calculator find $\mathbf{P} = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, so the respective eigenvectors are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

40 b We know that $\mathbf{A}=\mathbf{PDP}^{-1}$, so:

$$\mathbf{A}=\begin{pmatrix}-1 & 0 & -1\\ 0 & 2 & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -4\end{pmatrix}\begin{pmatrix}1 & -1 & 2\\ 1 & 0 & 1\\ -2 & 1 & -2\end{pmatrix}$$

$$=\begin{pmatrix}-3 & 0 & 4\\ 0 & 0 & -4\\ 3 & 0 & -4\end{pmatrix}\begin{pmatrix}1 & -1 & 2\\ 1 & 0 & 1\\ -2 & 1 & -2\end{pmatrix}$$

$$=\begin{pmatrix}-11 & 7 & -14\\ 8 & -4 & 8\\ 11 & -7 & 14\end{pmatrix}$$

41 a Cayley–Hamilton theorem states that every square matrix satisfies its own characteristic equation.

For $\mathbf{A}=\begin{pmatrix}1 & 1 & 2\\ 0 & 2 & 1\\ 1 & 0 & 2\end{pmatrix}$, we have:

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda & 1 & 2\\ 0 & 2-\lambda & 1\\ 1 & 0 & 2-\lambda\end{vmatrix}$$

$$=(2-\lambda)^2(1-\lambda)+1-2(2-\lambda)$$

$$=-\lambda^3+\lambda^2+4\lambda^2-4\lambda-4\lambda+4+1-4+2\lambda$$

$$=-\lambda^3+5\lambda^2-6\lambda+1$$

Thus we know that $\lambda^3-5\lambda^2+6\lambda-1=0$ and hence, by the Cayley–Hamilton theorem, we have $\mathbf{A}^3-5\mathbf{A}^2+6\mathbf{A}-\mathbf{I}=0$ as required.

b Rearranging the above equation we get

$$\mathbf{A}^3-5\mathbf{A}^2+6\mathbf{A}=\mathbf{I}$$

$$\mathbf{A}(\mathbf{A}^2-5\mathbf{A}+6\mathbf{I})=\mathbf{I}$$

$$\mathbf{A}(\mathbf{A}-3\mathbf{I})(\mathbf{A}-2\mathbf{I})=\mathbf{I}$$

As required. Multiplying both sides by $\mathbf{A}^{-1}$ from the left, we get $\mathbf{A}^{-1}=(\mathbf{A}-3\mathbf{I})(\mathbf{A}-2\mathbf{I})$.

Hence:

$$\mathbf{A}^{-1}=\begin{pmatrix}-2 & 1 & 2\\ 0 & -1 & 1\\ 1 & 0 & -1\end{pmatrix}\begin{pmatrix}-1 & 1 & 2\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}4 & -2 & -3\\ 1 & 0 & -1\\ -2 & 1 & 2\end{pmatrix}$$

42 a We know that the characteristic polynomial of **M** is $\lambda^3 - 14\lambda - 19 = 0$ and that

$\mathbf{M}^2 = \begin{pmatrix} 18 & -2 & -3 \\ 1 & 4 & 4 \\ -2 & 1 & 6 \end{pmatrix}$. By Cayley–Hamilton Theorem, we know that **M** satisfies its own characteristic equation, i.e. $\mathbf{M}^3 - 14\mathbf{M} - 19\mathbf{I} = 0$. Rearranging this gives:

$\mathbf{M}^3 - 14\mathbf{M} = 19\mathbf{I}$

$\mathbf{M}^2 - 14\mathbf{I} = 19\mathbf{M}^{-1}$

$\mathbf{M}^{-1} = \frac{1}{19}\left(\mathbf{M}^2 - 14\mathbf{I}\right)$

Thus $\mathbf{M}^{-1} = \frac{1}{19}\begin{pmatrix} 4 & -2 & -3 \\ 1 & -10 & 4 \\ -2 & 1 & -8 \end{pmatrix}$

b We know that $\mathbf{M}^{-1}\mathbf{Q}\mathbf{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and so $\mathbf{Q} = \mathbf{M}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}\mathbf{M}^{-1}$. We already found $\mathbf{M}^{-1}$,

hence we can use a calculator to find $\mathbf{M} = \begin{pmatrix} 4 & -1 & -2 \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{pmatrix}$

Thus:

$$\mathbf{Q} = \frac{1}{19}\begin{pmatrix} 4 & -1 & -2 \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 4 & -2 & -3 \\ 1 & -10 & 4 \\ -2 & 1 & -8 \end{pmatrix}$$

$$= \frac{1}{19}\begin{pmatrix} 8 & -5 & 2 \\ 0 & -10 & 1 \\ -2 & 0 & 2 \end{pmatrix}\begin{pmatrix} 4 & -2 & -3 \\ 1 & -10 & 4 \\ -2 & 1 & -8 \end{pmatrix}$$

$$= \frac{1}{19}\begin{pmatrix} 23 & 36 & -60 \\ -12 & 101 & -48 \\ -12 & 6 & -10 \end{pmatrix}$$

$$\mathbf{Q} = \frac{1}{19}\begin{pmatrix} 4 & -1 & -2 \\ 0 & -2 & -1 \\ -1 & 0 & -2 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 4 & -2 & -3 \\ 1 & -10 & 4 \\ -2 & 1 & -8 \end{pmatrix}$$

$$= \frac{1}{19}\begin{pmatrix} 8 & -5 & -6 \\ 0 & -10 & -3 \\ -2 & 0 & -6 \end{pmatrix}\begin{pmatrix} 4 & -2 & -3 \\ 1 & -10 & 4 \\ -2 & 1 & -8 \end{pmatrix}$$

$$= \frac{1}{19}\begin{pmatrix} 39 & 28 & 4 \\ -4 & 97 & -16 \\ 4 & -2 & 54 \end{pmatrix}$$

43 a Let $I_n = \int \sec^n x\,\mathrm{d}x,\ n \geqslant 0,$ we want to show that $(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}, n \geqslant 0.$

Let $u = \sec^{n-2} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = (n-2)\sec^{n-3} x \tan x$ and $v = \tan x$, so:

$$\begin{aligned} I_n &= \int \sec^n x\,\mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x \tan^2 x\,\mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2)\int \sec^n x\,\mathrm{d}x + (n-2)\int \sec^{n-2} x\,\mathrm{d}x \\ &= \tan x \sec^{n-2} x - (n-2)I_n + (n-2)I_{n-2} \end{aligned}$$

Rearranging gives:

$$I_n + (n-2)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$$

$$(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$$

As required.

b We want to find $I_4 = \int \sec^4 x\,\mathrm{d}x$. We have:

$$\begin{aligned} \int \sec^4 x\,\mathrm{d}x &= \tfrac{1}{3}\left(\tan x \sec^2 x + 2\int \sec^2 x\,\mathrm{d}x\right) \\ &= \tfrac{1}{3}\tan x \sec^2 x + \tfrac{2}{3}\tan x + c \end{aligned}$$

44 a Let $I_n = \int_0^{\frac{\pi}{4}} x^n \cos x\,\mathrm{d}x$. We want to show that for $n \geqslant 2$, $I_n = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right)^{n-1}\left(\frac{\pi}{4} + n\right) - n(n-1)I_{n-2}$.

Let $u = x^n$ and $\frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$ and $v = \sin x$, so:

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{4}} x^n \cos x\mathrm{d}x \\ &= \left[x^n \sin x\right]_0^{\frac{\pi}{4}} - n\int_0^{\frac{\pi}{4}} x^{n-1} \sin x\mathrm{d}x \\ &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^n - n\int_0^{\frac{\pi}{4}} x^{n-1} \sin x\mathrm{d}x \end{aligned}$$

Now, let $u = x^{n-1}$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sin x$. Then $\frac{\mathrm{d}u}{\mathrm{d}x} = (n-1)x^{n-2}$ and $v = -\cos x$, so:

$$\begin{aligned} \int_0^{\frac{\pi}{4}} x^{n-1} \sin x\mathrm{d}x &= -\left[x^{n-1}\cos x\right]_0^{\frac{\pi}{4}} + (n-1)\int_0^{\frac{\pi}{4}} x^{n-2}\cos x\,\mathrm{d}x \\ &= -\tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^{n-1} + (n-1)I_{n-2} \end{aligned}$$

Substituting this into the expression for I_n we get:

$$\begin{aligned} I_n &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^n - n\int_0^{\frac{\pi}{4}} x^{n-1} \sin x\mathrm{d}x \\ &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^n - n\left(-\tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^{n-1} + (n-1)I_{n-2}\right) \\ &= \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^{n-1}\left(\tfrac{\pi}{4} + n\right) - n(n-1)I_{n-2} \end{aligned}$$

As required.

44 b We want to find the value of $I_4 = \int_0^{\frac{\pi}{4}} x^4 \cos x \, dx$. Use part **a**:

$$I_n = \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^3\left(\tfrac{\pi}{4}+4\right) - 12I_2$$
$$= \tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)^3\left(\tfrac{\pi}{4}+4\right) - 12\left(\tfrac{1}{\sqrt{2}}\left(\tfrac{\pi}{4}\right)\left(\tfrac{\pi}{4}+2\right)\right)$$
$$\approx 0.0471$$

45 a Let $I_n = \int_0^a x^n (a-x)^{\frac{1}{3}} dx$, $n \geq 0$, $a > 0$. We want to show that $I_n = \dfrac{3an}{3n+4} I_{n-1}, n \geqslant 1$. Let $u = x^n$ and $\dfrac{dv}{dx} = (a-x)^{\frac{1}{3}}$. Then $\dfrac{du}{dx} = nx^{n-1}$ and $v = -\tfrac{3}{4}(a-x)^{\frac{4}{3}}$. So:

$$I_n = \int_0^a x^n (a-x)^{\frac{1}{3}} dx$$
$$= \left[-\tfrac{3}{4}x^n (a-x)^{\frac{4}{3}}\right]_0^a + \tfrac{3}{4}n\int_0^a x^{n-1}(a-x)^{\frac{4}{3}} dx$$
$$= \tfrac{3}{4}n\int_0^a (a-x)x^{n-1}(a-x)^{\frac{1}{3}} dx$$
$$= \tfrac{3}{4}an\int_0^a x^{n-1}(a-x)^{\frac{1}{3}} dx - \tfrac{3}{4}n\int_0^a x^n (a-x)^{\frac{1}{3}} dx$$
$$= \frac{3an}{4} I_{n-1} - \frac{3n}{4} I_n$$

Rearranging gives:

$$I_n + \frac{3n}{4} I_n = \frac{3an}{4} I_{n-1}$$
$$\frac{3n+4}{4} I_n = \frac{3an}{4} I_{n-1}$$
$$I_n = \frac{3an}{3n+4} I_{n-1}$$

As required.

b We know that $I_2 = \tfrac{27}{49}a^{\frac{4}{3}}$ and we want to find the value of a. Using part **a** we have that:

$$I_2 = \frac{6a}{10} I_1 = \frac{3a}{5}\cdot\frac{3a}{7} I_0$$
$$= \frac{9a^2}{35}\int_0^a (a-x)^{\frac{1}{3}} dx$$
$$= -\frac{27a^2}{140}\left[(a-x)^{\frac{4}{3}}\right]_0^a$$
$$= \tfrac{27}{140}a^2 \cdot a^{\frac{4}{3}} = \tfrac{27}{140}a^{\frac{10}{3}}$$

So we need:

$$\tfrac{27}{140}a^{\frac{10}{3}} = \tfrac{27}{49}a^{\frac{4}{3}}$$
$$49a^2 = 140$$
$$a^2 = \tfrac{140}{49}$$

Since $a > 0$, it must be $a = \tfrac{2}{7}\sqrt{35}$

46 a We want to find the length of an arc defined by $y=\left(ax^3\right)^{\frac{1}{2}}$ between $x=0$ and $x=4$.

We have $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}ax^2\left(ax^3\right)^{-\frac{1}{2}}=\frac{3\left(ax^3\right)^{\frac{1}{2}}}{2x}$. So:

$$s=\int_0^4\sqrt{1+\left(\frac{3\left(ax^3\right)^{\frac{1}{2}}}{2x}\right)^2}\,\mathrm{d}x$$

$$=\int_0^4\sqrt{1+\frac{9ax^3}{4x^2}}\,\mathrm{d}x$$

$$=\int_0^4\sqrt{1+\tfrac{9}{4}ax}\,\mathrm{d}x$$

$$=\int_0^4\left(1+\tfrac{9}{4}ax\right)^{\frac{1}{2}}\mathrm{d}x$$

$$=\left[\frac{\left(9ax+4\right)^{\frac{3}{2}}}{27a}\right]_0^4$$

$$=\frac{\left(36a+4\right)^{\frac{2}{3}}}{27a}-\frac{4^{\frac{2}{3}}}{27a}$$

$$=\frac{8}{27a}\left(\left(9a+1\right)^{\frac{3}{2}}-1\right)$$

b We know that $s=16$ and we want to find the value of a:

$$\frac{8}{27a}\left(\left(9a+1\right)^{\frac{3}{2}}-1\right)=16$$

$$\left(9a+1\right)^{\frac{3}{2}}-1=54a$$

$$\left(9a+1\right)^3=\left(54a+1\right)^2$$

$$729a^3+243a^2+27a+1=2916a^2+108a+1$$

$$729a^3-2673a^2-81a=0$$

$$9a^2-33a-1=0$$

$$a_1=\frac{11-5\sqrt{5}}{6},\ a_2=\frac{11+5\sqrt{5}}{6}$$

But we know that $a>0$, so $a=\frac{11+5\sqrt{5}}{6}\approx 3.6967$

47 a We have $y^2=2x+16$. Arc L is a section of this curve between $y=0$ and $y=3$. We want to show that $L=\int_0^3\sqrt{1+y^2}\,\mathrm{d}y$. We know that $L=\int_0^3\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$. $x=\frac{y^2}{2}-8$, so $\frac{\mathrm{d}x}{\mathrm{d}y}=y$.

Thus $L=\int_0^3\sqrt{1+y^2}\,\mathrm{d}y$ as required.

47 b We now need to calculate L:

$L=\int_0^3 \sqrt{1+y^2}\,\mathrm{d}y$

Let $y=\tan u$. Then $\frac{\mathrm{d}y}{\mathrm{d}u}=\sec^2 u$ and $\sqrt{1+y^2}=\sqrt{1+\tan^2 u}=\sec u$. So we can write

$L=\int_0^{\arctan(3)} \sec^3 u\mathrm{d}u$. Using the reduction formula, $\int \sec^n u\mathrm{d}u=\frac{\tan u\sec^{n-2}u}{n-1}+\frac{n-2}{n-1}\int \sec^{n-2}u\,\mathrm{d}u$

(see question **43**).

$$
\begin{aligned}
L&=\int_0^{\arctan(3)} \sec^3 u\mathrm{d}u\\
&=\left[\frac{\tan u\sec u}{2}\right]_0^{\arctan(3)}+\tfrac{1}{2}\int_0^{\arctan(3)}\sec u\mathrm{d}u\\
&=\tfrac{3\sqrt{10}}{2}+\tfrac{1}{2}\Big[\ln\left(\tan u+\sec u\right)\Big]_0^{\arctan(3)}\\
&=\tfrac{3\sqrt{10}}{2}+\tfrac{1}{2}\ln\left(3+\sqrt{10}\right)-\tfrac{1}{2}\ln 1\\
&=\tfrac{3\sqrt{10}}{2}+\tfrac{1}{2}\ln\left(3+\sqrt{10}\right)
\end{aligned}
$$

48 We have a curve given parametrically $x=t^2-1$ and $y=\frac{1}{3}t^3-2$. We want to calculate the length of an arc for $0\leqslant t\leqslant 2$. We have $s=\int_0^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$. Here, $\frac{\mathrm{d}x}{\mathrm{d}t}=2t$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=t^2$. Thus:

$$
\begin{aligned}
s&=\int_0^2\sqrt{(2t)^2+t^4}\,\mathrm{d}t\\
&=\int_0^2\left(4t^2+t^4\right)^{\frac{1}{2}}\mathrm{d}t\\
&=\int_0^2 t\left(t^2+4\right)^{\frac{1}{2}}\mathrm{d}t
\end{aligned}
$$

Let $u=t^2+4$. Then $\frac{\mathrm{d}u}{\mathrm{d}t}=2t$, so

$$
\begin{aligned}
s&=\tfrac{1}{2}\int_4^8 u^{\frac{1}{2}}\,\mathrm{d}u\\
&=\tfrac{1}{3}\left[u^{\frac{3}{2}}\right]_4^8\\
&=\frac{8^{\frac{3}{2}}-8}{3}
\end{aligned}
$$

As required.

49 a A spiral is given by the polar equation $r=\theta$ for $0 \leqslant \theta \leqslant 4\pi$. We want to use substitution of the form $\theta = f(x)$ to show that the length of this spiral is given by $W = \int_0^{\arctan(4\pi)} \sec^3 x \, dx$. We know that for polar equations the length of the arc is given by $W = \int_0^{4\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$. We have $\frac{dr}{d\theta} = 1$ so $W = \int_0^{4\pi} \sqrt{1+\theta^2} \, d\theta$. Let $\theta = \tan x$ and $\frac{d\theta}{dx} = \sec^2 x$.

Then:

$$W = \int_0^{\arctan 4\pi} \sec^2 x \sqrt{1+\tan^2 x} \, dx$$
$$= \int_0^{\arctan 4\pi} \sec^3 x \, dx$$

as required.

b We want to calculate the value of W. Using the reduction formula $\int \sec^n u du = \frac{\tan u \sec^{n-2} u}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} u \, du$ (see exercise **43**) we obtain:

$$W = \int_0^{\arctan 4\pi} \sec^3 x \, dx$$
$$= \left[\frac{\tan u \sec u}{2}\right]_0^{\arctan 4\pi} + \frac{1}{2} \int_0^{\arctan 4\pi} \sec u \, du$$
$$= \frac{4\pi\sqrt{1+16\pi^2}}{2} + \frac{1}{2}[\ln(\tan u + \sec u)]_0^{\arctan 4\pi} \approx 80.82$$

50 We are given a parametric curve $x = 2\sqrt{t}, y = 1-t, \; t \geqslant 0$. We want to calculate the area of the surface of revolution of the arc $1 \leqslant t \leqslant 4$ around the y-axis. We know that $S = 2\pi \int_{t_A}^{t_B} x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$.

Here, $\frac{dx}{dt} = t^{-\frac{1}{2}}, \; \frac{dy}{dt} = -1$, so:

$$S = 2\pi \int_1^4 2\sqrt{t} \cdot \sqrt{\left(t^{-\frac{1}{2}}\right)^2 + (-1)^2} \, dt$$
$$= 4\pi \int_1^4 \sqrt{1+t} \, dt$$
$$= 4\pi \cdot \tfrac{2}{3}\left[(t+1)^{\frac{3}{2}}\right]_1^4$$
$$= \tfrac{8\pi}{3}\left(5^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$
$$= \frac{8\pi\left(5\sqrt{5} - 2\sqrt{2}\right)}{3}$$

51 a We have $y=\sqrt{a-x^2}, -1\leqslant x\leqslant 1$. The area created by rotating this arc around x-axis is 24π.

We want to find a. We know that $S=2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

Here, $\frac{dy}{dx}=\frac{-x}{\sqrt{a-x^2}}$, so:

$S=2\pi\int_{-1}^{1}\sqrt{a-x^2}\sqrt{1+\frac{x^2}{a-x^2}}dx$

$=2\pi\int_{-1}^{1}\sqrt{a}dx$

$=2\pi\sqrt{a}\left[x\right]_{-1}^{1}$

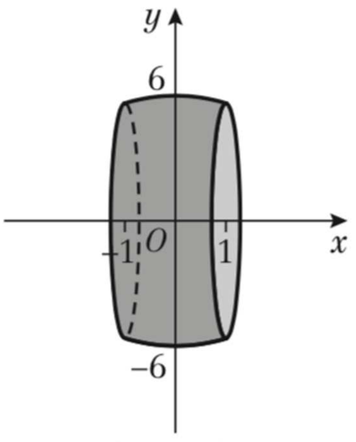
$=4\pi\sqrt{a}$

Now we need $24\pi=4\pi\sqrt{a}$:

$6=\sqrt{a}$

$a=36$

b Begin by sketching the curve $y=\sqrt{a-x^2}, -1\leqslant x\leqslant 1$. Then imagine it rotating around the x-axis:



52 We are given a polar equation $r=\sqrt{\cos 2\theta}$. An arc $0 \leqslant \theta \leqslant \frac{\pi}{4}$ is rotated around the initial line. We want to find the area of the surface created in this way. Recall that when rotation around the initial line $\theta=0$ and we use the formula $S=2\pi\int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2+\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$. Here we have

$\frac{\mathrm{d}r}{\mathrm{d}\theta}=-\frac{\sin 2\theta}{\sqrt{\cos 2\theta}}$, so:

$$S=2\pi\int_0^{\frac{\pi}{4}}\sqrt{\cos 2\theta}\sin\theta\sqrt{\cos 2\theta+\frac{\sin^2 2\theta}{\cos 2\theta}}\mathrm{d}\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sqrt{\cos 2\theta}\sin\theta\sqrt{\frac{\sin^2 2\theta+\cos^2 2\theta}{\cos 2\theta}}\mathrm{d}\theta$$

$$=2\pi\int_0^{\frac{\pi}{4}}\sin\theta\mathrm{d}\theta$$

$$=-2\pi\left[\cos\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\pi\left(1-\tfrac{\sqrt{2}}{2}\right)$$

53 We want to find the surface area when the arc $\mathrm{f}(x)=\mathrm{e}^x,\ 0\leqslant x\leqslant 1$ is rotated around the x-axis. Recall that when rotating this kind of function around the x-axis we use

$S=2\pi\int_{x_A}^{x_B}\mathrm{f}(x)\sqrt{1+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$. Here, $\frac{\mathrm{d}f}{\mathrm{d}x}=\mathrm{e}^x$, so:

$S=2\pi\int_0^1\mathrm{e}^x\sqrt{1+\mathrm{e}^{2x}}\,\mathrm{d}x$. Let $\mathrm{e}^x=u$ and $\mathrm{e}^x\mathrm{d}x=\mathrm{d}u$. Then:

$S=2\pi\int_1^{\mathrm{e}}\sqrt{1+u^2}\,\mathrm{d}u$. Now, let $u=\tan s$ and $\mathrm{d}u=\sec^2 s\mathrm{d}s$. Thus:

$S=2\pi\int_{\arctan 1}^{\arctan \mathrm{e}}\sec^3 s\mathrm{d}s$. We can now use the reduction formula (see exercise **43**)

$$\int\sec^n x\,\mathrm{d}x=\frac{\tan x\sec^{n-2}x}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}x\,\mathrm{d}x:$$

$$S=2\pi\left(\left[\frac{\tan s\sec s}{2}\right]_{\arctan 1}^{\arctan \mathrm{e}}+\tfrac{1}{2}\int_{\arctan 1}^{\arctan \mathrm{e}}\sec s\mathrm{d}s\right)$$

$$=2\pi\left(\frac{\mathrm{e}\sqrt{1+\mathrm{e}^2}-\sqrt{2}}{2}+\tfrac{1}{2}\left[\ln\left(\tan s+\sec s\right)\right]_{\arctan 1}^{\arctan \mathrm{e}}\right)$$

$$=\pi\left(\mathrm{e}\sqrt{1+\mathrm{e}^2}-\sqrt{2}+\ln\left(\mathrm{e}+\sqrt{1+\mathrm{e}^2}\right)-\ln\left(1+\sqrt{2}\right)\right)$$

$$\approx 22.943$$

Challenge

1 a We want use induction to show that $\mathbf{A}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$ where $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ describes correctly the recurrence relation $a_{n+1} = 3a_n - 2a_{n-1}$, $a_0 = 0$, $a_1 = 1$. Begin by checking the basis. According to the recurrence formula, we have $a_2 = 3a_1 - 2a_0 = 3$. Using the matrix form for $n = 1$ gives:

$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}$ as required. Now assume that the matrix formula works for $n = k$.

We want to show that then it also works for $n = k + 1$:

$$\begin{aligned}
\mathbf{A}^{k+1}\begin{pmatrix} a_1 \\ a_0 \end{pmatrix} &= \mathbf{A}\left(\mathbf{A}^k\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}\right) \\
&= \mathbf{A}\begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} \\
&= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k+1} \\ a_k \end{pmatrix} \\
&= \begin{pmatrix} 3a_{k+1} - 2a_k \\ a_{k+1} \end{pmatrix} \\
&= \begin{pmatrix} a_{k+2} \\ a_{k+1} \end{pmatrix}
\end{aligned}$$

This shows that if the formula works for $n = k$, then it also works for $n = k + 1$.

Thus, by mathematical induction, $\mathbf{A}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix}$ for all n.

Challenge

1 b To find **P** and **D** we begin by finding the eigenvalues and eigenvectors of **A**. The characteristic equation is:

$$\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -2 \\ 1 & -\lambda \end{vmatrix}$$
$$= -\lambda(3-\lambda)+2$$
$$= \lambda^2 - 3\lambda + 2$$
$$= (\lambda-2)(\lambda-1)$$

So the eigenvalues of **A** are 2 and 1. To find the eigenvector corresponding to $\lambda = 1$, consider:

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$3x - 2y = x$$
$$x = y$$

Letting $x = 1$ gives the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\lambda = 1$ consider:

$$\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$
$$3x - 2y = 2x$$
$$x = 2y$$

Letting $y = 1$ gives the eigenvector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Thus we have $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$.

Challenge

1 c We have $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$, so $\mathbf{A} = \mathbf{PDP}^{-1}$ and $\mathbf{A}^n = \mathbf{PD}^n\mathbf{P}^{-1}$. We want to find $\mathbf{A}^{100}$. Begin by finding $\mathbf{P}^{-1}$:

$\det \mathbf{P} = \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} = -\frac{1}{\sqrt{10}}$

$\mathbf{P}^{-1} = -\sqrt{10}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix}.$

Thus:

$$\begin{aligned}\mathbf{A}^{100} &= \mathbf{PD}^{100}\mathbf{P}^{-1} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{2^{101}}{\sqrt{5}} \\ \frac{1}{\sqrt{2}} & \frac{2^{100}}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -\sqrt{2} & 2\sqrt{2} \\ \sqrt{5} & -\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} -1+2^{101} & 2-2^{101} \\ -1+2^{100} & 2-2^{100} \end{pmatrix}\end{aligned}$$

Now, to find a_{100}, consider:

$\mathbf{A}^{100}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{101} \\ a_{100} \end{pmatrix}$. We only need to find the bottom coefficient, so $a_{100} = 2^{100} - 1$.

d To solve $a_{n+1} = 3a_n - 2a_{n-1}$, $a_0 = 0$, $a_1 = 1$, let $a_n = k^n$.

Then:

$k^{n+1} = 3k^n - 2k^{n-1}$

$k^2 - 3k + 2 = 0$

$(k-1)(k-2) = 0$

So write $a_n = A + B\cdot 2^n$. Use initial conditions to find A and B:

$0 = A + B \quad \Rightarrow \quad A = -B$

$1 = A + 2B$

$B = 1,\ A = -1$

Thus $a_n = -1 + 2^n$ and $a_{100} = -1 + 2^{100}$, which is the same as the answer to part **c**.

2 a To find the volume we use $V = \int_{x_A}^{x_B} \pi\left(f(x)\right)^2 dx$. Here $f(x) = \frac{1}{x}$, so

$$\begin{aligned}V &= \int_1^{\infty} \frac{\pi}{x^2} dx \\ &= -\left[\frac{\pi}{x}\right]_1^{\infty} \\ &= \pi\end{aligned}$$

Challenge

2 b The surface of revolution can be calculated as $S = 2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$. Here $\frac{dy}{dx} = -\frac{1}{x^2}$, so:

$$S = 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\left(-\frac{1}{x^2}\right)^2}dx$$

$$= 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx$$

As required.

c We want to show that $2\pi\int_1^{a} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx > 2\pi\int_1^{a} \frac{1}{x}dx$ for all $x > 0$. Note that:

$x > 0 \quad \Rightarrow \quad \frac{1}{x^4} > 0$. Thus

$$1+\frac{1}{x^4} > 1$$

$$\sqrt{1+\frac{1}{x^4}} > 1$$

For all $x > 0$. This means that $\frac{1}{x}\sqrt{1+\frac{1}{x^4}} > \frac{1}{x}$ for all $x > 0$. Hence we deduce that

$2\pi\int_1^{a} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx > 2\pi\int_1^{a} \frac{1}{x}dx$ for all $x > 0$.

d We know that the surface area of the Torricelli trumpet can be expressed as $2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx$.

Moreover, from part **c** we know that $2\pi\int_1^{a} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx > 2\pi\int_1^{a} \frac{1}{x}dx$.

Now, $2\pi\int_1^{a} \frac{1}{x}dx = 2\pi[\ln x]_1^a = 2\pi\ln a$. So for all $a > 1,\ x > 0$ we have $2\pi\int_1^{a} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx > 2\pi\ln a$.

As $a \to \infty$, we see that $S = 2\pi\int_1^{\infty} \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \geqslant 2\pi\lim_{a\to\infty}(\ln a)$ which diverges to infinity. Thus the surface area of the Torricelli trumpet is infinite.