Integration techniques Mixed exercise

1 **a** Using integration by parts gives

$$I_n = \int (\ln x)^n \, dx$$
$$= \int 1 \times (\ln x)^n \, dx$$
$$= x(\ln x)^n - \int x \frac{n}{x} (\ln x)^{n-1} \, dx$$
$$= x(\ln x)^n - n\int (\ln x)^{n-1} \, dx$$
$$= x(\ln x)^n - nI_{n-1}$$

b $I_3 = \int_1^2 (\ln x)^3 \, dx$

$$= \left[x(\ln x)^3 \right]_1^2 - 3I_2$$
$$= 2(\ln 2)^3 - 3\left(\left[x(\ln x)^2 \right]_1^2 - 2I_1 \right)$$
$$= 2(\ln 2)^3 - 3\left(2(\ln 2)^2 - 2\left(\left[x(\ln x) \right]_1^2 - I_0 \right) \right)$$
$$= 2(\ln 2)^3 - 3\left(2(\ln 2)^2 - 2\left(2(\ln 2) - I_0 \right) \right)$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6I_0$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6\int_1^2 (\ln x)^0 \, dx$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6\int_1^2 1 \, dx$$
$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6$$

2 We have $y = 3x^{\frac{3}{2}} - 1$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9}{2}\sqrt{x}$.

Now

$$\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \sqrt{1+\left(\frac{9\sqrt{x}}{2}\right)^2}$$
$$= \sqrt{1+\frac{81}{4}x}$$
$$= \frac{1}{2}\sqrt{4+81x}$$

Now we use the equation for arc length

$$s = \int_0^1 \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$
$$= \frac{1}{2}\int_0^1 \sqrt{4+81x}\,\mathrm{d}x$$
$$= \frac{1}{243}\left[(4+81x)^{\frac{3}{2}}\right]_0^1$$
$$= \frac{1}{243}\left(85^{\frac{3}{2}} - 4^{\frac{3}{2}}\right)$$
$$= \frac{1}{243}\left(85^{\frac{3}{2}} - 8\right)$$
$$= \frac{85\sqrt{85} - 8}{243}$$

3 In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{x_A}^{x_B} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

We find $\frac{dy}{dx} = -\sin x$ and substitute into the surface area equation to find

$$S = 2\pi\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+(-\sin x)^2}\,dx$$

$$= 2\pi\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin^2 x}\,dx$$

We now use the substitution

$u = \sin x$,

$\frac{du}{dx} = \cos x$

which makes our integral

$$S = 2\pi\int_0^{\frac{\pi}{2}} \cos x\sqrt{1+\sin^2 x}\,dx$$

$$= 2\pi\int_0^1 \sqrt{1+u^2}\,du$$

We now use a second substitution of

$u = \sinh v$,

$\frac{du}{dv} = \cosh v$

which makes our integral

$$S = 2\pi\int_0^1 \sqrt{1+u^2}\,du$$

$$= 2\pi\int_0^{\sinh^{-1}1} \sqrt{1+\sinh^2 v}\cosh v\,dv$$

$$= 2\pi\int_0^{\sinh^{-1}1} \cosh^2 v\,dv$$

$$= \pi\int_0^{\sinh^{-1}1} (\cosh 2v+1)\,dv$$

$$= \pi\left[\frac{\sinh 2v}{2}+v\right]_0^{\sinh^{-1}1}$$

$$= \pi\left[\sinh v\cosh v+v\right]_0^{\sinh^{-1}1}$$

$$= \pi\left[\sinh v\sqrt{1+\sinh^2 v}+v\right]_0^{\sinh^{-1}1}$$

$$= \pi\left(\sqrt{2}+\sinh^{-1}1\right)$$

$$= \pi\left(\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right)$$

4 $y = 4\cosh\left(\frac{x}{4}\right)$, so $\frac{dy}{dx} = \frac{4}{4}\sinh\left(\frac{x}{4}\right) = \sinh\left(\frac{x}{4}\right)$

$$\text{arc length} = \int_{-20}^{20}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\int_0^{20}\sqrt{1+\sinh^2\left(\frac{x}{4}\right)}\,dx$$

Using the symmetry of the catenary

$$= 2\int_0^{20}\cosh\left(\frac{x}{4}\right)dx$$

$$= 2\left[4\sinh\left(\frac{x}{4}\right)\right]_0^{ka}$$

$$= 8\sinh 5$$

$$= 594 \text{ (3 s.f.)}$$

5 In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{t_A}^{t_B} y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

We find $\frac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 6t$ and substitute into the surface area equation to find

$$S = 2\pi\int_0^2 3t^2\sqrt{\left(3t^2\right)^2 + \left(6t\right)^2}\,\mathrm{d}t$$

$$= 2\pi\int_0^2 3t^2\sqrt{\left(9t^4\right) + \left(36t^2\right)}\,\mathrm{d}t$$

$$= 18\pi\int_0^2 t^3\sqrt{t^2+4}\,\mathrm{d}t$$

Now use the substitution

$u = t^2 + 4,$

$\frac{\mathrm{d}u}{\mathrm{d}t} = 2t$

which makes our integral

$$S = 18\pi\int_0^2 t^3\sqrt{t^2+4}\,\mathrm{d}t$$

$$= 9\pi\int_4^8 (u-4)\sqrt{u}\,\mathrm{d}u$$

$$= 9\pi\int_4^8 \left(u^{\frac{3}{2}} - 4\sqrt{u}\right)\mathrm{d}u$$

$$= 9\pi\left[\frac{2u^{\frac{5}{2}}}{5} - \frac{8u^{\frac{3}{2}}}{3}\right]_4^8$$

$$= 9\pi\left(\left(\frac{2}{5}8^{\frac{5}{2}} - \frac{8}{3}8^{\frac{3}{2}}\right) - \left(\frac{64}{5} - \frac{64}{3}\right)\right)$$

$$= 9\pi\left(\left(\frac{8}{15}8^{\frac{3}{2}}\right) + \left(\frac{128}{15}\right)\right)$$

$$= \frac{384\pi}{5}\left(\sqrt{2}+1\right)$$

6 a i $I_0 = \int_0^{\frac{\pi}{2}} \cos x\,\mathrm{d}x = [\sin x]_0^{\frac{\pi}{2}} = 1$

ii $I_1 = \int_0^{\frac{\pi}{2}} x\cos x\,\mathrm{d}x = [x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\,\mathrm{d}xc$

Using integration by parts

$$= \frac{\pi}{2} + [\cos x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} + [0-1] = \frac{\pi}{2} - 1$$

6 b Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \cos x$

$\frac{du}{dx} = nx^{n-1}, \quad v = \sin x$

So $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, dx = \left[x^n \sin x \right]_0^{\frac{\pi}{2}} - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$

$= \left(\frac{\pi}{2}\right)^n - n\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx \quad *$

Integrating by parts on $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx$ with $u = x^{n-1}$ and $\frac{dv}{dx} = \sin x$

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = -\cos x$

gives $\int_0^{\frac{\pi}{2}} x^{n-1} \sin x \, dx = \left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \cos x \, dx$

$= (n-1)I_{n-2}$ as $\left[-x^{n-1} \cos x \right]_0^{\frac{\pi}{2}} = 0$

Substituting in *

$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

c $\int_0^{\frac{\pi}{2}} x^3 \cos x \, dx = I_3 = \left(\frac{\pi}{2}\right)^3 - 3(2)I_1$

$= \left(\frac{\pi}{2}\right)^3 - 6\left(\frac{\pi}{2} - 1\right)$ Using **a ii**

$= \frac{\pi^3}{8} - 3\pi + 6$

$= \frac{1}{8}\left(\pi^3 - 24\pi + 48\right)$

d $\int_0^{\frac{\pi}{2}} x^4 \cos x \, dx = I_4 = \left(\frac{\pi}{2}\right)^4 - 4(3)I_2$

$= \left(\frac{\pi}{2}\right)^4 - 12\left\{\left(\frac{\pi}{2}\right)^2 - 2(1)I_0\right\}$

$= \frac{\pi^4}{16} - 3\pi^2 + 24$ as $I_0 = 1$ from **a i**

7 Since we have $x = a\left(\cos\theta + \theta\sin\theta\right)$ and $y = r\left(\sin\theta - \theta\cos\theta\right)$, we have $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a\theta\cos\theta$ and $\frac{\mathrm{d}y}{\mathrm{d}\theta} = r\theta\sin\theta$.

We substitute into the equation

$$s = \int_0^{\pi}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

$$= \int_0^{\pi}\sqrt{\left(a\theta\cos\theta\right)^2 + \left(a\theta\sin\theta\right)^2}\,\mathrm{d}\theta$$

$$= \int_0^{\pi} a\theta\mathrm{d}\theta$$

$$= \left[\frac{|a|\theta^2}{2}\right]_0^{\pi}$$

$$= \frac{\pi^2|a|}{2}$$

8 a Using integration by parts on I_n, with $u = x^n$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = (1-x)^{\frac{1}{3}}$

so $\frac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}$ and $v = -\frac{3}{4}(1-x)^{\frac{4}{3}}$

$$I_n = -\frac{3}{4}\left[x^n(1-x)^{\frac{4}{3}}\right]_0^8 + \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{4}{3}}\,\mathrm{d}x$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{4}{3}}\,\mathrm{d}x$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)(1-x)^{\frac{1}{3}}\,\mathrm{d}x$$

$$= \frac{3n}{4}\int_0^8 x^{n-1}(1-x)^{\frac{1}{3}}\,\mathrm{d}x - \frac{3n}{4}\int_0^8 x^n(1-x)^{\frac{1}{3}}\,\mathrm{d}x$$

$$\Rightarrow I_n = \frac{3n}{4}(I_{n-1} - I_n) \Rightarrow I_n = \frac{3n}{3n+4}I_{n-1}$$

b $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,\mathrm{d}x = \int_0^1 (1+x^2)(1-x)^{\frac{1}{3}}\,\mathrm{d}x = I_0 - I_2$

$I_0 = \int_0^1 (1+x)^{\frac{1}{3}}\,\mathrm{d}x = \left[-\frac{3}{4}(1-x)^{\frac{4}{3}}\right]_0^1 = \frac{3}{4}$

Using **a** $I_2 = \frac{3}{5}I_1 = \frac{3}{5}\left(\frac{3}{7}I_0\right)\left(= \frac{27}{140}\right)$

So $\int_0^1 (1+x)(1-x)^{\frac{4}{3}}\,\mathrm{d}x = \frac{3}{4} - \frac{27}{140} = \frac{78}{140} = \frac{39}{70}$

9 $x = t - \ln t$, so $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 1 - \dfrac{1}{t}$

$y = 4\sqrt{t}$, so $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{2}{\sqrt{t}}$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 1 - \frac{2}{t} + \frac{1}{t^2} + \frac{4}{t} = 1 + \frac{2}{t} + \frac{1}{t^2} = \left(1 + \frac{1}{t}\right)^2$$

a Arc length $= \int_1^4 \sqrt{\left(1 + \frac{1}{t}\right)^2}\,\mathrm{d}t = \int_1^4 \left(1 + \frac{1}{t}\right)\mathrm{d}t = \left[t + \ln t\right]_1^4 = (4 + \ln 4) - 1 = 3 + \ln 4$

b Using $\int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$,

the area of the surface is $2\pi \int_1^4 4\sqrt{t}\left(1 + \frac{1}{t}\right)\mathrm{d}t$

$$= 8\pi \int_1^4 \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)\mathrm{d}t$$

$$= 8\pi \left[\frac{2}{3}t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right]_1^4$$

$$= 8\pi \left[\left(\frac{16}{3} + 4\right) - \left(\frac{2}{3} + 2\right)\right]$$

$$= \frac{160\pi}{3}$$

10 We have $r = 1 - \cos\theta$, $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \sin\theta$ and substitute into the equation $s = \int_\alpha^\beta \sqrt{(r)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$

in order to get

$$s = \int_0^{2\pi} \sqrt{(1 - \cos\theta)^2 + \sin^2\theta}\,\mathrm{d}\theta$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\,\mathrm{d}\theta$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta$$

$$= 2\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right)\mathrm{d}\theta$$

$$= 2\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi}$$

$$= 2(2 - -2)$$

$$= 8 \text{ units}$$

$$= 80 \text{ cm}$$

11 $y = 2\sqrt{x}$ represents the section of curve for $x \geqslant 0, y \geqslant 0$, so $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$

a Using $2\pi \int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$\text{area of surface} = 2\pi \int_0^1 2\sqrt{x}\sqrt{1+\frac{1}{x}}\, dx$$
$$= 4\pi \int_0^1 \sqrt{x}\sqrt{\frac{x+1}{x}}\, dx$$
$$= 4\pi \int_0^1 \sqrt{1+x}\, dx$$

b $4\pi \int_0^1 \sqrt{1+x}\, dx = 4\pi \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^1$

$$= \frac{8\pi}{3}\left(2\sqrt{2}-1\right)$$

c Using the symmetry of the parabola, arc length is $2\times$ the length of arc from origin to (1, 2)

$$\text{so arc length} = 2\int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= 2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\, dx$$

d Using $x = \sinh^2\theta, dx = 2\sinh\theta\cosh\theta d\theta$

$$2\sqrt{\left(\frac{x+1}{x}\right)}\, dx = 2\int \sqrt{\left(\frac{\sinh^2\theta+1}{\sinh^2\theta}\right)} \times 2\sinh\theta\cosh\theta d\theta$$
$$= 4\int \cosh^2\theta d\theta$$
$$= 2\int (1+\cosh 2\theta)\, d\theta$$
$$= 2\left(\theta + \frac{\sinh 2\theta}{2}\right) + C$$
$$= 2(\theta + \sinh\theta\cosh\theta) + C$$
$$= 2\left\{\text{arsinh}\sqrt{x} + \sqrt{x}\sqrt{1+x}\right\} + C$$

$$\text{So arc length} = 2\int_0^1 \sqrt{\left(\frac{x+1}{x}\right)}\, dx = 2\left(\text{arsinh}1 + \sqrt{2}\right)$$
$$= 2\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

$\text{arsinh}\, x = \ln\left\{x + \sqrt{1+x^2}\right\}$

12 Since we have $x = \cos\theta$ and $y = \ln(\sec\theta + \tan\theta) - \sin\theta$, we have $\frac{dx}{d\theta} = -\sin\theta$ and

$$\frac{dy}{d\theta} = \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta} - \cos\theta$$
$$= \sec\theta - \cos\theta$$

We substitute into the equation

$$L = \int_0^{\frac{\pi}{3}} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$
$$= \int_0^{\frac{\pi}{3}} \sqrt{(-\sin\theta)^2 + (\sec\theta - \cos\theta)^2}\, d\theta$$
$$= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2\theta - 1}\, d\theta$$
$$= \int_0^{\frac{\pi}{3}} \tan\theta\, d\theta$$
$$= \left[\ln(\sec\theta)\right]_0^{\frac{\pi}{3}}$$
$$= \ln 2 - \ln 1$$
$$= \ln 2$$

13 a $$I_n - I_{n-1} = \int \frac{[\sin(2n+1)x - \sin(2n-1)x]}{\sin x}\, dx$$
$$= \int \frac{2\cos 2nx \sin x}{\sin x}\, dx$$
$$= \int 2\cos 2nx\, dx$$
$$= \frac{\sin 2nx}{n}$$

$$\boxed{\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}$$

b $I_5 - I_4 = \frac{\sin 10x}{5}, I_4 - I_3 = \frac{\sin 8x}{4}, I_3 - I_2 = \frac{\sin 6x}{3}, I_2 - I_1 = \frac{\sin 4x}{2}$

$I_1 - I_0 = \sin 2x$

Adding: $I_5 = \frac{\sin 10x}{5} + \frac{\sin 8x}{4} + \frac{\sin 6x}{3} + \frac{\sin 4x}{2} + \sin 2x + I_0$

where $I_0 = \int 1\, dx = x + C$

$$= \frac{\sin 10x}{5} + \frac{\sin 8x}{4} + \frac{\sin 6x}{3} + \frac{\sin 4x}{2} + \sin 2x + x + C$$

c $$\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\, dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\, dx = \left[\frac{\sin 2nx}{n}\right]_0^{\frac{\pi}{2}} = \frac{\sin(n\pi)}{n}$$

So, if n is any a positive integer $\int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\, dx - \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\, dx = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin(2n+1)x}{\sin x}\, dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2n-1)x}{\sin x}\, dx = \ldots = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x}\, dx = \frac{\pi}{2}$$

14 a The point A on the curve has coordinates (1, 0).

Using symmetry, the length of the loop is $2\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$.

As $y^2 = \frac{1}{3}x(x-1)^2 = \frac{1}{3}(x^3-2x^2+x)$

$2y\frac{dy}{dx} = \frac{1}{3}(3x^2-4x+1) = \frac{1}{3}(3x-1)(x-1)$

So $\frac{dy}{dx} = \frac{\frac{1}{3}(3x-1)(x-1)}{\pm 2\sqrt{\frac{x}{3}}(x-1)} = \pm\frac{1}{2\sqrt{3}}\frac{(3x-1)}{\sqrt{x}}$

and $1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{9x^2-6x+1}{12x} = \frac{9x^2-6x+1}{12x} = \frac{(3x+1)^2}{12x}$

Therefore, arc length $= 2\int_0^1 \frac{3x+1}{2\sqrt{3}\sqrt{x}}\,dx$

$= \frac{1}{\sqrt{3}}\int_0^1 \left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$

$= \frac{1}{\sqrt{3}}\left[2x^{\frac{3}{2}}+2\sqrt{x}\right]_0^1$

$= \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

b Using $2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ for area of surface generated about the x-axis

Area of surface $= 2\pi\int_0^1 \frac{1}{\sqrt{3}}\sqrt{x}(1-x)\frac{(3x+1)}{\sqrt{12x}}\,dx$

$= \frac{\pi}{3}\int_0^1 (1-x)(3x+1)\,dx$

$= \frac{\pi}{3}\int_0^1 (1+2x-3x^2)\,dx$

$= \frac{\pi}{3}\left[x+x^2-x^3\right]_0^1$

$= \frac{\pi}{3}$

Note: y is +ve for OA, so you need to take $y = -\frac{\sqrt{x}(x-1)}{\sqrt{3}} = \frac{\sqrt{x}(1-x)}{\sqrt{3}}$

15 By Pythagoras' Theorem, $r^2 = x^2 + y^2$ and we want to use the equation $S = 2\pi\int_{y_A}^{y_B} x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$

So we rearrange the Pythagoras' Theorem to express $x = \sqrt{r^2 - y^2}$ and obtain $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{-y}{\sqrt{r^2 - y^2}}$

Substituting these expressions into the equation $S = 2\pi\int_{y_A}^{y_B} x\sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y$

gives

$$
\begin{aligned}
S &= 2\pi\int_{r-h}^{r} x\sqrt{1+\left(\frac{-y}{\sqrt{r^2-y^2}}\right)^2}\,\mathrm{d}y \\
&= 2\pi\int_{r-h}^{r} x\sqrt{1+\left(\frac{y}{x}\right)^2}\,\mathrm{d}y \\
&= 2\pi\int_{r-h}^{r} \sqrt{x^2+y^2}\,\mathrm{d}y \\
&= 2\pi\int_{r-h}^{r} \sqrt{r^2}\,\mathrm{d}y \\
&= 2\pi\int_{r-h}^{r} r\,\mathrm{d}y \\
&= 2\pi\left[ry\right]_{r-h}^{r} \\
&= 2\pi\left(r^2 - r(r-h)\right) \\
&= 2\pi rh
\end{aligned}
$$

16 a $\int \sec^n x\,\mathrm{d}x = \int \sec^{n-2} x \sec^2 x\,\mathrm{d}x$

Let $u = \sec^{n-2} x$ and $\frac{\mathrm{d}v}{\mathrm{d}x} = \sec^2 x$

$\frac{\mathrm{d}u}{\mathrm{d}x} = (n-2)\sec^{n-2} x(\sec x \tan x) = (n-2)\sec^{n-2} x \tan x$ and $v = \tan x$

Integrating by parts

$$
\begin{aligned}
\int \sec^n x\,\mathrm{d}x = I_n &= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\,\mathrm{d}x \\
&= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x\left(\sec^2 x - 1\right)\mathrm{d}x \\
&= \sec^{n-2} x \tan x - (n-2)\int \sec^{n} x\,\mathrm{d}x + (n-2)\int \sec^{n-2} x\,\mathrm{d}x \\
I_n &= \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}
\end{aligned}
$$

So $(n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}, n \geqslant 2,$ *

16 b $\int \sec^5 x\,dx = I_5 = \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}I_3$ ← Substituting $n = 5$ in *

$= \frac{1}{4}\sec^3 x\tan x + \frac{3}{4}\left(\frac{1}{2}\sec x\tan x + \frac{1}{2}I_1\right)$ ← Substituting $n = 3$ in *

But $I_1 = \int \sec x\,dx = \ln|\sec x + \tan x| + C$ ← On Edexcel formula sheet

So $\int \sec^5 x\,dx = I_5 = \frac{1}{4}\sec^3 x\tan x + \frac{3}{8}\sec x\tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$

c $\int_0^{\frac{\pi}{4}} \sec^5 x\,dx = \frac{1}{4}\left(\sqrt{2}\right)^3 + \frac{3}{8}\left(\sqrt{2}\right) + \frac{3}{8}\ln\left(\sqrt{2}+1\right)$

$= \frac{1}{8}\left\{7\sqrt{2} + 3\ln\left(\sqrt{2}+1\right)\right\}$

Challenge

We use the parametric version of the arc length equation which uses the parametric derivatives $\frac{dx}{dt} = \cos\left(\frac{\pi t^2}{2}\right)$ and $\frac{dy}{dt} = \sin\left(\frac{\pi^2}{2}\right)$.

Substituting these values into the equation for arc length gives

$$s = \int_0^a \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^a \sqrt{\cos^2\left(\frac{\pi t^2}{2}\right) + \sin^2\left(\frac{\pi t^2}{2}\right)}\,dt$$

$$= \int_0^a 1\,dt$$

$$= [t]_0^a$$

$$= a$$