Integration techniques 6C

1 a $y=\frac{3}{4}x \Rightarrow \frac{dy}{dx}=\frac{3}{4} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2=\frac{25}{16}$

$$\text{Surface area} = \int_4^8 2\pi\left(\frac{3}{4}x\right)\left(\frac{5}{4}\right)dx$$

Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$=\frac{15}{8}\pi\int_4^8 x\,dx$$

$$=\frac{15}{8}\pi\left[\frac{x^2}{2}\right]_4^8=45\pi$$

b Rotating about the y-axis:

From the work in **a** $1+\left(\frac{dx}{dy}\right)^2=1+\frac{16}{9}=\frac{25}{9}$

As integration is w.r.t. y, the integrand must be in terms of y

The limits for y are 3 (when $x = 4$) and 6 (when $x=8$),

so area of surface is $\int_3^6 2\pi\left(\frac{4}{3}y\right)\left(\frac{5}{3}\right)dy$,

$$=\frac{40}{9}\pi\left[\frac{y^2}{2}\right]_3^6$$

$$=\frac{40\times 27}{9\times 2}\pi=60\pi$$

Although it is quicker to use

$\int_u^8 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

here $\int_{y_1}^{y_2} 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

is used to give an example of its use.

2 $y=x^3$ so $\frac{dy}{dx}=3x^2$

Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $\int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx$

$$=\frac{2\pi}{36}\int_0^1 36x^3\sqrt{1+9x^4}\,dx$$

$$=\frac{2\pi}{36}\left[\frac{2}{3}\left(1+9x^4\right)^{\frac{3}{2}}\right]_0^1$$

$$=\frac{\pi}{27}\left[10\sqrt{10}-1\right] \quad (3.56, 3\text{ s.f.})$$

3 $y=\frac{1}{2}x^2$, so $\frac{dy}{dx}=x$

Using $\int_{x_1}^{x_2} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $\int_0^2 2\pi x\sqrt{1+x^2}\,dx$

$$=\pi\int_0^2 2x\sqrt{1+x^2}\,dx$$

$$=\pi\left[\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]_0^2$$

$$=\frac{2\pi}{3}\left[5\sqrt{5}-1\right]$$

4 In order to calculate the area of the generated surface we want to use the equation

$$S=2\pi\int_{t_A}^{t_B} x\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

We find $\frac{dx}{dt}=2\sin t\cos t$ and $\frac{dy}{dt}=-2\cos t\sin t$ and substitute into the surface area equation to find

$$S=2\pi\int_0^{\frac{\pi}{2}}\sin^2 t\sqrt{\left(2\sin t\cos t\right)^2+\left(-2\cos t\sin t\right)^2}\,dt$$

$$=2\pi\int_0^{\frac{\pi}{2}}\sin^2 t\sqrt{2}\left(2\sin t\cos t\right)dt$$

$$=2^{\frac{5}{2}}\pi\int_0^{\frac{\pi}{2}}\cos t\sin^3 t\,dt$$

$$=2^{\frac{5}{2}}\pi\left[\frac{\sin^4 t}{4}\right]_0^{\frac{\pi}{2}}$$

$$=\sqrt{2}\pi$$

5 $y = \cosh x$, so $\dfrac{dy}{dx} = \sinh x$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2 x = \cosh^2 x$$

a Using $\int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\int_0^1 2\pi \cosh^2 x \; dx$

$$= \pi \int_0^1 (\cosh 2x + 1)\; dx$$

$$= \pi \left[\frac{\sinh 2x}{2} + x\right]_0^1$$

$$= \pi [\sinh x \cosh x + x]_0^1$$

$$= \pi [\sinh 1 \cosh 1 + 1]$$

$$= 8.84 \text{ (3 s.f.)}$$

b Using $\int_{x_1}^{x_2} 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$,

the area of the surface is $\int_0^1 2\pi x \cosh x \; dx$

$$= 2\pi \left\{ [x \sinh x]_0^1 - \int_0^1 \sinh x \; dx \right\}$$

Using integration by parts

$$= 2\pi \left\{ \sinh 1 - [\cosh x]_0^1 \right\}$$

$$= 2\pi \{ \sinh 1 - \cosh 1 + 1 \}$$

$$= 2\pi \left\{ \frac{1}{2}\left(e - \frac{1}{e} - e - \frac{1}{e} \right) + 1 \right\}$$

$$= 2\pi \left(1 - \frac{1}{e} \right)$$

$$= 2\pi \left(\frac{e - 1}{e} \right)$$

6 a $y = \dfrac{1}{2x} + \dfrac{x^3}{6}$, so $\dfrac{dy}{dx} = -\dfrac{1}{2x^2} + \dfrac{x^2}{2} = \dfrac{1}{2}\left(x^2 - \dfrac{1}{x^2}\right)$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right) = \frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2$$

So $\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{1}{2}\left(x^2 + \dfrac{1}{x^2}\right)$

6 b Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $\pi\int_1^3\left(\frac{1}{2x}+\frac{x^3}{6}\right)\left(x^2+\frac{1}{x^2}\right)dx$

$=\pi\int_1^3\left(\frac{2x}{3}+\frac{x^5}{6}+\frac{1}{2x^3}\right)dx$

$=\pi\left[\frac{x^2}{3}+\frac{x^6}{36}-\frac{1}{4x^2}\right]_1^3$

$=\frac{208}{9}\pi=23\frac{1}{9}\pi=72.6$ (3 s.f.)

7 $x^{\frac{2}{3}}+y^{\frac{2}{3}}=4$, so $\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}=-\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$

So $1+\left(\frac{dy}{dx}\right)^2=1+\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{4}{x^{\frac{2}{3}}}$

Using $\int_{x_1}^{x_2} 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $2\pi\int_0^8 x\left(\frac{2}{x^{\frac{1}{3}}}\right)dx$

$=2\pi\int_0^8 2x^{\frac{2}{3}}\,dx$

$=2\pi\left[\frac{6}{5}x^{\frac{5}{3}}\right]_0^8$

$=\frac{12\pi}{5}[32]$

$=\frac{384\pi}{5}=241$ (3s.f.)

8 In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{\alpha}^{\beta} r\cos\theta\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

We find $\frac{dr}{d\theta} = -\sin\theta$ and substitute into the surface area equation to find

$$S = 2\pi\int_0^{\pi}\cos^2\theta\sqrt{\cos^2\theta+\sin^2\theta}\,d\theta$$

$$= 2\pi\int_0^{\pi}\cos^2\theta\,d\theta$$

$$= \pi\int_0^{\pi}(\cos 2\theta+1)\,d\theta$$

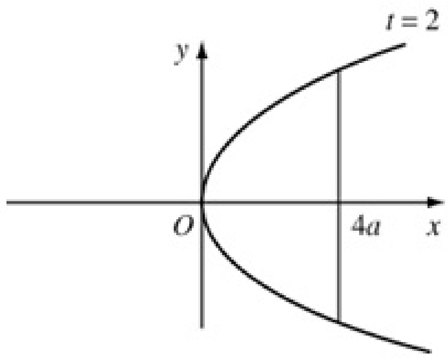
$$= \pi\left[\frac{1}{2}\sin 2\theta+\theta\right]_0^{\pi}$$

$$= \pi^2$$

9 $x = at^2, y = 2at$, so $\frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$

So $\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 = 4a^2t^2+4a^2 = 4a^2(1+t^2)$

$x = 4a$ when $t = \pm 2$ (See diagram.)



A rotation of π radians gives a surface which would be found by rotating the section $y \geqslant 0$, i.e. $t = 0$ to $t = 2$ through 2π radians.

Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$,

the area of the surface is $2\pi\int_0^2 4a^2t\sqrt{1+t^2}\,dt$

$$= 8\pi a^2\left[\frac{1}{3}(1+t^2)^{\frac{3}{2}}\right]_0^2$$

$$= \frac{8}{3}\pi a^2\left[5^{\frac{3}{2}}-1\right]$$

$$= \frac{8}{3}\pi a^2\left(5\sqrt{5}-1\right)$$

10 $x = \text{sech}t, y = \tanh t$, so $\frac{dx}{dt} = -\text{sech}t\tanh t, \frac{dy}{dt} = \text{sech}^2 t$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \text{sech}^2 t\tanh^2 t + \text{sech}^4 t = \text{sech}^2 t\left(\tanh^2 t + \text{sech}^2 t\right) = \text{sech}^2 t$$

Using $\int_{t_1}^{t_2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi\int_0^{\ln 2} \tanh t\,\text{sech}t\, dt$

$$= 2\pi\left[-\text{sech}t\right]_0^{\ln 2}$$

$$= 2\pi\left[-\frac{2}{e^t + e^{-t}}\right]_0^{\ln 2}$$

$$= 2\pi\left(\frac{-2}{2.5} + 1\right)$$

$$= \frac{2\pi}{5}$$

11 a $x = 3t^2, y = 2t^3$, so $\frac{dx}{dt} = 6t, \frac{dy}{dt} = 6t^2$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36t^2\left(t^2 + 1\right)$$

Using $\int_{t_1}^{t_2} 2\pi x\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$,

the area of the surface is $2\pi\int_0^2 3t^2 \times 6t\sqrt{1+t^2}\, dt$

$$= 36\pi\int_0^2 t^3\sqrt{1+t^2}\, dt$$

b Let $u = t^2, \frac{dv}{dt} = t\sqrt{1+t^2}$

So $\frac{du}{dt} = 2t, v = \frac{1}{3}\left(1+t^2\right)^{\frac{3}{2}}$

$$36\pi\int_0^2 t^2\left(t\sqrt{1+t^2}\right)dt = 36\pi\left\{\left[\frac{1}{3}t^2\left(1+t^2\right)^{\frac{3}{2}}\right]_0^2 - \int_0^2 \frac{2}{3}t\left(1+t^2\right)^{\frac{3}{2}}\, dt\right\}$$

$$= 12\pi\left[t^2\left(1+t^2\right)^{\frac{3}{2}} - \frac{2}{5}\left(1+t^2\right)^{\frac{5}{2}}\right]_0^2$$

$$= 12\pi\left[4\left(5\sqrt{5}\right) - \frac{2}{5}\left(25\sqrt{5}\right) + \frac{2}{5}\right]$$

$$= 12\pi\left[10\sqrt{5} + \frac{2}{5}\right]$$

$$= \frac{24\pi}{5}\left[25\sqrt{5} + 1\right]$$

12 In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{t_A}^{t_B} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

We find $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = 1 - t^2$ and substitute into the surface area equation to find

$$S = 2\pi\int_0^1 \left(t - \frac{1}{3}t^3\right)\sqrt{(2t)^2 + \left(1 - t^2\right)^2}\,dt$$

$$= 2\pi\int_0^1 \left(t - \frac{1}{3}t^3\right)\left(1 + t^2\right)dt$$

$$= 2\pi\int_0^1 t + \frac{2t^3}{3} - \frac{t^5}{3}\,dt$$

$$= 2\pi\left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^1$$

$$= \frac{11\pi}{9}$$

13 a In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{t_A}^{t_B} y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

We find $\frac{dx}{dt} = -3a\sin t\cos^2 t$ and $\frac{dy}{dt} = 3a\cos t\sin^2 t$ and substitute into the surface area equation to find

$$S = 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(a\sin^3 t\right)\sqrt{\left(-3a\sin t\cos^2 t\right)^2 + \left(3a\cos t\sin^2 t\right)^2}\,dt$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(a\sin^3 t\right)\sqrt{\left(3a\cos t\sin t\right)^2\left(\cos^2 t + \sin^2 t\right)}\,dt$$

$$= 2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(a\sin^3 t\right)\left(3a\cos t\sin t\right)dt$$

$$= 6a^2\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\cos t\sin^4 t\right)dt$$

$$= 6a^2\pi\left[\frac{\sin^5 t}{5}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 6a^2\pi\left(\frac{1}{5} - \frac{1}{2^5 \times 5}\right)$$

$$= \frac{6a^2\pi}{5} \times \frac{31}{32}$$

$$= \frac{93a^2\pi}{80}$$

13 b We find the value of a by using the equation for y. Since we have that the diameter of the small hole is 3cm, that means $2y$ evaluated at $t = \frac{\pi}{6}$ is equal to 3.

That is $3 = 2a\sin^3\left(\frac{\pi}{6}\right) = \frac{a}{4}$ and so $a = 12$.

14 $y = e^x, \frac{dy}{dx} = e^x$

Using $\int_{x_1}^{x_2} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

the area of the surface is $2\pi\int_0^{\ln 2} e^x\sqrt{1+e^{2x}}\,dt$

Make the substitution $e^x = \sinh u$, so $e^x dx = \cosh u\,du$

Limits: when $x = \ln 2, u = \text{arsinh}\,e^{\ln 2} = \text{arsinh}\,2$

when $x = 0, u = \text{arsinh}\,e^0 = \text{arsinh}\,1$

Then the area of the surface is $2\pi\int_{\text{arsinh}1}^{\text{arsinh}2} \cosh^2 u\,du$

$= \pi\int_{\text{arsinh}1}^{\text{arsinh}2} (1+\cosh 2u)\,du$

$= \pi\left[u + \frac{\sinh 2u}{2}\right]_{\text{arsinh}1}^{\text{arsinh}2}$

$= \pi\left[u + \sinh u\cosh u\right]_{\text{arsinh}1}^{\text{arsinh}2}$

$\cosh u = \sqrt{1+\sinh^2 u}$

$= \pi\left[\text{arsinh}2 + 2\sqrt{5} - \left(\text{arsinh}1 + (1)\sqrt{2}\right)\right]_{\text{arsinh}1}^{\text{arsinh}2}$

$= \pi\left(\text{arsinh}2 - \text{arsinh}1 + 2\sqrt{5} - \sqrt{2}\right)$

15 In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{\alpha}^{\beta} r\cos\theta\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

We find $\frac{\mathrm{d}r}{\mathrm{d}\theta} = e^{\theta}$ and substitute into the surface area equation to find

$$S = 2\pi\int_0^{\frac{\pi}{2}} e^{\theta}\cos\theta\sqrt{e^{2\theta} + e^{2\theta}}\,\mathrm{d}\theta$$

$$= 2^{\frac{3}{2}}\pi\int_0^{\frac{\pi}{2}} e^{2\theta}\cos\theta\,\mathrm{d}\theta.$$

We set $I = \int e^{2\theta}\cos\theta d\theta$ and try to solve by parts.

$$I = \int e^{2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= e^{2\theta}\sin\theta - \int 2e^{2\theta}\sin\theta\,\mathrm{d}\theta$$

$$= e^{2\theta}\sin\theta + 2e^{2\theta}\cos\theta - \int 4e^{2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= e^{2\theta}\sin\theta + 2e^{2\theta}\cos\theta - 4I$$

so now we have a solution for the integral

$$I = \int e^{2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= \frac{e^{2\theta}}{5}(\sin\theta + 2\cos\theta).$$

Thus we have

$$S = 2\pi\int_0^{\frac{\pi}{2}} e^{\theta}\cos\theta\sqrt{e^{2\theta} + e^{2\theta}}\,\mathrm{d}\theta$$

$$= 2^{\frac{3}{2}}\pi\int_0^{\frac{\pi}{2}} e^{2\theta}\cos\theta\,\mathrm{d}\theta$$

$$= 2^{\frac{3}{2}}\pi\left[\frac{e^{2\theta}}{5}(\sin\theta + 2\cos\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2^{\frac{3}{2}}\pi}{5}(e^{\pi} - 2)$$

16 In order to calculate the area of the generated surface we want to use the equation

$$S = 2\pi\int_{\alpha}^{\beta} r\sin\theta\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$$

We find $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -3\sin\theta$ and substitute into the surface area equation to find

$$S = 2\pi\int_0^{\pi}(3+3\cos\theta)\sin\theta\sqrt{(3+3\cos\theta)^2 + (-3\sin\theta)^2}\,\mathrm{d}\theta$$
$$= 6\pi\int_0^{\pi}(1+\cos\theta)\sin\theta\sqrt{18+18\cos\theta}\,\mathrm{d}\theta$$

at this point let us introduce a substitution to stop it from getting out of hand.

Let

$u = 1+\cos\theta,$

$\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta.$

Then

$$S = -6\pi\int_2^0 u\sqrt{18u}\,\mathrm{d}u$$
$$= 6\sqrt{18}\pi\int_0^2 u^{\frac{3}{2}}\,\mathrm{d}u$$
$$= 6\sqrt{18}\pi\left[\frac{2u^{\frac{5}{2}}}{5}\right]_0^2$$
$$= \frac{12\sqrt{18}\pi}{5}\left(\left(2^{\frac{5}{2}}\right) - 0\right)$$
$$= \frac{288}{5}\pi$$
$$\approx 181.0\ \mathrm{cm}^2\ (1\text{ d.p.})$$

17 In order to calculate the area of the generated (not base) surface we want to use the equation

$$S = 2\pi\int_{y_A}^{y_B} x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

We find

$$x = \sqrt{5y},$$

$$\frac{dx}{dy} = \frac{\sqrt{5}}{2\sqrt{y}}$$

and substitute into the surface area equation to find

$$S = 2\pi\int_{5}^{10} x\sqrt{1+\left(\frac{\sqrt{5}}{2\sqrt{y}}\right)^2}\,dy$$

$$= 2\pi\int_{5}^{10} \sqrt{5y}\sqrt{1+\left(\frac{5}{4y}\right)}dy$$

$$= \pi\int_{5}^{10} \sqrt{20y+25}dy$$

$$= \frac{\pi}{30}\left[(20y+25)^{\frac{3}{2}}\right]_5^{10}$$

$$= \frac{\pi}{30}\left((200+25)^{\frac{3}{2}} - (100+25)^{\frac{3}{2}}\right)$$

$$= \frac{\pi}{30}\left(225^{\frac{3}{2}} - 125^{\frac{3}{2}}\right)$$

The circular base has area of $A = \pi\times 5^2 = 25\pi$.

Thus we have a total area of 285.619 cm^2 and so have a total cost of £5.71.