## Matrix algebra Mixed exercise

**1 a** For $\lambda_1 = 5$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$x+8y=5x \Rightarrow x=2y$

Let $y=1$, then $x=2$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

For $\lambda_2 = -15$

$$\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -15\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+8y \\ 8x-11y \end{pmatrix} = \begin{pmatrix} -15x \\ -15y \end{pmatrix}$$

Equating the upper elements

$x+8y=-15x \Rightarrow y=-2x$

Let $x=1$, then $y=-2$

An eigenvector corresponding to the eigenvalue –15 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

**b** The magnitude of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\sqrt{(2^2+1^2)} = \sqrt{5}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $\sqrt{(1^2+(-2)^2)} = \sqrt{5}$

Hence $\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \end{pmatrix}$

**2 a** $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} -5-\lambda & 8 \\ 3 & -7-\lambda \end{pmatrix}$

$$\begin{vmatrix} -5-\lambda & 8 \\ 3 & -7-\lambda \end{vmatrix} = (5+\lambda)(7+\lambda) - 24 = \lambda^2 + 12\lambda + 11 = (\lambda+1)(\lambda+11)$$

$\det(\mathbf{A}-\lambda\mathbf{I}) = 0 \Rightarrow (\lambda+1)(\lambda+11) = 0 \Rightarrow \lambda = -1, -11$

The eigenvalues of **A** are –1 and –11.

**2 b** For $\lambda = -1$

$$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Equating the upper elements

$$-5x+8y=-x \Rightarrow y=\frac{1}{2}x$$

For $\lambda = -11$

$$\begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -11\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} -5x+8y \\ 3x-7y \end{pmatrix} = \begin{pmatrix} -11x \\ -11y \end{pmatrix}$$

Equating the upper elements

$$-5x+8y=-11x \Rightarrow y=-\frac{3}{4}x$$

Cartesian equations of the lines through the origin which are invariant under $T$ are $y=\frac{1}{2}x$ and $y=-\frac{3}{4}x$.

**3 a** $A = \begin{pmatrix} 4 & k \\ 2 & -2 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 4-\lambda & k \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-2-\lambda)-2k=0$$

$$\lambda^2-2\lambda-8-2k=0$$

Repeated roots implies $b^2-4ac=0$

$$4-4(-8-2k)=0$$

$$8k+36=0$$

$$k=-\frac{9}{2}$$

**3 b** $k=-\frac{9}{2}\Rightarrow\lambda^2-2\lambda+1=0$

$(\lambda-1)^2=0$

$\lambda=1$

$$\begin{pmatrix}4 & -\frac{9}{2}\\ 2 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

Equating lower elements:

$2x-2y=y$

$2x=3y$

Choosing $y=2$ gives $x=3$ and so one possible eigenvector is $\begin{pmatrix}3\\ 2\end{pmatrix}$

**c** Since $\begin{pmatrix}3\\ 2\end{pmatrix}$ is an eigenvector, $y=\frac{2}{3}x$ is invariant under $T$

**4 a** $\mathbf{M}=\begin{pmatrix}a & a\\ 2 & 1\end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix}a-\lambda & a\\ 2 & 1-\lambda\end{vmatrix}=0$$

$(a-\lambda)(1-\lambda)-2a=0$

$\lambda^2-(a+1)\lambda-a=0$

Eigenvalues are complex, so $b^2-4ac<0$

$(a+1)^2+4a<0$

$a^2+6a+1<0$

Critical values are given by $a=\frac{-6\pm\sqrt{32}}{2}=\frac{-6\pm4\sqrt{2}}{2}=-3\pm2\sqrt{2}$

Therefore the solution to $a^2+6a+1<0$ is $-3-2\sqrt{2}<a<-3+2\sqrt{2}$

**4 b** $a = -1 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$

When $\lambda = i$

$$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = i\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$-x - y = ix$

$-y = x(1+i)$

$x = -\dfrac{y}{1+i}$

Choosing $y = 2 \Rightarrow x = -\dfrac{2}{1+i}\left(\dfrac{1-i}{1-i}\right) = \dfrac{-2+2i}{2} = -1+i$

Therefore one possible eigenvector is $\begin{pmatrix} -1+i \\ 2 \end{pmatrix}$

When $\lambda = -i$

$$\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -i\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$-x - y = -ix$

$x + y = ix$

$x - ix = -y$

$x = -\dfrac{y}{1-i}$

Choosing $y = 2 \Rightarrow x = -\dfrac{2}{1-i}\left(\dfrac{1+i}{1+i}\right) = \dfrac{-2-2i}{2} = -1-i$

Therefore one possible eigenvector is $\begin{pmatrix} -1-i \\ 2 \end{pmatrix}$

**c** There are no real eigenvectors, therefore no invariant lines under the transformation $T$.

**5 a** $\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-1-\lambda)+6=0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda-2)(\lambda-1)=0$$

So 1 and 2 are both eigenvalues.

If $\lambda = 2$

$$\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating lower elements:

$$2x - y = 2y$$

$$2x = 3y$$

Choosing $x = 3$ gives $y = 2$ and so $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is a possible eigenvector.

**b** If $\lambda = 1$

$$\begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating lower elements:

$$2x - y = y$$

$$x = y$$

Choosing $x = 1$ gives $y = 1$ and so $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a possible eigenvector.

Therefore $\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

**6 a** $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & -1 \\ 4 & -3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(-3-\lambda)+4=0$$
$$\lambda^2+\lambda-2=0$$
$$(\lambda-1)(\lambda+2)=0$$
$$\lambda=1 \text{ or } \lambda=-2$$

If $\lambda = 1$

$$\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:
$2x-y=x$
$x=y$

Choosing $x=1$ gives $y=1$, giving an eigenvector of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

If $\lambda = -2$

$$\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:
$2x-y=-2x$
$4x=y$

Choosing $y=4$ gives $x=1$, giving an eigenvector of $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

**b** Therefore $P = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$

**c** $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = P^{-1}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = P^{-1}A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = DP^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = D\begin{pmatrix} u_n \\ v_n \end{pmatrix}$

**6 d** $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$

$u_{n+1} = u_n$ so $u_{n+1} = u_1$

$v_{n+1} = -2v_n$ so $v_{n+1} = v_1 \times (-2)^n$

$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ so $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = P\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$

$x_n = u_n + v_n = u_1 + v_1 \times (-2)^{n-1}$

$y_n = u_n + 4v_n = u_1 + 4v_1 \times (-2)^{n-1}$

When $n = 1$, $\quad 2 = u_1 + v_1$

$3 = u_1 + 4v_1$

Solving simultaneously gives $u_1 = \dfrac{5}{3}$ and $v_1 = \dfrac{1}{3}$

Therefore $x_n = \dfrac{5}{3} + \dfrac{1}{3} \times (-2)^{n-1}$ and $y_n = \dfrac{5}{3} + \dfrac{4}{3} \times (-2)^{n-1}$

**7 a** $\mathbf{A}=\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}-1-\lambda & 2\\ 0 & 1-\lambda\end{vmatrix}=0$$

$(-1-\lambda)(1-\lambda)=0$

$\lambda^2-1=0$

$\lambda=1$ or $\lambda=-1$

If $\lambda=1$

$$\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x\\ y\end{pmatrix}$$

Equating upper elements:

$-x+2y=x$

$y=x$

Choosing $x=1$ gives $y=1$, giving an eigenvector of $\begin{pmatrix}1\\ 1\end{pmatrix}$

If $\lambda=-1$

$$\begin{pmatrix}-1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=-\begin{pmatrix}x\\ y\end{pmatrix}$$

Equating lower elements:

$y=-y$

$y=0$

Choosing $x=1$ gives an eigenvector of $\begin{pmatrix}1\\ 0\end{pmatrix}$

Therefore $\mathbf{P}=\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

**7 b** $\mathbf{D}^{50} = \left(\mathbf{P}^{-1}\mathbf{AP}\right)^{50} = \mathbf{P}^{-1}\mathbf{A}^{50}\mathbf{P}$

Therefore $\mathbf{A}^{50} = \mathbf{PD}^{50}\mathbf{P}^{-1}$

Using a calculator gives $\mathbf{P}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

Therefore $\mathbf{A}^{50} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1^{50} & 0 \\ 0 & (-1)^{50} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

**8 a** $\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix}$

The characteristic equation is given by:

$$\begin{vmatrix} 4-\lambda & 5 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$(4-\lambda)(2-\lambda)+5=0$
$\lambda^2 - 6\lambda + 13 = 0$

**b** By Cayley-Hamilton

$\mathbf{A}^2 - 6\mathbf{A} + 13\mathbf{I} = \mathbf{0}$

$\mathbf{A}^3 - 6\mathbf{A}^2 + 13\mathbf{A} = \mathbf{0}$
$\mathbf{A}^3 = 6\mathbf{A}^2 - 13\mathbf{A}$
$\mathbf{A}^3 = 6(6\mathbf{A} - 13\mathbf{I}) - 13\mathbf{A}$
$\mathbf{A}^3 = (36\mathbf{A} - 78\mathbf{I}) - 13\mathbf{A}$
$\mathbf{A}^3 = 23\mathbf{A} - 78\mathbf{I}$

**9** $\mathbf{A} = \begin{pmatrix} 7 & 1 \\ -1 & 2 \end{pmatrix}$

The characteristic equation is given by:

$$\begin{vmatrix} 7-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(2-\lambda)+1=0$$
$$\lambda^2 - 9\lambda + 15 = 0$$

By Cayley-Hamilton

$$\mathbf{A}^2 - 9\mathbf{A} + 15\mathbf{I} = \mathbf{0}$$
$$9\mathbf{A} = \mathbf{A}^2 + 15\mathbf{I}$$
$$\mathbf{A} = \frac{1}{9}\mathbf{A}^2 + \frac{15}{9}\mathbf{I}$$

Therefore $p = \frac{1}{9}$ and $q = \frac{5}{3}$

**10 a** For $\lambda = 1$

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x+y \\ 2x+4y \\ x+z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating the top elements
$3x + y = x \Rightarrow 2x + y = 0$ **(1)**
Equating the middle elements
$2x + 4y = y \Rightarrow 2x + 3y = 0$ **(2)**
**(2) − (1)**
$2y = 0 \Rightarrow y = 0$
Substituting $y = 0$ into **(1)**
$2x = 0 \Rightarrow x = 0$
$z$ can take any non-zero value
Let $z = 1$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

**10 b** Let $\mathbf{A}=\begin{pmatrix}3&1&0\\2&4&0\\1&0&1\end{pmatrix}$, then $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda&1&0\\2&4-\lambda&0\\1&0&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda&1&0\\2&4-\lambda&0\\1&0&1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}4-\lambda&0\\0&1-\lambda\end{vmatrix}-1\begin{vmatrix}2&0\\1&1-\lambda\end{vmatrix}+0\begin{vmatrix}2&4-\lambda\\1&0\end{vmatrix}$$

$$=(3-\lambda)(4-\lambda)(1-\lambda)-2(1-\lambda)$$

$$=(1-\lambda)\big((3-\lambda)(4-\lambda)-2\big)=(1-\lambda)(\lambda^2-7\lambda+10)$$

$$=(1-\lambda)(\lambda-2)(\lambda-5)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(1-\lambda)(\lambda-2)(\lambda-5)=0\Rightarrow\lambda=1,2,5$$

The other eigenvalues are 2 and 5.

**11 a** $\mathbf{A}=\begin{pmatrix}0&1&1\\2&1&-1\\-6&-5&-3\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}-\lambda&1&1\\2&1-\lambda&-1\\-6&-5&-3-\lambda\end{vmatrix}=0$$

$$(-\lambda)\big[(1-\lambda)(-3-\lambda)-5\big]-1\big[2(-3-\lambda)-6\big]+1\big[-10+6(1-\lambda)\big]=0$$

$$(-\lambda)(\lambda^2+2\lambda-8)-1(-12-2\lambda)+(-4-6\lambda)=0$$

$$-\lambda^3-2\lambda^2+8\lambda+12+2\lambda-4-6\lambda=0$$

$$-\lambda^3-2\lambda^2+4\lambda+8=0$$

$$\lambda^3+2\lambda^2-4\lambda-8=0$$

Let $f(\lambda)=\lambda^3+2\lambda^2-4\lambda-8$

$f(2)=2^3+2\times2^2-4\times2-8=0\Rightarrow(\lambda-2)$ is a factor

So $f(\lambda)=(\lambda-2)(\lambda^2+k\lambda+4)$

Equating coefficients of $\lambda^2$ gives

$-2+k=2$, so $k=4$

$$(\lambda-2)(\lambda^2+4\lambda+4)=0$$

$$(\lambda-2)(\lambda+2)^2=0$$

Therefore the required eigenvalues are 2 and −2.

**11 a (continued)**

Taking $\lambda = 2$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$y + z = 2x$

$5y + 5z = 10x$ **(1)**

Equating lower elements:

$-6x - 5y - 3z = 2z$

$-6x - 5y - 5z = 0$ **(2)**

**(1)** + **(2)** gives $-6x = 10x$ , so $x = 0$ and $y = -z$

Therefore choosing $z = 1$ gives an eigenvector of $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Taking $\lambda = -2$:

$$\begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$y + z = -2x$ **(1)**

Equating middle elements:

$2x + y - z = -2y$

$2x + 3y - z = 0$ **(2)**

**(1)** + **(2)** gives $2x + 4y = -2x$, so $x = -y$

Therefore choosing $y = 1$ gives $x = -1$ and $z = 1$, giving an eigenvector of $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

**b** The matrix representing the transformation will always have at least one real eigenvector, which defines an invariant line.

**c** Each invariant line must go through the origin.

The invariant lines are therefore $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

**12 a** $\mathbf{A}=\begin{pmatrix}2&0&2\\2&2&0\\0&1&3\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}2-\lambda&0&2\\2&2-\lambda&0\\0&1&3-\lambda\end{vmatrix}=0$$

$$(2-\lambda)\left[(2-\lambda)(3-\lambda)\right]+2[2]=0$$
$$(2-\lambda)\left(\lambda^2-5\lambda+6\right)+4=0$$
$$2\lambda^2-10\lambda+12-\lambda^3+5\lambda^2-6\lambda+4=0$$
$$-\lambda^3+7\lambda^2-16\lambda+16=0$$
$$\lambda^3-7\lambda^2+16\lambda-16=0$$

Let $f(\lambda)=\lambda^3-7\lambda^2+16\lambda-16$

$f(4)=4^3-7\times4^2+16\times4-16=0\Rightarrow(\lambda-4)$ is a factor

So $f(\lambda)=(\lambda-4)\left(\lambda^2+k\lambda+4\right)$

Equating coefficients of $\lambda^2$ gives

$-4+k=-7,$ so $k=-3$

$$(\lambda-2)\left(\lambda^2-3\lambda+4\right)=0$$

$$\lambda^2-3\lambda+4=0\Rightarrow\lambda=\frac{3\pm\sqrt{-7}}{2}=\frac{3\pm i\sqrt{7}}{2}$$

Therefore the required eigenvalues are $4,\frac{3+i\sqrt{7}}{2}$ and $\frac{3-i\sqrt{7}}{2}$

**12 b** Taking $\lambda = 4$:

$$\begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$2x + 2z = 4x$

$z = x$ **(1)**

Equating lower elements:

$y + 3z = 4z$

$y = z$ **(2)**

Therefore choosing $x = 1$ gives an eigenvector of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Taking $\lambda = \frac{3 + i\sqrt{7}}{2}$:

$$\begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\frac{3 + i\sqrt{7}}{2}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$2x + 2z = \left(\frac{3 + i\sqrt{7}}{2}\right)x$ **(1)**

Equating lower elements:

$y + 3z = \left(\frac{3 + i\sqrt{7}}{2}\right)z$ **(2)**

Choosing $z = 2$, equation **(1)** becomes

$2x + 4 = \left(\frac{3 + i\sqrt{7}}{2}\right)x$

$4x + 8 = \left(3 + i\sqrt{7}\right)x$

$8 = \left(-1 + i\sqrt{7}\right)x$

$x = \frac{8}{\left(-1 + i\sqrt{7}\right)} = -1 - i\sqrt{7}$

Equation **(2)** becomes

$y + 6 = 3 + i\sqrt{7}$

$y = -3 + i\sqrt{7}$

**12 b (continued)**

Therefore a corresponding eigenvector is given by $\begin{pmatrix} -1-i\sqrt{7} \\ -3+i\sqrt{7} \\ 2 \end{pmatrix}$

Taking $\lambda = \frac{3-i\sqrt{7}}{2}$:

$$\begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\frac{3-i\sqrt{7}}{2}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$$2x + 2z = \left(\frac{3-i\sqrt{7}}{2}\right)x \qquad \textbf{(1)}$$

Equating lower elements:

$$y + 3z = \left(\frac{3-i\sqrt{7}}{2}\right)z \qquad \textbf{(2)}$$

Choosing $z = 2$, equation **(1)** becomes

$$2x + 4 = \left(\frac{3-i\sqrt{7}}{2}\right)x$$

$$4x + 8 = \left(3-i\sqrt{7}\right)x$$

$$8 = \left(-1-i\sqrt{7}\right)x$$

$$x = \frac{8}{\left(-1-i\sqrt{7}\right)} = -1+i\sqrt{7}$$

Equation **(2)** becomes

$$y + 6 = 3 - i\sqrt{7}$$

$$y = -3 - i\sqrt{7}$$

Therefore a corresponding eigenvector is given by $\begin{pmatrix} -1+i\sqrt{7} \\ -3-i\sqrt{7} \\ 2 \end{pmatrix}$ whose entries are the complex conjugates of the eigenvector for $\lambda = \frac{3+i\sqrt{7}}{2}$ as expected.

**c** The only invariant line has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

**13 a** $\mathbf{A} = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 4-\lambda & 1 & -1 \\ 1 & -\lambda & 3 \\ 1 & 2 & 1-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\left[(-\lambda)(1-\lambda)-6\right]-1\left[1(1-\lambda)-3\right]-1\left[2+\lambda\right]=0$$
$$(4-\lambda)\left(\lambda^2-\lambda-6\right)-1(-2-\lambda)-2-\lambda=0$$
$$4\lambda^2-4\lambda-24-\lambda^3+\lambda^2+6\lambda+2+\lambda-2-\lambda=0$$
$$-\lambda^3+5\lambda^2+2\lambda-24=0$$
$$\lambda^3-5\lambda^2-2\lambda+24=0$$

Let $f(\lambda)=\lambda^3-5\lambda^2-2\lambda+24$

$f(-2)=(-2)^3-5\times(-2)^2-2\times(-2)+24=0 \Rightarrow (\lambda+2)$ is a factor

So $f(\lambda)=(\lambda+2)\left(\lambda^2+k\lambda+12\right)$

Equating coefficients of $\lambda^2$ gives
$2+k=-5,$ so $k=-7$

$$(\lambda+2)\left(\lambda^2-7\lambda+12\right)=0$$
$$(\lambda+2)(\lambda-3)(\lambda-4)=0$$

Therefore the required eigenvalues are −2, 3 and 4.

**13 b** Taking $\lambda = -2$:

$$\begin{pmatrix} 4 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:
$4x + y - z = -2x$
$6x + y - z = 0$ **(1)**

Equating middle elements:
$x + 3z = -2y$
$x + 2y + 3z = 0$ **(2)**

Choosing $z = 1$ gives equations
$6x + y = 1$
$x + 2y = -3$

Solving simultaneously gives $x = \frac{5}{11}$ and $y = -\frac{19}{11}$

Therefore one possible eigenvector is $\begin{pmatrix} \frac{5}{11} \\ -\frac{19}{11} \\ 1 \end{pmatrix}$, or by scaling, $\begin{pmatrix} 5 \\ -19 \\ 11 \end{pmatrix}$

Taking $\lambda = 4$:

$$\begin{pmatrix} 4 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:
$4x + y - z = 4x$
$y = z$ **(1)**

Equating middle elements:
$x + 3z = 4y$
$x - 4y + 3z = 0$ **(2)**

Choosing $z = 1$ gives $y = 1$ and $x = 1$

Therefore one possible eigenvector is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Taking $\lambda = 3$:

$$\begin{pmatrix} 4 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

**13 b (continued)**

Equating upper elements:
$4x+y-z=3x$
$x+y-z=0$ **(1)**

Equating middle elements:
$x+3z=3y$
$x-3y+3z=0$ **(2)**

Subtracting equation **(2)** from **(1)** gives $4y-4z=0$ , or $y=z$

Choosing $z=1$ gives $y=1$ and $x=0$

Therefore one possible eigenvector is $\begin{pmatrix}0\\1\\1\end{pmatrix}$

**c** $\mathbf{P}=\begin{pmatrix}5 & 1 & 0\\-19 & 1 & 1\\11 & 1 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}-2 & 0 & 0\\0 & 4 & 0\\0 & 0 & 3\end{pmatrix}$

**14 a** $$\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & 4 & -4\\4 & 5-\lambda & 0\\-4 & 0 & 1-\lambda\end{pmatrix}$$

$$\begin{vmatrix}3-\lambda & 4 & -4\\4 & 5-\lambda & 0\\-4 & 0 & 1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}5-\lambda & 0\\0 & 1-\lambda\end{vmatrix}-4\begin{vmatrix}4 & 0\\-4 & 1-\lambda\end{vmatrix}+(-4)\begin{vmatrix}4 & 5-\lambda\\-4 & 0\end{vmatrix}$$
$$=(3-\lambda)(5-\lambda)(1-\lambda)-16+16\lambda-80+16\lambda$$
$$=(3-\lambda)(5-\lambda)(1-\lambda)-96+32\lambda$$
$$=(3-\lambda)(5-\lambda)(1-\lambda)-32(3-\lambda)$$
$$=(3-\lambda)\big((5-\lambda)(1-\lambda)-32\big)=(3-\lambda)(\lambda^2-6\lambda-27)$$
$$=(3-\lambda)(\lambda+3)(\lambda-9)$$
$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(\lambda+3)(\lambda-9)=0\Rightarrow\lambda=3,-3,9$$
3 is an eigenvalue of **A** and the other eigenvalues are –3 and 9.

**14 b** $\begin{pmatrix} 3 & 5 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\begin{pmatrix} 3x+4y-4z \\ 4x+5y \\ -4x+z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$

Equating the middle elements

$4x+5y=3y \Rightarrow y=-2x$

Let $x=1$, then $y=-2$

Equating the lowest elements and substituting $x=1$

$-4+z=3z \Rightarrow z=-2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$

**c** The magnitudes of $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ are all

$\sqrt{\left(1^2+2^2+2^2\right)} = \sqrt{9} = 3$

Hence

$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

**15 a** $\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 4-6+0 \\ -4+3-2 \\ 0+6-5 \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix} = -1\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, is an eigenvalue of **A** corresponding to the eigenvalue 3.

$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4+2+0 \\ -4-1+2 \\ 0-2+5 \end{pmatrix} = \begin{pmatrix} 6 \\ -3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, is an eigenvalue of **A** corresponding to the eigenvalue 3.

**15 b** For $\lambda = 6$

$$\begin{pmatrix} 0 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x-2y \\ -2x+y+2z \\ 2y+5z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$2x-2y = 6x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

Equating the lowest elements and substituting $y = -2$

$-4+5z = 6z \Rightarrow z = -4$

$\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ is an eigenvalue of **A** corresponding to the eigenvalue 6.

**c** The magnitude of $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ is $\sqrt{\left(2^2+3^2+(-1)^2\right)} = \sqrt{14}$

The magnitude of $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(2^2+(-1)^2+1^2\right)} = \sqrt{6}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \\ -4 \end{pmatrix}$ is $\sqrt{\left(1^2+(-2)^2+(-4)^2\right)} = \sqrt{21}$

Hence

$$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{14}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{21}} \\ \frac{3}{\sqrt{14}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{21}} \\ -\frac{1}{\sqrt{14}} & \frac{1}{\sqrt{6}} & -\frac{4}{\sqrt{21}} \end{pmatrix}$$

**16 a** $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}\alpha-\lambda & 0 & 2\\ 4 & 3-\lambda & 0\\ -2 & -1 & 1-\lambda\end{pmatrix}$

$$\begin{vmatrix}\alpha-\lambda & 0 & 2\\ 4 & 3-\lambda & 0\\ -2 & -1 & 1-\lambda\end{vmatrix}=(\alpha-\lambda)\begin{vmatrix}3-\lambda & 0\\ -1 & 1-\lambda\end{vmatrix}-0\begin{vmatrix}4 & 0\\ -2 & 1-\lambda\end{vmatrix}+2\begin{vmatrix}4 & 3-\lambda\\ -2 & -1\end{vmatrix}$$
$$=(\alpha-\lambda)(3-\lambda)(1-\lambda)+2(-4+6-2\lambda)$$
$$=(\alpha-\lambda)(3-\lambda)(1-\lambda)+4(1-\lambda)$$
$$=(1-\lambda)\big((\alpha-\lambda)(3-\lambda)+4\big)$$

Hence, for all $\alpha$, $\lambda=1$ is a solution of $\det(\mathbf{A}-\lambda\mathbf{I})=0$, and, for all $\alpha$, an eigenvalue of **A** is 1.

**b** $\begin{pmatrix}\alpha & 0 & 2\\ 4 & 3 & 0\\ -2 & -1 & 1\end{pmatrix}\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}=\beta\begin{pmatrix}2\\ -2\\ 1\end{pmatrix}$

$$\begin{pmatrix}2\alpha+2\\ 8-6\\ -4+2+1\end{pmatrix}=\begin{pmatrix}2\beta\\ -2\beta\\ \beta\end{pmatrix}=\begin{pmatrix}2\alpha+2\\ 2\\ -1\end{pmatrix}$$

Equating the lowest elements

$\beta=-1$

Equating the top elements and substituting $\beta=-1$

$2\alpha+2=-2\Rightarrow\alpha=-2$

$$\alpha=-2,\beta=-1$$

**c** Substituting $\alpha=-2$ into * in part **a** and equating to 0

$(1-\lambda)\big((-2-\lambda)(3-\lambda)+4\big)=0$

$$(1-\lambda)(\lambda^2-\lambda-2)=(1-\lambda)(\lambda-2)(\lambda+1)$$
$$\lambda=1,2,-1$$

The third eigenvalue is 2.

**17 a** $\mathbf{M} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(2-\lambda)(3-\lambda)\right] = 0$$
$$(2-\lambda)\left(\lambda^2 - 5\lambda + 6\right) = 0$$
$$2\lambda^2 - 10\lambda + 12 - \lambda^3 + 5\lambda^2 - 6\lambda = 0$$
$$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$$
$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

Let $f(\lambda) = \lambda^3 - 7\lambda^2 + 16\lambda - 12$

$f(2) = 2^3 - 7 \times 2^2 + 16 \times 2 - 12 = 0 \Rightarrow (\lambda - 2)$ is a factor

So $f(\lambda) = (\lambda - 2)\left(\lambda^2 + k\lambda + 6\right)$

Equating coefficients of $\lambda^2$ gives

$-2 + k = -7,$ so $k = -5$

$$(\lambda-2)\left(\lambda^2 - 5\lambda + 6\right) = 0$$
$$(\lambda-2)(\lambda-2)(\lambda-3) = 0$$
$$(\lambda-2)^2(\lambda-3) = 0$$

Therefore the two distinct eigenvalues are 2 and 3.

**17 b** Taking $\lambda = 2$:

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating lower elements:

$y + 3z = 2z$

$y = -z$ **(1)**

Equating upper elements:

$2x + 2y + 2z = 2x$

$y = -z$, giving the same equation **(2)**

Choosing $z = 1$ gives $y = -1$, so one possible eigenvector is $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Choosing $z = 0$ gives $y = 0$, so another possible eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Taking $\lambda = 3$:

$$\begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating lower elements:

$y + 3z = 3z$

$y = 0$ **(1)**

Equating upper elements:

$2x + 2y + 2z = 3x$

$-x + 2z = 0$, giving the same equation **(2)**

Choosing $z = 1$ gives $x = 2$, so one possible eigenvector is $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

**18 a** $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda & -3 & 6\\ 0 & 2-\lambda & -8\\ 0 & 0 & -2-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}2-\lambda & -8\\ 0 & -2-\lambda\end{vmatrix}-(-3)\begin{vmatrix}0 & -8\\ 0 & -2-\lambda\end{vmatrix}+6\begin{vmatrix}0 & 2-\lambda\\ 0 & 0\end{vmatrix}$$

$$=(3-\lambda)(2-\lambda)(-2-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(2-\lambda)(-2-\lambda)=0\Rightarrow\lambda=-2,2,3$$

The eigenvalues are –2, 2 and 3.

**b** $\begin{pmatrix}3 & -3 & 6\\ 0 & 2 & -8\\ 0 & 0 & -2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=\begin{pmatrix}9-3\\2\\0\end{pmatrix}=\begin{pmatrix}6\\2\\0\end{pmatrix}=2\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 2.

**c** $\begin{pmatrix}7 & -6 & 2\\ 1 & 2 & 3\\ 1 & -3 & 2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix}=\begin{pmatrix}21-6\\3+2\\3-3\end{pmatrix}=\begin{pmatrix}15\\5\\0\end{pmatrix}=5\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of **B** corresponding to the eigenvalue 5.

**d** $AB\begin{pmatrix}3\\1\\0\end{pmatrix}=\mathbf{A}.\left[\mathbf{B}\begin{pmatrix}3\\1\\0\end{pmatrix}\right]=\mathbf{A}5\begin{pmatrix}3\\1\\0\end{pmatrix}=5\mathbf{A}\begin{pmatrix}3\\1\\0\end{pmatrix}=5\times2\begin{pmatrix}3\\1\\0\end{pmatrix}=10\begin{pmatrix}3\\1\\0\end{pmatrix}$

$\begin{pmatrix}3\\1\\0\end{pmatrix}$ is an eigenvector of **AB** corresponding to the eigenvalue 10.

**19 a** $\mathbf{A}=\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 2\\ 3 & 3 & -3\end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix}1-\lambda & 1 & 1\\ -1 & -\lambda & 2\\ 3 & 3 & -3-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\left[(-\lambda)(-3-\lambda)-6\right]-1\left[(-1)(-3-\lambda)-6\right]+1\left[-3+3\lambda\right]=0$$
$$(1-\lambda)\left(\lambda^2+3\lambda-6\right)+3-\lambda-3+3\lambda=0$$
$$\lambda^2+3\lambda-6-\lambda^3-3\lambda^2+6\lambda+3-\lambda-3+3\lambda=0$$
$$-\lambda^3-2\lambda^2+11\lambda-6=0$$
$$\lambda^3+2\lambda^2-11\lambda+6=0$$

**b** By Cayley-Hamilton

$$\mathbf{A}^3+2\mathbf{A}^2-11\mathbf{A}+6\mathbf{I}=\mathbf{0}$$
$$\mathbf{A}^3+2\mathbf{A}^2-11\mathbf{A}=-6\mathbf{I}$$
$$\mathbf{A}^2+2\mathbf{A}-11\mathbf{I}=-6\mathbf{A}^{-1}$$

**c** $\mathbf{A}^2+2\mathbf{A}-11\mathbf{I}=\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 2\\ 3 & 3 & -3\end{pmatrix}^2+2\begin{pmatrix}1 & 1 & 1\\ -1 & 0 & 2\\ 3 & 3 & -3\end{pmatrix}-11\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

$$=\begin{pmatrix}3 & 4 & 0\\ 5 & 5 & -7\\ -9 & -6 & 18\end{pmatrix}+\begin{pmatrix}2 & 2 & 2\\ -2 & 0 & 4\\ 6 & 6 & -6\end{pmatrix}+\begin{pmatrix}-11 & 0 & 0\\ 0 & -11 & 0\\ 0 & 0 & -11\end{pmatrix}$$
$$=\begin{pmatrix}-6 & 6 & 2\\ 3 & -6 & -3\\ -3 & 0 & 1\end{pmatrix}$$

Therefore $\mathbf{A}^{-1}=-\frac{1}{6}\begin{pmatrix}-6 & 6 & 2\\ 3 & -6 & -3\\ -3 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -1 & -\frac{1}{3}\\ -\frac{1}{2} & 1 & \frac{1}{2}\\ \frac{1}{2} & 0 & -\frac{1}{6}\end{pmatrix}$

**Challenge**

**a** $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

$\mathbf{AB} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$, so $\text{tr}(\mathbf{AB}) = ae+bg+cf+dh$

$\mathbf{BA} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$, so $\text{tr}(\mathbf{BA}) = ae+cf+bg+dh$

Therefore $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$

**b** From part a, $\text{tr}(\mathbf{P}^{-1}\mathbf{MP}) = \text{tr}(\mathbf{P}^{-1}(\mathbf{MP})) = \text{tr}((\mathbf{MP})\mathbf{P}^{-1}) = \text{tr}(\mathbf{M}(\mathbf{PP}^{-1})) = \text{tr}(\mathbf{MI}) = \text{tr}(\mathbf{M})$

So $\text{tr}(\mathbf{P}^{-1}\mathbf{MP}) = p+q \Rightarrow \text{tr}(\mathbf{M}) = p+q$ as required.