Matrix algebra 5D

1 **a** $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 3-\lambda & 4 \\ -1 & 2-\lambda \end{vmatrix} = 0$$
$$(3-\lambda)(2-\lambda)+4=0$$
$$\lambda^2-5\lambda+10=0$$

$$\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}^2 - 5\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix} + 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 5 & 20 \\ -5 & 0 \end{pmatrix} + \begin{pmatrix} -15 & -20 \\ 5 & -10 \end{pmatrix} + \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ as required.}$$

b $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} -2-\lambda & 1 \\ 3 & -\lambda \end{vmatrix} = 0$$
$$(-2-\lambda)(-\lambda)-3=0$$
$$\lambda^2+2\lambda-3=0$$

$$\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}^2 + 2\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 7 & -2 \\ -6 & 3 \end{pmatrix} + \begin{pmatrix} -4 & 2 \\ 6 & 0 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ as required.}$$

1 c $\begin{pmatrix} 7 & -4 \\ 0 & 3 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 7-\lambda & -4 \\ 0 & 3-\lambda \end{vmatrix} = 0$$
$$(7-\lambda)(3-\lambda) = 0$$
$$\lambda^2 - 10\lambda + 21 = 0$$

$$\begin{pmatrix} 7 & -4 \\ 0 & 3 \end{pmatrix}^2 - 10\begin{pmatrix} 7 & -4 \\ 0 & 3 \end{pmatrix} + 21\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 49 & -40 \\ 0 & 9 \end{pmatrix} + \begin{pmatrix} -70 & 40 \\ 0 & -30 \end{pmatrix} + \begin{pmatrix} 21 & 0 \\ 0 & 21 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ as required.}$$

2 $\mathbf{A} = \begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}$

a $\begin{vmatrix} 6-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$
$$(6-\lambda)(3-\lambda) + 2 = 0$$
$$\lambda^2 - 9\lambda + 20 = 0$$

b By Cayley-Hamilton:
$$\mathbf{A}^2 - 9\mathbf{A} + 20\mathbf{I} = 0$$
$$\mathbf{A}^2 = 9\mathbf{A} - 20\mathbf{I}$$
$$\mathbf{A}^3 = 9\mathbf{A}^2 - 20\mathbf{A}$$
$$\mathbf{A}^3 = 9(9\mathbf{A} - 20\mathbf{I}) - 20\mathbf{A}$$
$$\mathbf{A}^3 = 81\mathbf{A} - 180\mathbf{I} - 20\mathbf{A}$$
$$\mathbf{A}^3 = 61\mathbf{A} - 180\mathbf{I}$$

3 a $\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 0 & 6 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 3-\lambda & -2 \\ 0 & -\lambda \end{vmatrix} = 0$$
$$(4-\lambda)(6-\lambda) = 0$$
$$\lambda^2 - 10\lambda + 24 = 0$$

3 b By Cayley-Hamilton:

$$\mathbf{M}^2 - 10\mathbf{M} + 24\mathbf{I} = 0$$
$$24\mathbf{I} = 10\mathbf{M} - \mathbf{M}^2$$
$$24\mathbf{M}^{-1} = 10\mathbf{I} - \mathbf{M}$$
$$\mathbf{M}^{-1} = \frac{1}{24}(10\mathbf{I} - \mathbf{M})$$
$$\mathbf{M}^{-1} = \frac{1}{24}\left[\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} - \begin{pmatrix} 4 & -2 \\ 0 & 6 \end{pmatrix}\right]$$
$$\mathbf{M}^{-1} = \frac{1}{24}\begin{pmatrix} 6 & 2 \\ 0 & 4 \end{pmatrix}$$
$$\mathbf{M}^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{12} \\ 0 & \frac{1}{6} \end{pmatrix}$$

4 $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 0 & 4 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 6-\lambda & 3 \\ 0 & 4-\lambda \end{vmatrix} = 0$$
$$(6-\lambda)(4-\lambda) = 0$$
$$\lambda^2 - 10\lambda + 24 = 0$$

By Cayley-Hamilton:

$$A^2 - 10A + 24I = 0$$
$$10A = A^2 + 24I$$
$$A = \frac{1}{10}A^2 + \frac{24}{10}I$$

Therefore $p = \frac{1}{10}$ and $q = \frac{12}{5}$

5 a $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 3 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 1-\lambda & 0 & 1 \\ 2 & 2-\lambda & 2 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(3-\lambda)+2\right]+1[-2]=0$$
$$(1-\lambda)\left(\lambda^2-5\lambda+8\right)-2=0$$
$$\lambda^2-5\lambda+8-\lambda^3+5\lambda^2-8\lambda-2=0$$
$$6-13\lambda+6\lambda^2-\lambda^3=0$$
$$\lambda^3-6\lambda^2+13\lambda-6=0$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 3 \end{pmatrix}^3 - 6\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 3 \end{pmatrix}^2 + 13\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -6 & -11 \\ 10 & -8 & -46 \\ -12 & -17 & 9 \end{pmatrix} - 6\begin{pmatrix} 1 & -1 & 4 \\ 6 & 2 & 12 \\ -2 & -5 & 7 \end{pmatrix} + 13\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 3 \end{pmatrix} - 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -6 & -11 \\ 10 & -8 & -46 \\ -12 & -17 & 9 \end{pmatrix} + \begin{pmatrix} -6 & 6 & -24 \\ -36 & -12 & -72 \\ 12 & 30 & -42 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 13 \\ 26 & 13 & 26 \\ 0 & -13 & 39 \end{pmatrix} + \begin{pmatrix} -6 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ as required.}$$

5 b $\begin{pmatrix} 7 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 7-\lambda & 2 & -1 \\ 0 & -1-\lambda & 3 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)\left[(-1-\lambda)(2-\lambda)\right]-2[-3]-1\left[-1(-1-\lambda)\right]=0$$
$$(7-\lambda)\left(\lambda^2-\lambda-2\right)+6-1-\lambda=0$$
$$7\lambda^2-7\lambda-14-\lambda^3+\lambda^2+2\lambda+5-\lambda=0$$
$$-9-6\lambda+8\lambda^2-\lambda^3=0$$
$$\lambda^3-8\lambda^2+6\lambda+9=0$$

$$\begin{pmatrix} 7 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix}^3 - 8\begin{pmatrix} 7 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix}^2 + 6\begin{pmatrix} 7 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 333 & 84 & -18 \\ 24 & 5 & 6 \\ 66 & 16 & 3 \end{pmatrix} - 8\begin{pmatrix} 48 & 12 & -3 \\ 3 & 1 & 3 \\ 9 & 2 & 3 \end{pmatrix} + 6\begin{pmatrix} 7 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix} + 9\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 333 & 84 & -18 \\ 24 & 5 & 6 \\ 66 & 16 & 3 \end{pmatrix} + \begin{pmatrix} -384 & -96 & 24 \\ -24 & -8 & -24 \\ -72 & -16 & -24 \end{pmatrix} + \begin{pmatrix} 42 & 12 & -6 \\ 0 & -6 & 18 \\ 6 & 0 & 12 \end{pmatrix} + \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ as required.}$$

6 a $\mathbf{M} = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} 1-\lambda & 4 & 1 \\ 2 & -\lambda & -1 \\ 3 & 2 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[\lambda^2+2\right]-4\left[-2\lambda+3\right]+1\left[4+3\lambda\right]=0$$
$$\lambda^2+2-\lambda^3-2\lambda+8\lambda-12+4+3\lambda=0$$
$$-6+9\lambda+\lambda^2-\lambda^3=0$$
$$\lambda^3-\lambda^2-9\lambda+6=0$$
$$\lambda^3=\lambda^2+9\lambda-6 \text{ as required}$$

6 b By Cayley-Hamilton:

$\mathbf{M}^3 = \mathbf{M}^2 + 9\mathbf{M} - 6\mathbf{I}$

$\mathbf{M}^4 = \mathbf{M}^3 + 9\mathbf{M}^2 - 6\mathbf{M}$

$\mathbf{M}^4 = \left(\mathbf{M}^2 + 9\mathbf{M} - 6\mathbf{I}\right) + 9\mathbf{M}^2 - 6\mathbf{M}$

$\mathbf{M}^4 = 10\mathbf{M}^2 + 3\mathbf{M} - 6\mathbf{I}$ as required

7 a $\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 0 & 4 \\ 4 & -1 & 3 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} -1-\lambda & 1 & 1 \\ -2 & -\lambda & 4 \\ 4 & -1 & 3-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(-\lambda)(3-\lambda)+4\right] - 1\left[-2(3-\lambda)-16\right] + 1\left[2+4\lambda\right] = 0$

$(1-\lambda)\left[\lambda^2 - 3\lambda + 4\right] - 1\left[2\lambda - 22\right] + 2 + 4\lambda = 0$

$-\lambda^2 + 3\lambda - 4 - \lambda^3 + 3\lambda^2 - 4\lambda - 2\lambda + 22 + 2 + 4\lambda$

$20 - \lambda + 2\lambda^2 - \lambda^3 = 0$

$\lambda^3 - 2\lambda^2 - \lambda - 20 = 0$

b By Cayley-Hamilton:

$\mathbf{A}^3 - 2\mathbf{A}^2 - \mathbf{A} - 20\mathbf{I} = 0$

$\mathbf{A}^3 - 2\mathbf{A}^2 - \mathbf{A} = 20\mathbf{I}$

$\mathbf{A}^2 - 2\mathbf{A} - \mathbf{I} = 20\mathbf{A}^{-1}$ as required.

c $\mathbf{A}^2 - 2\mathbf{A} - \mathbf{I}$

$$= \begin{pmatrix} -1 & 1 & 1 \\ -2 & 0 & 4 \\ 4 & -1 & 3 \end{pmatrix}^2 - 2\begin{pmatrix} -1 & 1 & 1 \\ -2 & 0 & 4 \\ 4 & -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 6 \\ 18 & -6 & 10 \\ 10 & 1 & 9 \end{pmatrix} + \begin{pmatrix} 2 & -2 & -2 \\ 4 & 0 & -8 \\ -8 & 2 & -6 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & -4 & 4 \\ 22 & -7 & 2 \\ 2 & 3 & 2 \end{pmatrix} = 20\mathbf{A}^{-1}$$

Therefore $\mathbf{A}^{-1} = \dfrac{1}{20}\begin{pmatrix} 4 & -4 & 4 \\ 22 & -7 & 2 \\ 2 & 3 & 2 \end{pmatrix}$

8 $\mathbf{M} = \begin{pmatrix} -3 & 2 & -1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$

To find the characteristic equation:

$$\begin{vmatrix} -3-\lambda & 2 & -1 \\ 1 & 2-\lambda & 3 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(-2+\lambda^2)+2\lambda+6+2-\lambda=0$$
$$6\lambda-3\lambda^2+2\lambda^2-\lambda^3+2\lambda+6+2-\lambda$$
$$8+7\lambda-\lambda^2-\lambda^3=0$$
$$\lambda^3+\lambda^2-7\lambda-8=0$$

By Cayley-Hamilton:

$$\mathbf{M}^3+\mathbf{M}^2-7\mathbf{M}-8\mathbf{I}=0$$
$$7\mathbf{M}=\mathbf{M}^3+\mathbf{M}^2-8\mathbf{I}$$
$$\mathbf{M}=\frac{1}{7}\mathbf{M}^3+\frac{1}{7}\mathbf{M}^2-\frac{8}{7}\mathbf{I}$$

Therefore $a=b=\frac{1}{7}$ and $c=-\frac{8}{7}$

Challenge

$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

The characteristic equation is given by

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$
$$(a-\lambda)(d-\lambda) - bc = 0$$
$$\lambda^2 - (a+d)\lambda + ad - bc = 0$$

Now replacing λ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a^2+bc-(a+d)a+ad-bc & ab+bd-(a+d)b \\ ac+cd-(a+d)c & bc+d^2-(a+d)d+ad-bc \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Therefore the Cayley–Hamilton theorem holds for 2×2 matrices.