Matrix algebra 5C

1 a $\mathbf{A}=\begin{pmatrix}5 & 4\\3 & 6\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}5-\lambda & 4\\3 & 6-\lambda\end{vmatrix}=0$$

$$(5-\lambda)(6-\lambda)-12=0$$
$$\lambda^2-11\lambda+18=0$$
$$(\lambda-9)(\lambda-2)=0$$
$$\lambda=9 \text{ or } \lambda=2$$

If $\lambda=9$

$$\begin{pmatrix}5 & 4\\3 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=9\begin{pmatrix}x\\y\end{pmatrix}$$

Equating upper elements:

$$5x+4y=9x$$
$$y=x$$

Choosing $x=1$ gives $y=1$, giving an eigenvector of $\begin{pmatrix}1\\1\end{pmatrix}$

If $\lambda=2$

$$\begin{pmatrix}5 & 4\\3 & 6\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=2\begin{pmatrix}x\\y\end{pmatrix}$$

Equating upper elements:

$$5x+4y=2x$$
$$4y=-3x$$

Choosing $x=4$ gives $y=-3$, giving an eigenvector of $\begin{pmatrix}4\\-3\end{pmatrix}$

Therefore $\mathbf{P}=\begin{pmatrix}4 & 1\\-3 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2 & 0\\0 & 9\end{pmatrix}$

1 b $\mathbf{A} = \begin{pmatrix} -3 & -2 \\ 5 & 4 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} -3-\lambda & -2 \\ 5 & 4-\lambda \end{vmatrix} = 0$$

$$(-3-\lambda)(4-\lambda)+10=0$$
$$\lambda^2-\lambda-2=0$$
$$(\lambda+1)(\lambda-2)=0$$
$$\lambda=-1 \text{ or } \lambda=2$$

If $\lambda=-1$

$$\begin{pmatrix} -3 & -2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$$-3x-2y=-x$$
$$y=-x$$

Choosing $x=-1$ gives $y=1$, giving an eigenvector of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

If $\lambda=2$

$$\begin{pmatrix} -3 & -2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$$-3x-2y=2x$$
$$-2y=5x$$

Choosing $x=-2$ gives $y=5$, giving an eigenvector of $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$

Therefore $\mathbf{P} = \begin{pmatrix} -2 & -1 \\ 5 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$

2 **a** Using det $(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9 = 1 - 2\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = -2, 4$$

For $\lambda = -2$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$$

Equating the upper elements

$x + 3y = -2x \Rightarrow y = -x$

Let $x = 1$, then $y = 1$

An eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is $\sqrt{\left(1^2 + (-1)^2\right)} = \sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

For $\lambda = 4$

$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 4\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x+3y \\ 3x+y \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \end{pmatrix}$$

Equating the upper elements

$x + 3y = 4x \Rightarrow y = x$

Let $x = 1$, then $y = 1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(1^2 + 1^2\right)} = \sqrt{2}$.

2 a (continued)

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $\quad \mathbf{P}^{\mathrm{T}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} + \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} -1-1 & 2-2 \\ -1+1 & 2+2 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$$

b Using det $(\mathbf{A} - \lambda\mathbf{I}) = 0$

$$\begin{vmatrix} 1-\lambda & -2 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4 = 4 - 5\lambda + \lambda^2 - 4$$

$$= \lambda^2 - 5\lambda = \lambda(\lambda - 5) = 0$$

$$\lambda = 0, 5$$

For $\lambda = 5$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x - 2y \\ -2x + 4y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

Equating the upper elements

$x - 2y = 5x \Rightarrow y = -2x$

Let $x = 1$, then $y = -2$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2 + (-2)^2\right)} = \sqrt{5}$.

A normalised eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

2 b (continued)

For $\lambda = 0$

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x-2y \\ -2x+4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Equating the upper elements

$x - 2y = 0 \Rightarrow x = 2y$

Let $y = 1$, then $x = 2$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is $\sqrt{(2^2+1^2)} = \sqrt{5}$.

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}, \qquad \mathbf{P}^{\mathrm{T}} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{5}}+\frac{4}{\sqrt{5}} & \frac{2}{\sqrt{5}}-\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}}-\frac{8}{\sqrt{5}} & -\frac{4}{\sqrt{5}}+\frac{4}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} -\frac{5}{\sqrt{5}} & 0 \\ -\frac{10}{\sqrt{5}} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4 & 0 \\ 2-2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$$

3 a $\mathbf{Q}=\begin{pmatrix}4 & -2\\ -1 & 3\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}4-\lambda & -2\\ -1 & 3-\lambda\end{vmatrix}=0$$
$$(4-\lambda)(3-\lambda)-2=0$$
$$\lambda^2-7\lambda+10=0$$
$$(\lambda-5)(\lambda-2)=0$$
$$\lambda=5 \text{ or } \lambda=2$$

If $\lambda=5$

$$\begin{pmatrix}4 & -2\\ -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$$

Equating upper elements:

$$4x-2y=5x$$
$$x=-2y$$

Choosing $y=1$ gives $x=-2,$ giving an eigenvector of $\begin{pmatrix}-2\\ 1\end{pmatrix}$

If $\lambda=2$

$$\begin{pmatrix}4 & -2\\ -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=2\begin{pmatrix}x\\ y\end{pmatrix}$$

Equating upper elements:

$$4x-2y=2x$$
$$x=y$$

Choosing $x=1$ gives an eigenvector of $\begin{pmatrix}1\\ 1\end{pmatrix}$

So $\lambda=2$ has an associated normalised eigenvector of $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$

and $\lambda=5$ has an associated normalised eigenvector of $\begin{pmatrix}-\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}$

Therefore $\mathbf{P}=\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}2 & 0\\ 0 & 5\end{pmatrix}$

4 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3-\lambda & \sqrt{2}\\ \sqrt{2} & 4-\lambda\end{pmatrix}$

$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}3-\lambda & \sqrt{2}\\ \sqrt{2} & 4-\lambda\end{vmatrix}=(3-\lambda)(4-\lambda)-2=12-7\lambda+\lambda^2-2$

$=\lambda^2-7\lambda+10=(\lambda-2)(\lambda-5)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(\lambda-2)(\lambda-5)=0$

The eigenvalues of **A** are 2 and 5.

b For $\lambda=2$

$\begin{pmatrix}3 & \sqrt{2}\\ \sqrt{2} & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=2\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}3x+\sqrt{2}y\\ \sqrt{2}x+4y\end{pmatrix}=\begin{pmatrix}2x\\ 2y\end{pmatrix}$

Equating the upper elements

$3x+\sqrt{2}y=2x\Rightarrow x=-\sqrt{2}y$

Let $y=1$, then $x=-\sqrt{2}$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}$

The magnitude of $\begin{pmatrix}-\sqrt{2}\\ 1\end{pmatrix}$ is $\sqrt{\left(\left(-\sqrt{2}\right)^2+1^2\right)}=\sqrt{3}$.

A normalised eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}-\frac{\sqrt{2}}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$

For $\lambda=5$

$\begin{pmatrix}3 & \sqrt{2}\\ \sqrt{2} & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=5\begin{pmatrix}x\\ y\end{pmatrix}$

$\begin{pmatrix}3x+\sqrt{2}y\\ \sqrt{2}x+4y\end{pmatrix}=\begin{pmatrix}5x\\ 5y\end{pmatrix}$

Equating the upper elements

$3x+\sqrt{2}y=5x\Rightarrow\sqrt{2}y=2x\Rightarrow y=\sqrt{2}x$

Let $x=1$, then $y=\sqrt{2}$

4 b (continued)

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$

The magnitude of $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$ is $\sqrt{\left(1^2 + \left(\sqrt{2}\right)^2\right)} = \sqrt{3}$.

A normalised eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$

c $\mathbf{P} = \begin{pmatrix} -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}} \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

5 a $\mathbf{A} = \begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}$

$\begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \end{pmatrix} = 9\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ Therefore $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of A

$\begin{pmatrix} 7 & 4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -1\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ Therefore $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is an eigenvector of A

b Adam is incorrect, since the eigenvectors are not normalised.

Normalising $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ gives $\begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$

Normalising $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ gives $\begin{pmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

Therefore $\mathbf{Q} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$

6 a $\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(1-\lambda)-12=0$
$\lambda^2 - 3\lambda - 10 = 0$
$(\lambda-5)(\lambda+2)=0$
$\lambda = 5$ or $\lambda = -2$

If $\lambda = 5$

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$2x + 4y = 5x$
$4y = 3x$

Choosing $x = 4$ gives $y = 3$, giving an eigenvector of $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

If $\lambda = -2$

$$\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$2x + 4y = -2x$
$y = -x$

Choosing $y = 1$ gives an eigenvector of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b Therefore $P = \begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$

6 c

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$$
$$= \mathbf{P}^{-1}\mathbf{A}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
$$= \mathbf{P}^{-1}\mathbf{APP}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
$$= \mathbf{DP}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
$$= \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix} \text{ as required}$$

d $u_{n+1} = -2u_n$ so $u_{n+1} = u_1 \times (-2)^n$

$v_{n+1} = 5v_n$ so $v_{n+1} = v_1 \times 5^n$

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = P^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \text{ so } \begin{pmatrix} x_n \\ y_n \end{pmatrix} = P\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} -1 & 4 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$$

$$x_n = -u_n + 4v_n = -u_1 \times (-2)^{n-1} + 4v_1 \times 5^{n-1}$$
$$y_n = u_n + 3v_n = u_1 \times (-2)^{n-1} + 3v_1 \times 5^{n-1}$$

When $n = 1$, $3 = -u_1 + 4v_1$

$1 = u_1 + 3v_1$

Solving simultaneously gives $u_1 = -\frac{5}{7}$ and $v_1 = \frac{4}{7}$

Therefore $x_n = \frac{5}{7}\times(-2)^{n-1} + \frac{16}{7}\times 5^{n-1}$ and $y_n = -\frac{5}{7}\times(-2)^{n-1} + \frac{12}{7}\times 5^{n-1}$

7 a $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-\lambda)+2=0$$
$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda-2)(\lambda-1)=0$$
$$\lambda = 1 \text{ or } \lambda = 2$$

If $\lambda = 1$

$$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating lower elements:

$$2x = y$$

Choosing $x = 1$ gives $y = 2$, giving an eigenvector of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

If $\lambda = 2$

$$\begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating lower elements:

$$2x = 2y$$
$$y = x$$

Choosing $x = 1$ gives an eigenvector of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Therefore $P = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

7 b $\mathbf{D}^{100} = \left(\mathbf{P}^{-1}\mathbf{MP}\right)^{100} = \mathbf{P}^{-1}\mathbf{M}^{100}\mathbf{P}$

Therefore $\mathbf{M}^{100} = \mathbf{PD}^{100}\mathbf{P}^{-1}$

Using a calculator gives $\mathbf{P}^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Therefore $\mathbf{M}^{100} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2^{101} & -2^{100} \end{pmatrix}$

$= \begin{pmatrix} 2^{101}-1 & 1-2^{100} \\ 2^{101}-2 & 2-2^{100} \end{pmatrix}$

8 a $\mathbf{A} = \begin{pmatrix} 1 & 4 & -1 \\ -1 & 6 & -1 \\ 2 & -2 & 4 \end{pmatrix}$

To find the eigenvalues:

$\begin{vmatrix} 1-\lambda & 4 & -1 \\ -1 & 6-\lambda & -1 \\ 2 & -2 & 4-\lambda \end{vmatrix} = 0$

$(1-\lambda)\left[(6-\lambda)(4-\lambda)-2\right]-4\left[-1(4-\lambda)+2\right]-1\left[2-2(6-\lambda)\right]=0$

$(1-\lambda)\left(\lambda^2-10\lambda+22\right)-4(-2+\lambda)-(-10+2\lambda)=0$

$\lambda^2-10\lambda+22-\lambda^3+10\lambda^2-22\lambda+8-4\lambda+10-2\lambda=0$

$-\lambda^3+11\lambda^2-38\lambda+40=0$

$\lambda^3-11\lambda^2+38\lambda-40=0$

Let $f(\lambda)=\lambda^3-11\lambda^2+38\lambda-40$

$f(2)=2^3-11\times2^2+38\times2-40=0 \Rightarrow (\lambda-2)$ is a factor

So $f(\lambda)=(\lambda-2)\left(\lambda^2+k\lambda+20\right)$

Equating coefficients of λ^2 gives

$-2+k=-11$, so $k=-9$

$(\lambda-2)\left(\lambda^2-9\lambda+20\right)=0$

$(\lambda-2)(\lambda-4)(\lambda-5)=0$

Therefore the required eigenvalues are 2, 4 and 5.

Taking $\lambda=2$:

$\begin{pmatrix} 1 & 4 & -1 \\ -1 & 6 & -1 \\ 2 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

8 a (continued)

Equating upper elements:
$x + 4y - z = 2x$
$x - 4y + z = 0$ **(1)**

Equating lower elements:
$2x - 2y + 4z = 2z$
$2x - 2y + 2z = 0$
$x - y + z = 0$ **(2)**

(1)– (2) gives $-3y = 0$, so $y = 0$ and $x = -z$

Therefore choosing $x = 1$ gives an eigenvector of $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Taking $\lambda = 4$:
$$\begin{pmatrix} 1 & 4 & -1 \\ -1 & 6 & -1 \\ 2 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:
$x + 4y - z = 4x$
$3x - 4y + z = 0$ **(1)**

Equating lower elements:
$2x - 2y + 4z = 4z$
$2x - 2y = 0$
$x = y$ **(2)**

Substituting into (1) gives $x = z$

Therefore choosing $z = 1$ gives an eigenvector of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Taking $\lambda = 5$:
$$\begin{pmatrix} 1 & 4 & -1 \\ -1 & 6 & -1 \\ 2 & -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:
$x + 4y - z = 5x$
$4x - 4y + z = 0$ **(1)**

Equating lower elements:
$2x - 2y + 4z = 5z$
$2x - 2y - z = 0$ **(2)**
(1)+(2) gives $6x - 6y = 0$, so $x = y$

8 a (continued)

Therefore choosing $x=1$ gives $y=1$ and $z=0$, so an eigenvector is $\begin{pmatrix}1\\1\\0\end{pmatrix}$

Therefore $P=\begin{pmatrix}1&1&1\\0&1&1\\-1&1&0\end{pmatrix}$ and $D=\begin{pmatrix}2&0&0\\0&4&0\\0&0&5\end{pmatrix}$

b $\mathbf{A}=\begin{pmatrix}1&2&1\\6&-1&0\\-1&-2&-1\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}1-\lambda&2&1\\6&-1-\lambda&0\\-1&-2&-1-\lambda\end{vmatrix}=0$$

$$(1-\lambda)\left[(-1-\lambda)(-1-\lambda)-0\right]-2\left[6(-1-\lambda)-0\right]+1\left[-12+(-1-\lambda)\right]=0$$
$$(1-\lambda)\left(\lambda^2+2\lambda+1\right)-12(-1-\lambda)+(-13-\lambda)=0$$
$$\lambda^2+2\lambda+1-\lambda^3-2\lambda^2-\lambda+12+12\lambda-13-\lambda=0$$
$$-\lambda^3+\lambda^2+12\lambda=0$$
$$\lambda^3+\lambda^2-12\lambda=0$$
$$\lambda\left(\lambda^2+\lambda-12\right)=0$$
$$\lambda(\lambda+4)(\lambda-3)=0$$

Therefore the required eigenvalues are $-4, 0$ and 3.

Taking $\lambda=-4$:

$$\begin{pmatrix}1&2&1\\6&-1&0\\-1&-2&-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-4\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating middle elements:

$$6x-y=-4y$$
$$6x=-3y$$
$$y=-2x$$

Equating upper elements:

$$x+2y+z=-4x$$
$$5x+2y-z=0$$

Therefore choosing $x=-1$ gives $y=2$ and $z=1$, so an eigenvector is $\begin{pmatrix}-1\\2\\1\end{pmatrix}$

8 b (continued)

Taking $\lambda = 0$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:

$6x - y = 0$

$y = 6x$

Equating upper elements:

$x + 2y + z = 0$

Therefore choosing $x = -1$ gives $y = -6$ and $z = 13$, so an eigenvector is $\begin{pmatrix} -1 \\ -6 \\ 13 \end{pmatrix}$

Taking $\lambda = 3$:

$$\begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:

$6x - y = 3y$

$6x = 4y$

$3x = 2y$

Equating upper elements:

$x + 2y + z = 3x$

$-2x + 2y + z = 0$

Therefore choosing $x = -2$ gives $y = -3$ and $z = 2$, so an eigenvector is $\begin{pmatrix} -2 \\ -3 \\ 2 \end{pmatrix}$

Therefore $P = \begin{pmatrix} -1 & -1 & -2 \\ 2 & -6 & -3 \\ 1 & 13 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

9 $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 0 \end{pmatrix}$

a $\mathbf{M}$ is not a symmetric matrix, therefore it is not orthogonally diagonalisable.

b To find the eigenvalues:

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 3 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(1-\lambda)(-\lambda)-6\right]=0$$
$$(1-\lambda)\left(-6-\lambda+\lambda^2\right)=0$$
$$-6-\lambda+\lambda^2+6\lambda+\lambda^2-\lambda^3=0$$
$$-6+5\lambda+2\lambda^2-\lambda^3=0$$

$$\lambda^3-2\lambda^2-5\lambda+6=0$$

Let $f(\lambda)=\lambda^3-2\lambda^2-5\lambda+6$

$f(1)=0 \Rightarrow (\lambda-1)$ is a factor

So $f(\lambda)=(\lambda-1)\left(\lambda^2+k\lambda-6\right)$

Equating coefficients of λ^2 gives

$-1+k=-2$, so $k=-1$

$$(\lambda-1)\left(\lambda^2-\lambda-6\right)=0$$
$$(\lambda-1)(\lambda+2)(\lambda-3)=0$$

Therefore the required eigenvalues are −2, 1 and 3.

9 c Taking $\lambda = 1$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:

$y + 2z = y$

$z = 0$

Equating lower elements:

$3y = z$

$y = 0$

Therefore one possible eigenvector is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Taking $\lambda = -2$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:

$y + 2z = -2y$

$3y + 2z = 0$

Equating upper elements:

$x + 2y + 3z = -2x$

$3x + 2y + 3z = 0$

Choosing $z = 1$ gives $y = -\dfrac{2}{3}$ and $x = -\dfrac{5}{9}$

Multiplying each value by 9 gives an eigenvector of $\begin{pmatrix} -5 \\ -6 \\ 9 \end{pmatrix}$

Taking $\lambda = 3$:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:

$y + 2z = 3y$

$y = z$

Equating upper elements:

$x + 2y + 3z = 3x$

$2y + 3z = 2x$

Choosing $z = 1$ gives $y = 1$ and $x = \dfrac{5}{2}$

Multiplying each value by 2 gives an eigenvector of $\begin{pmatrix} 5 \\ 5 \\ 2 \end{pmatrix}$

9 d Forming a matrix of eigenvectors gives $P=\begin{pmatrix}1 & -5 & 5\\ 0 & -6 & 5\\ 0 & 9 & 2\end{pmatrix}$

10 a $\mathbf{PP}^{\mathrm{T}}=\begin{pmatrix}\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}$

$$=\begin{pmatrix}\frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{2}{6}-\frac{1}{3}\\ \frac{1}{6}+\frac{1}{3}-\frac{1}{2} & \frac{1}{6}+\frac{1}{3}+\frac{1}{2} & \frac{2}{6}-\frac{1}{3}\\ \frac{2}{6}-\frac{1}{3} & \frac{2}{6}-\frac{1}{3} & \frac{4}{6}+\frac{1}{3}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\mathbf{I}$$

Hence **P** is an orthogonal matrix.

10 b $\mathbf{P}^{\mathrm{T}}\mathbf{AP}=\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}\frac{3}{2} & -\frac{3}{2} & 1\\ -\frac{3}{2} & \frac{3}{2} & 1\\ 1 & 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0\end{pmatrix}$

$$=\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}\frac{3}{2\sqrt{6}}-\frac{3}{2\sqrt{6}}+\frac{2}{\sqrt{6}} & -\frac{3}{2\sqrt{3}}+\frac{3}{2\sqrt{3}}+\frac{1}{\sqrt{3}} & \frac{3}{2\sqrt{2}}+\frac{3}{2\sqrt{2}}\\ -\frac{3}{2\sqrt{6}}+\frac{3}{2\sqrt{6}}+\frac{2}{\sqrt{6}} & \frac{3}{2\sqrt{3}}-\frac{3}{2\sqrt{3}}+\frac{1}{\sqrt{3}} & -\frac{3}{2\sqrt{2}}-\frac{3}{2\sqrt{2}}\\ \frac{1}{\sqrt{6}}+\frac{1}{\sqrt{6}}+\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}\begin{pmatrix}\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{3}{\sqrt{2}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{3}{\sqrt{2}}\\ \frac{4}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & 0\end{pmatrix}=\begin{pmatrix}\frac{2}{6}+\frac{2}{6}+\frac{8}{6} & \frac{1}{\sqrt{18}}+\frac{1}{\sqrt{18}}-\frac{2}{\sqrt{18}} & -\frac{3}{\sqrt{12}}-\frac{3}{\sqrt{12}}\\ -\frac{2}{18}-\frac{2}{18}+\frac{4}{\sqrt{18}} & -\frac{1}{3}-\frac{1}{3}-\frac{1}{3} & -\frac{3}{\sqrt{6}}+\frac{3}{\sqrt{6}}\\ \frac{2}{\sqrt{12}}-\frac{2}{\sqrt{12}} & \frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}} & \frac{3}{2}+\frac{3}{2}\end{pmatrix}$$

$$=\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{pmatrix}$$, a diagonal matrix.

11 $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & 2-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 0 & 2\\ 0 & 2-\lambda & 0\\ 2 & 0 & 2-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda & 0\\ 0 & 2-\lambda\end{vmatrix}-0\begin{vmatrix}0 & 0\\ 2 & 2-\lambda\end{vmatrix}+2\begin{vmatrix}0 & 2-\lambda\\ 2 & 0\end{vmatrix}$$

$$=(2-\lambda)^3-4(2-\lambda)=(2-\lambda)\left((2-\lambda)^2-4\right)=(2-\lambda)(-\lambda)(4-\lambda)$$

$$=-\lambda(2-\lambda)(4-\lambda)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-\lambda(\lambda-2)(\lambda-4)=0\Rightarrow\lambda=0,2,4$

For $\lambda=0$

$$\begin{pmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=0\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x+2z\\ 2y\\ 2x+2z\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

Equating the top elements

$2x+2z=0\Rightarrow z=-x$

Let $x=1$, then $z=-1$

Equating the middle elements

$2y=0\Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$

The magnitude of $\begin{pmatrix}1\\ 0\\ -1\end{pmatrix}$ is $\sqrt{\left(1^2+0^2+(-1)^2\right)}=\sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{pmatrix}$

For $\lambda=2$

$$\begin{pmatrix}2 & 0 & 2\\ 0 & 2 & 0\\ 2 & 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x+2z\\ 2y\\ 2x+2z\end{pmatrix}=\begin{pmatrix}2x\\ 2y\\ 2z\end{pmatrix}$$

Equating the top elements

$2x+2z=2x\Rightarrow z=0$

11 (continued)

Equating the lowest elements
$2x+2z=2z \Rightarrow x=0$
y can any value
Let $y=1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}0\\1\\0\end{pmatrix}$

The magnitude of this vector is 1, so it is already normalised.
For $\lambda=4$

$$\begin{pmatrix}2&0&2\\0&2&0\\2&0&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=4\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}2x+2z\\2y\\2x+2z\end{pmatrix}=\begin{pmatrix}4x\\4y\\4z\end{pmatrix}$$

Equating the top elements
$2x+2z=4x \Rightarrow z=x$
Let $x=1$, then $z=1$
Equating the middle elements
$2y=4y \Rightarrow 2y=0 \Rightarrow y=0$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}1\\0\\1\end{pmatrix}$

The magnitude of $\begin{pmatrix}1\\0\\1\end{pmatrix}$ is $\sqrt{\left(1^2+0^2+1^2\right)}=\sqrt{2}$

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$

11 (continued)

Let $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

Then $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \\ 0 & 2 & 0 \\ \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} & 0 & \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{4}{\sqrt{2}} \\ 0 & 2 & 0 \\ 0 & 0 & \frac{4}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 2-2 \\ 0 & 2 & 0 \\ 0 & 0 & 2+2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

12 a For $\lambda = 0$

$$\begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 5x+3y+2z \\ 3x+y+z \\ 3x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Equating the top elements

$5x+3y+3z=0$ **(1)**

Equating the middle elements

$3x+y+z=0$ **(2)**

$3\times$**(2)**$-$**(1)**

$x=0$

Substituting $x=0$ into **(2)**

$y+z=0 \Rightarrow z=-y$

Let $y=1$, then $z=-1$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

The magnitude of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is $\sqrt{\left(0^2+1^2+(-1)^2\right)} = \sqrt{2}$

A normalised eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

12 b The magnitude of $\begin{pmatrix}-1\\1\\1\end{pmatrix}$ is $\sqrt{\left((-1)^2+1^2+1^2\right)}=\sqrt{3}$

A normalised eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix}-\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\\\frac{1}{\sqrt{3}}\end{pmatrix}$

The magnitude of $\begin{pmatrix}2\\1\\1\end{pmatrix}$ is $\sqrt{\left(2^2+1^2+1^2\right)}=\sqrt{6}$

A normalised eigenvector corresponding to the eigenvalue 8 is $\begin{pmatrix}\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix}$

$$\mathbf{P}=\begin{pmatrix}0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{pmatrix},\quad \mathbf{D}=\begin{pmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 8\end{pmatrix}$$

13 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}7-\lambda & 0 & -2\\ 0 & 5-\lambda & -2\\ -2 & -2 & 6-\lambda\end{vmatrix}$$

$$=(7-\lambda)\begin{vmatrix}5-\lambda & -2\\ -2 & 6-\lambda\end{vmatrix}-0\begin{vmatrix}0 & -2\\ -2 & 6-\lambda\end{vmatrix}+(-2)\begin{vmatrix}0 & 5-\lambda\\ -2 & -2\end{vmatrix}$$

$$=(7-\lambda)\big((5-\lambda)(6-\lambda)-4\big)-2(10-2\lambda)$$

$$=(7-\lambda)(26-11\lambda+\lambda^2)-20+4\lambda$$

$$=182-103\lambda+18\lambda^2-\lambda^3-20+4\lambda=-(\lambda^3-18\lambda^2+99\lambda-162)$$

Let $\lambda^3-18\lambda^2+99\lambda-162=(\lambda-9)(\lambda^2+k\lambda+18)$

Equating coefficients of λ^2

$-18=-9+k\Rightarrow k=-9$

Hence

$\lambda^3-18\lambda^2+99\lambda-162=(\lambda-9)(\lambda^2-9\lambda+18)=(\lambda-9)(\lambda-6)(\lambda-3)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-3)(\lambda-6)(\lambda-9)=0\Rightarrow\lambda=3,6,9$

The other two eigenvalues of **A** are 3 and 6.

13 b For $\lambda = 3$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the top elements

$7x-2z=3x \Rightarrow z=2x$

Let $x=1$, then $z=2$

Equating the middle elements and substituting $z=2$

$5y-4=3y \Rightarrow y=2$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

For $\lambda = 6$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$7x-2z=6x \Rightarrow x=2z$

Let $z=1$, then $x=2$

Equating the middle elements and substituting $z=1$

$5y-2=6y \Rightarrow y=-2$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

For $\lambda = 9$

$$\begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 9\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x-2z \\ 5y-2z \\ -2x-2y+6z \end{pmatrix} = \begin{pmatrix} 9x \\ 9y \\ 9z \end{pmatrix}$$

13 b (continued)

Equating the top elements
$7x - 2z = 9x \Rightarrow z = -x$
Let $x = 2$, then $z = -2$
Equating the middle elements and substituting $z = -2$
$5y + 4 = 9y \Rightarrow y = 1$

An eigenvector corresponding to the eigenvalue 9 is $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

c The eigenvectors are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 2 - 4 + 2 = 0$$

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 2 + 2 - 4 = 0$$

$$\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} . \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = 4 - 2 - 2 = 0$$

Therefore the eigenvectors are mutually orthogonal.

d The magnitudes of the vector $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ are all $\sqrt{\left(1^2 + 2^2 + 2^2\right)} = \sqrt{9} = 3$

$$\mathbf{P} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

14 a $\det(\mathbf{A}-\lambda\mathbf{I}) = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & \sqrt{5} \\ 0 & \sqrt{5} & 1-\lambda \end{vmatrix}$

Substituting $\lambda = 4$,

$$\begin{vmatrix} 1-4 & 2 & 0 \\ 2 & 1-4 & \sqrt{5} \\ 0 & \sqrt{5} & 1-4 \end{vmatrix} = \begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & \sqrt{5} \\ 0 & \sqrt{5} & -3 \end{vmatrix} = (-3)\begin{vmatrix} -3 & \sqrt{5} \\ \sqrt{5} & -3 \end{vmatrix} - 2\begin{vmatrix} 2 & \sqrt{5} \\ 0 & -3 \end{vmatrix} + 0\begin{vmatrix} 2 & -3 \\ 0 & \sqrt{5} \end{vmatrix}$$

$$= (-3)(9-5) - 2(-6-0) = -12 + 12 = 0$$

Hence, by the factor theorem, 4 is an eigenvalue of **A**.

$$\begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & \sqrt{5} \\ 0 & \sqrt{5} & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & \sqrt{5} \\ \sqrt{5} & 1-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & \sqrt{5} \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 1-\lambda \\ 0 & \sqrt{5} \end{vmatrix}$$

$$= (1-\lambda)\left((1-\lambda)^2 - 5\right) - 4 + 4\lambda$$

$$= (1-\lambda)(\lambda^2 - 2\lambda - 4) - 4 + 4\lambda = -\lambda^3 + 3\lambda^2 + 6\lambda - 8$$

$$= -\lambda^3 + 4\lambda^2 - \lambda^2 + 4\lambda + 2\lambda - 8 = -\lambda^2(\lambda-4) - \lambda(\lambda-4) + 2(\lambda-4)$$

$$= -(\lambda-4)(\lambda^2 + \lambda - 2) = -(\lambda-4)(\lambda+2)(\lambda-1)$$

$$\det(\mathbf{A}-\lambda\mathbf{I}) = -(\lambda-4)(\lambda+2)(\lambda-1) = 0 \Rightarrow \lambda = 4, -2, 1$$

The other two eigenvalue of **A** are –2 and 1.

14 b For $\lambda = 4$

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+2y \\ 2x+y+\sqrt{5}z \\ \sqrt{5}y+z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$x+2y = 4x \Rightarrow 2y = 3x$

Let $x = 2$, then $y = 3$

Equating the lowest elements and substituting $y = 3$

$3\sqrt{5} + z = 4z \Rightarrow z = \sqrt{5}$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 2 \\ 3 \\ \sqrt{5} \end{pmatrix}$

The magnitude of $\begin{pmatrix} 2 \\ 3 \\ \sqrt{5} \end{pmatrix}$ is $\sqrt{\left(2^2 + 3^2 + (\sqrt{5})^2\right)} = \sqrt{18}$

A normalised eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} \frac{2}{\sqrt{18}} \\ \frac{3}{\sqrt{18}} \\ \frac{\sqrt{5}}{\sqrt{18}} \end{pmatrix}$

14 c $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix} = \begin{pmatrix} -2+6 \\ -4+3-5 \\ 3\sqrt{5}-\sqrt{5} \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 2\sqrt{5} \end{pmatrix} = (-2)\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$

An eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$

The magnitude of $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ is $\sqrt{\left((-2)^2+3^2+(-\sqrt{5})^2\right)} = \sqrt{18}$

A normalised eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} -\frac{2}{\sqrt{18}} \\ \frac{3}{\sqrt{18}} \\ -\frac{5}{\sqrt{18}} \end{pmatrix}$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$

The magnitude of $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ is $\sqrt{\left(\left(\sqrt{5}\right)^2+0^2+2^2\right)} = \sqrt{9} = 3$

A normalised eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \frac{\sqrt{5}}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix}$

$\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{18}} & -\frac{2}{\sqrt{18}} & \frac{\sqrt{5}}{3} \\ \frac{3}{\sqrt{18}} & \frac{3}{\sqrt{18}} & 0 \\ \frac{\sqrt{5}}{\sqrt{18}} & -\frac{\sqrt{5}}{\sqrt{18}} & -\frac{2}{3} \end{pmatrix}$, $\mathbf{D} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

15 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2-\lambda & 2 & -3\\ 2 & 2-\lambda & 3\\ -3 & 3 & 3-\lambda\end{pmatrix}$

$$\det(\mathbf{A}-\lambda\mathbf{I})=\begin{vmatrix}2-\lambda & 2 & -3\\ 2 & 2-\lambda & 3\\ -3 & 3 & 3-\lambda\end{vmatrix}$$

$$=(2-\lambda)\begin{vmatrix}2-\lambda & 3\\ 3 & 3-\lambda\end{vmatrix}-2\begin{vmatrix}2 & 3\\ -3 & 3-\lambda\end{vmatrix}+(-3)\begin{vmatrix}2 & 2-\lambda\\ -3 & 3\end{vmatrix}$$

$$=(2-\lambda)\big((2-\lambda)(3-\lambda)-9\big)-2(6-2\lambda+9)-3(6+6-3\lambda)$$

$$=(2-\lambda)(\lambda^2-5\lambda-3)-30+4\lambda-36+9\lambda$$

$$=-\lambda^3+7\lambda^2-7\lambda-6-66+13\lambda=-\lambda^3+7\lambda^2+6\lambda-72$$

$$=-\lambda^3+6\lambda^2+-\lambda^2-6\lambda+12\lambda-72$$

$$=-\lambda^2(\lambda-6)+\lambda(\lambda-6)+12(\lambda-6)=-(\lambda-6)(\lambda^2-\lambda-12)$$

$$=-(\lambda-6)(\lambda-4)(\lambda+3)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-6)(\lambda-4)(\lambda+3)=0\Rightarrow\lambda=6,4,-3$$

As $\lambda_1>\lambda_2>\lambda_3, \lambda_1=6$ as required, $\lambda_2=4$ and $\lambda_3=-3$.

b $\det(\mathbf{A})=\begin{vmatrix}2 & 2 & -3\\ 2 & 2 & 3\\ -3 & 3 & 3\end{vmatrix}=2\begin{vmatrix}2 & 3\\ 3 & 3\end{vmatrix}-2\begin{vmatrix}2 & 3\\ -3 & 3\end{vmatrix}+(-3)\begin{vmatrix}2 & 2\\ -3 & 3\end{vmatrix}$

$$=2(6-9)-2(6+9)-3(6+6)=-6-30-36$$

$$=-72=6\times4\times(-3)=\lambda_1\lambda_2\lambda_3,\ \text{as required.}$$

15 c For $\lambda = 6$

$$\begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y-3z \\ 2x+2y+3z \\ -3x+3y+3z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$2x+2y-3z = 6x \Rightarrow -4x+2y-3z = 0$ **(1)**

Equating the middle elements

$2x+2y+3z = 6y \Rightarrow 2x-4y+3z = 0$ **(2)**

(1) + **(2)**

$-2x-2y = 0 \Rightarrow y = -x$

Let $x = 1$, then $y = -1$

Substitute $x = 1$ and $y = -1$ into **(1)**

$-4-2-3z = 0 \Rightarrow z = -2$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

d The magnitude of $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2+(-1)^2+(-2)^2\right)} = \sqrt{6}$

The magnitude of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is $\sqrt{\left(1^2+1^2+0^2\right)} = \sqrt{2}$

The magnitude of $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is $\sqrt{\left(1^2+(-1)^2+1^2\right)} = \sqrt{3}$

Hence $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$

Challenge

a $\mathbf{A} = \begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 0.3-\lambda & -0.2 \\ -0.1 & 0.4-\lambda \end{vmatrix} = 0$$

$$(0.3-\lambda)(0.4-\lambda) - 0.02 = 0$$
$$\lambda^2 - 0.7\lambda + 0.10 = 0$$
$$(\lambda - 0.2)(\lambda - 0.5) = 0$$

So the eigenvalues are 0.2 and 0.5

Taking $\lambda = 0.2$

$$\begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.2\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$$0.3x - 0.2y = 0.2x$$
$$0.1x - 0.2y = 0$$
$$x = 2y$$

Choosing $y = 1$ gives an eigenvector of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Taking $\lambda = 0.5$

$$\begin{pmatrix} 0.3 & -0.2 \\ -0.1 & 0.4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.5\begin{pmatrix} x \\ y \end{pmatrix}$$

Equating upper elements:

$$0.3x - 0.2y = 0.5x$$
$$-0.2x - 0.2y = 0$$
$$x = -y$$

Choosing $y = 1$ gives an eigenvector of $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Challenge

b Therefore $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}$

c $\begin{pmatrix} u' \\ v' \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \mathbf{P}^{-1}\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{D}\mathbf{P}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{D}\begin{pmatrix} u \\ v \end{pmatrix}$

d $\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0.5u \\ 0.2v \end{pmatrix}$

Equating the top elements gives $u' = 0.5u$

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{u}{2}$$
$$\int \frac{\mathrm{d}u}{u} = \int \frac{\mathrm{d}t}{2}$$
$$\ln u = \frac{t}{2} + \ln c_1$$
$$\ln u - \ln c_1 = \frac{t}{2}$$
$$\ln\left(\frac{u}{c_1}\right) = \frac{t}{2}$$
$$\frac{u}{c_1} = e^{\frac{t}{2}}$$
$$u = c_1 e^{\frac{t}{2}}$$

Equating the lower elements gives $v' = 0.2v$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{v}{5}$$
$$\int \frac{\mathrm{d}v}{v} = \int \frac{\mathrm{d}t}{5}$$
$$\ln v = \frac{t}{5} + \ln c_2$$
$$\ln v - \ln c_2 = \frac{t}{5}$$
$$\ln\left(\frac{v}{c_2}\right) = \frac{t}{5}$$
$$\frac{v}{c_2} = e^{\frac{t}{5}}$$
$$v = c_2 e^{\frac{t}{5}}$$

Challenge

e From a calculator, $\mathbf{P}^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$

When $t = 0$, $x = 5$ and $y = 20$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So $u = -\frac{x}{3} + \frac{2y}{3}$ and $v = \frac{x}{3} + \frac{y}{3}$

So when $t = 0$, $u = -\frac{5}{3} + \frac{40}{3} = \frac{35}{3}$ and $v = \frac{5}{3} + \frac{20}{3} = \frac{25}{3}$

Therefore $u = \frac{35e^{\frac{t}{2}}}{3}$ and $v = \frac{25e^{\frac{t}{5}}}{3}$

Now $\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{P}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{35e^{\frac{t}{2}}}{3} \\ \frac{25e^{\frac{t}{5}}}{3} \end{pmatrix}$

Therefore the solutions are:

$$x = -\frac{35e^{\frac{t}{2}}}{3} + \frac{50e^{\frac{t}{5}}}{3}$$

$$y = \frac{35e^{\frac{t}{2}}}{3} + \frac{25e^{\frac{t}{5}}}{3}$$