Matrix algebra 5B

1 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}3-\lambda&0&0\\2&4-\lambda&2\\-2&0&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}3-\lambda&0&0\\2&4-\lambda&2\\-2&0&1-\lambda\end{vmatrix}=(3-\lambda)\begin{vmatrix}4-\lambda&2\\0&1-\lambda\end{vmatrix}-0\begin{vmatrix}2&2\\-2&1-\lambda\end{vmatrix}+0\begin{vmatrix}2&4-\lambda\\-2&0\end{vmatrix}$$

$$=(3-\lambda)(4-\lambda)(1-\lambda)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(3-\lambda)(4-\lambda)(1-\lambda)=0\Rightarrow\lambda=3,4,1$$

The eigenvalues are 1, 3 and 4.

For $\lambda=1$

$$\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3x\\2x+4y+2z\\-2x+z\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating the top elements

$3x=x\Rightarrow x=0$

Equating the middle elements and substituting $x=0$

$0+4y+2z=y\Rightarrow 3y=-2z$

Let $z=3$, then $y=-2$

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix}0\\-2\\3\end{pmatrix}$.

For $\lambda=3$

$$\begin{pmatrix}3&0&0\\2&4&2\\-2&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}3x\\2x+4y+2z\\-2x+z\end{pmatrix}=\begin{pmatrix}3x\\3y\\3z\end{pmatrix}$$

Equating the lowest elements

$-2x+z=3z\Rightarrow z=-x$

Let $x=1$, then $z=-1$

1 a (continued)

Equating the middle elements and substituting $x = 1$ and $z = -1$

$2 + 4y - 2 = 3y \Rightarrow y = 0$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

For $\lambda = 4$

$$\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 3x \\ 2x + 4y + 2z \\ -2x + z \end{pmatrix} = \begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the top elements

$3x = 4x \Rightarrow x = 0$

Equating the lowest elements and substituting $x = 0$

$0 + z = 4z \Rightarrow z = 0$

As y can take any non-zero value, let $y = 1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

1 b $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4&-2&-4\\2&3&0\\2&-5&-4\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}4-\lambda&-2&-4\\2&3-\lambda&0\\2&-5&-4-\lambda\end{pmatrix}$

$$\begin{vmatrix}4-\lambda&-2&-4\\2&3-\lambda&0\\2&-5&-4-\lambda\end{vmatrix}=(4-\lambda)\begin{vmatrix}3-\lambda&0\\-5&-4-\lambda\end{vmatrix}-(-2)\begin{vmatrix}2&0\\2&-4-\lambda\end{vmatrix}+(-4)\begin{vmatrix}2&3-\lambda\\2&-5\end{vmatrix}$$

$$=(4-\lambda)(3-\lambda)(-4-\lambda)+2(-8-2\lambda)-4-(-10-6+2\lambda)$$

$$=(\lambda^2-16)(3-\lambda)-16-4\lambda+64-8\lambda$$

$$=3\lambda^2-\lambda^3-48+16\lambda-12\lambda+48$$

$$=-\lambda^3+3\lambda^2+4\lambda=-\lambda(\lambda^2-3\lambda-4)=-\lambda(\lambda-4)(\lambda+1)$$

$$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-\lambda(\lambda-4)(\lambda+1)=0\Rightarrow\lambda=0,4,-1$$

The eigenvalues are –1, 0 and 4.

For $\lambda=-1$

$$\begin{pmatrix}4&-2&-4\\2&3&0\\2&-5&-4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}4x-2y-4z\\2x+3y\\2x-5y-4z\end{pmatrix}=\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$$

Equating the middle elements

$2x+3y=-y\Rightarrow x=-2y$

Let $y=1$, then $x=-2$

Equating the top elements and substituting $y=1$ and $x=-2$

$-8-2-4z=2\Rightarrow z=-3$

An eigenvector corresponding to the eigenvalue –1 is $\begin{pmatrix}-2\\1\\-3\end{pmatrix}$.

For $\lambda=0$

$$\begin{pmatrix}4&-2&-4\\2&3&0\\2&-5&-4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=0\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}4x-2y-4z\\2x+3y\\2x-5y-4z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

1 b (continued)

Equating the middle elements

$2x+3y=0 \Rightarrow 3y=-2x$

Let $x=3$, then $y=-2$

Equating the top elements and substituting $x=3$ and $y=-2$

$12+4-4z=0 \Rightarrow z=0$

An eigenvector corresponding to the eigenvalue 0 is $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$.

For $\lambda=4$

$$\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x-2y-4z \\ 2x+3y \\ 2x-5y-4z \end{pmatrix}=\begin{pmatrix} 4x \\ 4y \\ 4z \end{pmatrix}$$

Equating the middle elements

$2x+3y=4y \Rightarrow y=2x$

Let $x=1$, then $y=2$

Equating the top elements and substituting $x=1$ and $y=2$

$4-4-4z=4 \Rightarrow z=-1$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

2 a $\mathbf{M}=\begin{pmatrix}7 & 0 & -3\\ -9 & -2 & 3\\ 18 & 0 & -8\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}7-\lambda & 0 & -3\\ -9 & -2-\lambda & 3\\ 18 & 0 & -8-\lambda\end{vmatrix}=0$$

$$(7-\lambda)\left[(-2-\lambda)(-8-\lambda)\right]-3\left[0-18(-2-\lambda)\right]=0$$

$$(7-\lambda)\left(\lambda^2+10\lambda+16\right)+54(-2-\lambda)=0$$

$$7\lambda^2+70\lambda+112-\lambda^3-10\lambda^2-16\lambda-108-54\lambda=0$$

$$\lambda^3+3\lambda^2-4=0$$

Let $f(\lambda)=\lambda^3+3\lambda^2-4$

$f(1)=0\Rightarrow(\lambda-1)$ is a factor

So $\lambda^3+3\lambda^2-4=(\lambda-1)\left(\lambda^2+k\lambda+4\right)$

Equating coefficients of λ^2 gives

$-1+k=3$, so $k=4$

$$(\lambda-1)\left(\lambda^2+4\lambda+4\right)=0$$

$$(\lambda-1)(\lambda+2)^2=0$$

Therefore -2 is a repeated eigenvalue and the other eigenvalue is 1.

b $\begin{pmatrix}7 & 0 & -3\\ -9 & -2 & 3\\ 18 & 0 & -8\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$

Equating upper elements:

$$7x-3z=-2x$$
$$9x-3z=0$$
$$z=3x$$

So choosing $x=1$ gives an eigenvector of $\begin{pmatrix}1\\ 0\\ 3\end{pmatrix}$

So choosing $x=0, y=1$ gives an eigenvector of $\begin{pmatrix}0\\ 1\\ 0\end{pmatrix}$

3 $\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix} 3-\lambda & -1 & 2 \\ 3 & -1-\lambda & 6 \\ -2 & 2 & -2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-1-\lambda)(-2-\lambda)-12\right]+1\left[3(-2-\lambda)+12\right]+2\left[6+2(-1-\lambda)\right]=0$$

$$(3-\lambda)(\lambda^2+3\lambda-10)+6-3\lambda+2(4-2\lambda)=0$$

$$3\lambda^2+9\lambda-30-\lambda^3-3\lambda^2+10\lambda+6-3\lambda+8-4\lambda=0$$

$$\lambda^3-12\lambda+16=0$$

Let $f(\lambda)=\lambda^3-12\lambda+16$

$f(2)=0 \Rightarrow (\lambda-2)$ is a factor

So $f(\lambda)=(\lambda-2)(\lambda^2+k\lambda-8)$

Equating coefficients of λ^2 gives

$-2+k=0$, so $k=2$

$$(\lambda-2)(\lambda^2+2\lambda-8)=0$$

$$(\lambda-2)(\lambda-2)(\lambda+4)=0$$

$$(\lambda-2)^2(\lambda+4)=0$$

Therefore 2 is a repeated eigenvalue and the other eigenvalue is −4.

3 Taking $\lambda = 2$:

$$\begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$3x - y + 2z = 2x$

$x - y + 2z = 0$

So choosing $x = 1$ and $y = 1$ gives an eigenvector of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Choosing $x = -2,\ y = 0$ gives an eigenvector of $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

Taking $\lambda = -4$:

$$\begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -4\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$3x - y + 2z = -4x$

$7x - y + 2z = 0$

So choosing $x = -1$ and $y = -3$ gives an eigenvector of $\begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$

4 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2 & 2 & -2\\ -3 & 2 & 0\\ 1 & 4 & -3\end{pmatrix}-\begin{pmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}=\begin{pmatrix}2-\lambda & 2 & -2\\ -3 & 2-\lambda & 0\\ 1 & 4 & -3-\lambda\end{pmatrix}$

$$\begin{vmatrix}2-\lambda & 2 & -2\\ -3 & 2-\lambda & 0\\ 1 & 4 & -3-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda & 0\\ 4 & -3-\lambda\end{vmatrix}-2\begin{vmatrix}-3 & 0\\ 1 & -3-\lambda\end{vmatrix}+(-2)\begin{vmatrix}-3 & 2-\lambda\\ 1 & 4\end{vmatrix}$$

$$=(2-\lambda^2)(-3-\lambda)-2(9+3\lambda)-2(-12-2+\lambda)$$

$$=(\lambda^2-4\lambda+4)(-3-\lambda)-18-6\lambda+28-2\lambda$$

$$=-\lambda^3+\lambda^2+8\lambda-12-8\lambda+10$$

$$=-\lambda^3+\lambda^2-2=-(\lambda^3-\lambda^2+2)$$

$$\lambda^3-\lambda^2+2=\lambda^3+\lambda^2-2\lambda^2-2\lambda+2\lambda+2$$

$$=\lambda^2(\lambda+1)-2\lambda(\lambda+1)+2(\lambda+1)=(\lambda+1)(\lambda^2-2\lambda+2)$$

$$=(\lambda+1)\left((\lambda-1)^2+1\right)$$

As $(\lambda-1)^2+1\geqslant 1$ for all real $\lambda, (\lambda-1)^2+1=0$ has no real solutions.

Hence $\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow -(\lambda+1)\left((\lambda-1)^2+1\right)=0\Rightarrow \lambda=-1$

The only real eigenvalue of **A** is -1.

b For $\lambda=-1$

$$\begin{pmatrix}2 & 2 & -2\\ -3 & 2 & 0\\ 1 & 4 & -3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=-1\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\begin{pmatrix}2x+2y-2z\\ -3x+2y\\ x+4y-3z\end{pmatrix}=\begin{pmatrix}-x\\ -y\\ -z\end{pmatrix}$$

Equating the middle elements

$-3x+2y=-y\Rightarrow y=x$

Let $x=1$, then $y=1$

Equating the top elements and substituting $x=1$ and $y=1$

$2+2-2z=-1\Rightarrow 2z=5\Rightarrow z=\dfrac{5}{2}$

An eigenvector corresponding to the eigenvalue -1 is $\begin{pmatrix}2\\ 2\\ 5\end{pmatrix}$.

4 c The characteristic equation is $(\lambda+1)(\lambda^2-2\lambda+2)=0$

So the complex eigenvalues are found from solving

$\lambda^2-2\lambda+2=0$

$\lambda=\dfrac{2\pm\sqrt{-4}}{2}=1\pm i$

Taking $\lambda=1+i$

$$\begin{pmatrix}2 & 2 & -2\\ -3 & 2 & 0\\ 1 & 4 & -3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=(1+i)\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

Equating middle elements:

$-3x+2y=(1+i)y$

$-3x+2y=y+iy$

$y-iy=3x$

$y(1-i)=3x$

$y=\dfrac{3x}{1-i}$

$y=\dfrac{3x}{1-i}\left(\dfrac{1+i}{1+i}\right)$

$y=\dfrac{3(1+i)x}{2}$

Equating upper elements:

$2x+2y-2z=(1+i)x$

$2x+2y-(1+i)x=2z$

$z=x+y-\dfrac{(1+i)x}{2}$

$z=x+\dfrac{3(1+i)x}{2}-\dfrac{(1+i)x}{2}$

$z=x+(1+i)x$

$z=(2+i)x$

So choosing $x=1$, you obtain the eigenvector $\begin{pmatrix}1\\ \dfrac{3(1+i)}{2}\\ 2+i\end{pmatrix}$

Note: Any scalar multiple of this is also an eigenvector, so the following would also be valid:

$$(4-2i)\begin{pmatrix}1\\ \dfrac{3(1+i)}{2}\\ 2+i\end{pmatrix}=\begin{pmatrix}4-2i\\ 9+3i\\ 10\end{pmatrix}$$

Taking the conjugate of these elements we obtain the eigenvector for $\lambda=1-i=\begin{pmatrix}4+2i\\ 9-3i\\ 10\end{pmatrix}$

5 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&-1&3\\0&2&4\\0&2&0\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}2-\lambda&-1&3\\0&2-\lambda&4\\0&2&-\lambda\end{pmatrix}$

$$\begin{vmatrix}2-\lambda&-1&3\\0&2-\lambda&4\\0&2&-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}2-\lambda&4\\2&-\lambda\end{vmatrix}-(-1)\begin{vmatrix}0&4\\0&-\lambda\end{vmatrix}+3\begin{vmatrix}0&2-\lambda\\0&2\end{vmatrix}$$

$$=(2-\lambda)(-2\lambda+\lambda^2-8)+0+0$$

$$=(2-\lambda)(\lambda^2-2\lambda-8)=(2-\lambda)(\lambda-4)(\lambda+2)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(2-\lambda)(\lambda-4)(\lambda+2)=0\Rightarrow\lambda=2,4,-2$

The eigenvalues of **A** are 4, as required, 2 and –2.

b For $\lambda=4$

$$\begin{pmatrix}2&-1&3\\0&2&4\\0&2&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{4}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}2x-y+3z\\2y+4y\\2y\end{pmatrix}=\begin{pmatrix}4x\\4y\\4z\end{pmatrix}$$

Equating the lowest elements

$2y=4z\Rightarrow y=2z$

Let $z=1$, then $y=2$

Equating the top elements and substituting $y=2$ and $z=1$

$2x-2+3=4x\Rightarrow 2x=1\Rightarrow x=\dfrac{1}{2}$

An eigenvector corresponding to the eigenvalue 4 is $\begin{pmatrix}1\\4\\2\end{pmatrix}$.

6 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{pmatrix}$

$$\begin{vmatrix}1-\lambda&1&3\\2&4-\lambda&-1\\4&4&3-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}4-\lambda&-1\\4&3-\lambda\end{vmatrix}-1\begin{vmatrix}2&-1\\4&3-\lambda\end{vmatrix}+3\begin{vmatrix}2&4-\lambda\\4&4\end{vmatrix}$$

$$=(1-\lambda)\left((4-\lambda)(3-\lambda)+4\right)-(6-2\lambda+4)+3(8-16+4\lambda)$$

$$=(1-\lambda)(\lambda^2-7\lambda+16)+14\lambda-34$$

$$=-\lambda^3+8\lambda^2-23\lambda+16+14\lambda-34$$

$$=-\lambda^3+8\lambda^2-9\lambda-18$$

Let $\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2+k\lambda-6)$

Equating the coefficients of λ^2

$-8=-3+k\Rightarrow k=-5$

Hence $\lambda^3-8\lambda^2+9\lambda+18=(\lambda-3)(\lambda^2+5\lambda-6)=(\lambda-3)(\lambda-6)(\lambda+1)$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda-3)(\lambda-6)(\lambda+1)=0\Rightarrow\lambda=3,6,-1$

The other eigenvalues of **A** are –1 and 6.

b For $\lambda=-1$

$$\begin{pmatrix}1&1&3\\2&4&-1\\4&4&3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}x+y+3z\\2x+4y-z\\4x+4y+3z\end{pmatrix}=\begin{pmatrix}-x\\-y\\-z\end{pmatrix}$$

Equating the top elements

$x+y+3z=-x$

$2x+y+3z=0$ **(1)**

Equating the middle elements

$2x+4y-z=-y$

$2x+5y-z=0$ **(2)**

(2) – (1)

$4y-4z=0\Rightarrow y=z$

Let $z=1$, then $y=1$

Substituting $y=1$ and $z=1$ into **(1)**

$2x+1+3=0\Rightarrow x=-2$

An eigenvector corresponding to the eigenvalue –1 is $\begin{pmatrix}-2\\1\\1\end{pmatrix}$.

6 b (continued)

For $\lambda = 3$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \\ 3z \end{pmatrix}$$

Equating the lowest elements

$4x+4y+3z = 3z \Rightarrow y = -x$

Let $x = 1$, then $y = -1$

Equating the top elements and substituting $x = 1$ and $y = -1$

$1-1+3z = 3 \Rightarrow z = 1$

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

For $\lambda = 6$

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x+y+3z \\ 2x+4y-z \\ 4x+4y+3z \end{pmatrix} = \begin{pmatrix} 6x \\ 6y \\ 6z \end{pmatrix}$$

Equating the top elements

$x+y+3z = 6x$

$-5x+y+3z = 0$ **(1)**

Equating the lowest elements

$4x+4y+3z = 6z$

$4x+4y-3z = 0$ **(2)**

(1) + **(2)**

$-x+5y = 0 \Rightarrow x = 5y$

Let $y = 1$, then $x = 5$

Substituting $x = 5$ and $y = 1$ into **(1)**

$-25+1+3z = 0 \Rightarrow z = 8$

An eigenvector corresponding to the eigenvalue 6 is $\begin{pmatrix} 5 \\ 1 \\ 8 \end{pmatrix}$.

7 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}2&2&1\\-2&4&0\\4&2&5\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}2-\lambda&2&1\\-2&4-\lambda&0\\4&2&5-\lambda\end{pmatrix}$

When $\lambda=2$

$$\mathbf{A}-2\mathbf{I}=\begin{pmatrix}0&2&1\\-2&2&0\\4&2&3\end{pmatrix}$$

$$\det(\mathbf{A}-2\mathbf{I})=\begin{vmatrix}0&2&1\\-2&2&0\\4&2&3\end{vmatrix}=0\begin{vmatrix}2&0\\2&3\end{vmatrix}-2\begin{vmatrix}-2&0\\4&3\end{vmatrix}+1\begin{vmatrix}-2&2\\4&2\end{vmatrix}$$

$$=0-2\times(-6)+1(-4-8)=12-12=0$$

Hence 2 is an eigenvalue of **A**.

b $\begin{vmatrix}2-\lambda&2&1\\-2&4-\lambda&0\\4&2&5-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}4-\lambda&0\\2&5-\lambda\end{vmatrix}-2\begin{vmatrix}-2&0\\4&5-\lambda\end{vmatrix}+1\begin{vmatrix}-2&4-\lambda\\4&2\end{vmatrix}$

$$=(2-\lambda)(4-\lambda)(5-\lambda)+20-4\lambda+(-4-16+4\lambda)$$

$$=(2-\lambda)(4-\lambda)(5-\lambda)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=(2-\lambda)(4-\lambda)(5-\lambda)=0\Rightarrow\lambda=2,4,5$

The other eigenvalues of **A** are 4 and 5.

7 c For $\lambda = 2$

$$\begin{pmatrix} 2 & 2 & 1 \\ -2 & 4 & 0 \\ 4 & 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 2x+2y+z \\ -2x+4y \\ 4x+2y+5z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the middle elements

$-2x+4y = 2y \Rightarrow y = x$

Let $x = 1$, then $y = -1$

Equating the top elements and substituting $x = 1$ and $y = 1$

$2+2+z = 2 \Rightarrow z = -2$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ is $\sqrt{\left(1^2+1^2+(-2)^2\right)} = \sqrt{6}$

A normalised eigenvector corresponding to the eigenvalue 2 is $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix}$.

8 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4&2&1\\-2&0&5\\0&3&4\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}4-\lambda&2&1\\-2&-\lambda&5\\0&3&4-\lambda\end{pmatrix}$

$$\begin{vmatrix}4-\lambda&2&1\\-2&-\lambda&5\\0&3&4-\lambda\end{vmatrix}=(4-\lambda)\begin{vmatrix}-\lambda&5\\3&4-\lambda\end{vmatrix}-2\begin{vmatrix}-2&5\\3&4-\lambda\end{vmatrix}+1\begin{vmatrix}-2&-\lambda\\0&3\end{vmatrix}$$

$$=(4-\lambda)(-4\lambda+\lambda^2-15)-2(-8+2\lambda)-6$$

$$=-\lambda^3+8\lambda^2-\lambda-60+16-4\lambda-6$$

$$=-\lambda^3+8\lambda^2-5\lambda-50$$

$$\lambda^3-8\lambda^2+5\lambda+50=\lambda^3+2\lambda^2-10\lambda^2-20\lambda+25\lambda+50$$

$$=\lambda^2(\lambda+2)-10\lambda(\lambda+2)+25(\lambda+2)=(\lambda+2)(\lambda^2-10\lambda+25)$$

$$=(\lambda+2)(\lambda-5)^2$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow-(\lambda+2)(\lambda-5)^2=0\Rightarrow\lambda=-2,5$ repeated

–2 is one eigenvalue of **A** and the only other distinct eigenvalue is 5, which is repeated.

8 b For $\lambda = -2$

$$\begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x+2y+z \\ -2x+5z \\ 3y+4z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

Equating the lowest elements

$4y+4z = -2z \Rightarrow y = -2z$

Let $z = 1$, then $y = -2$

Equating the top elements and substituting $y = -2$ and $z = 1$

$4x - 4 + 1 = -2x \Rightarrow 6x = 3 \Rightarrow x = \frac{1}{2}$

An eigenvector corresponding to the eigenvalue –2 is $\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$.

For $\lambda = 5$

$$\begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 4x+2y+z \\ -2x+5z \\ 3y+4z \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \\ 5z \end{pmatrix}$$

Equating the lowest elements

$3y+4z = 5z \Rightarrow z = 3y$

Let $y = 1$, then $z = 3$

Equating the top elements and substituting $y = 1$ and $z = 3$

$4x + 2 + 3 = 5x \Rightarrow x = 5$

An eigenvector corresponding to the eigenvalue 5 is $\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$.

9 a $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}1&-1&0\\-1&0&1\\1&2&1\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}1-\lambda&-1&0\\-1&-\lambda&1\\1&2&1-\lambda\end{pmatrix}$

$$\begin{vmatrix}1-\lambda&-1&0\\-1&-\lambda&1\\1&2&1-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}-\lambda&1\\2&1-\lambda\end{vmatrix}-(-1)\begin{vmatrix}-1&1\\1&1-\lambda\end{vmatrix}+0\begin{vmatrix}-1&-\lambda\\1&2\end{vmatrix}$$

$$=(1-\lambda)\left(-\lambda+\lambda^2-2\right)+1(-1+\lambda-1)+0$$

$$=(1-\lambda)(\lambda-2)(\lambda+1)+1(\lambda-2)$$

$$=(\lambda-2)\left((1-\lambda)(1+\lambda)+1\right)=(\lambda-2)(2-\lambda^2)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0\Rightarrow(2-\lambda)(2-\lambda^2)=0\Rightarrow\lambda=2,\pm\sqrt{2}$

The other eigenvalues of **A** are $\pm\sqrt{2}$.

9 b For $\lambda = \sqrt{2}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} \sqrt{2}x \\ \sqrt{2}y \\ \sqrt{2}z \end{pmatrix}$$

Equating the top elements

$x - y = \sqrt{2x} \Rightarrow y = \left(1-\sqrt{2}\right)x$

Let $x = 1$, then $y = 1-\sqrt{2}$

Equating the middle elements and substituting $x = 1$ and $y = 1-\sqrt{2}$

$-1 + z = \sqrt{2}\left(1-\sqrt{2}\right) = \sqrt{2} - 2 \Rightarrow z = \sqrt{2} - 1$

An eigenvector corresponding to the eigenvalue $\sqrt{2}$ is $\begin{pmatrix} 1 \\ 1-\sqrt{2} \\ \sqrt{2}-1 \end{pmatrix}$.

For $\lambda = -\sqrt{2}$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\sqrt{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} -\sqrt{2}x \\ -\sqrt{2}y \\ -\sqrt{2}z \end{pmatrix}$$

Equating the top elements

$x - y = -\sqrt{2x} \Rightarrow y = \left(\sqrt{2}+1\right)x$

Equating the middle elements and substituting $x = 1$ and $y = 1+\sqrt{2}$

$-1 + z = -\sqrt{2}\left(1+\sqrt{2}\right) = -\sqrt{2} - 2 \Rightarrow z = -1 - \sqrt{2}$

An eigenvector corresponding to the eigenvalue $-\sqrt{2}$ is $\begin{pmatrix} 1 \\ 1+\sqrt{2} \\ -1-\sqrt{2} \end{pmatrix}$.

For $\lambda = 2$

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x-y \\ -x+z \\ x+2y+z \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

Equating the top elements

$x - y = 2x \Rightarrow y = -x$

9 b (continued)

Let $x=1$, then $y=-1$

Equating the middle elements and substituting $x=1$ and $y=-1$

$-1+z=-2 \Rightarrow z=-1$

An eigenvector corresponding to the eigenvalue 2 is $\begin{pmatrix}1\\-1\\-1\end{pmatrix}$.

10 a $\begin{pmatrix}4&1&2\\1&a&0\\-1&1&b\end{pmatrix}\begin{pmatrix}2\\2\\-1\end{pmatrix}=\lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$

$\begin{pmatrix}8+2-2\\2+2a\\-2+2-b\end{pmatrix}=\begin{pmatrix}2\lambda\\2\lambda\\-1\lambda\end{pmatrix}$

Equating the top elements

$8=2\lambda \Rightarrow \lambda=4$

The eigenvalue is 4.

b Equating the middle elements and substituting $\lambda=4$

$2+2a=8 \Rightarrow a=3$

Equating the lowest elements and substituting $\lambda=4$

$-b=-\lambda=-4 \Rightarrow b=4$

$a=3$ and $b=4$

c $\mathbf{A}-\lambda\mathbf{I}=\begin{pmatrix}4&1&2\\1&3&0\\-1&1&4\end{pmatrix}-\begin{pmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{pmatrix}=\begin{pmatrix}4-\lambda&1&2\\1&3-\lambda&0\\-1&1&4-\lambda\end{pmatrix}$

$$\begin{vmatrix}4-\lambda&1&2\\1&3-\lambda&0\\-1&1&4-\lambda\end{vmatrix}=(4-\lambda)\begin{vmatrix}3-\lambda&0\\1&4-\lambda\end{vmatrix}-1\begin{vmatrix}1&0\\-1&4-\lambda\end{vmatrix}+2\begin{vmatrix}1&3-\lambda\\-1&1\end{vmatrix}$$

$$=(4-\lambda)^2(3-\lambda)-1(4-\lambda)+2(1+3-\lambda)$$

$$=(4-\lambda)^2(3-\lambda)+1(4-\lambda)=(4-\lambda)\big((4-\lambda)(3-\lambda)+1\big)$$

$$=(4-\lambda)(\lambda^2-7\lambda+13)$$

$\det(\mathbf{A}-\lambda\mathbf{I})=0 \Rightarrow (4-\lambda)(\lambda^2-7\lambda+13)=0 \Rightarrow \lambda=4$ or $\lambda^2-7\lambda+13=0$

The discriminant of $\lambda^2-7\lambda+13=0$ is given by

$b^2-4ac=49-52=-3<0$

There are no real solutions of $\lambda^2-7\lambda+13=0$

4 is the only real eigenvalue of **A**.

10 d The characteristic equation is $(\lambda-4)(\lambda^2-7\lambda+13)=0$

So the complex eigenvalues are found from solving

$\lambda^2-7\lambda+13=0$

$$\lambda=\frac{7\pm\sqrt{49-52}}{2}=\frac{7\pm\sqrt{-3}}{2}=\frac{7\pm i\sqrt{3}}{2}$$

Taking $\lambda=\dfrac{7+i\sqrt{3}}{2}$

$$\begin{pmatrix}4&1&2\\1&3&0\\-1&1&4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\left(\frac{7+i\sqrt{3}}{2}\right)\begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating middle elements:

$$x+3y=\left(\frac{7+i\sqrt{3}}{2}\right)y$$

$$2x+6y=\left(7+i\sqrt{3}\right)y$$

$$2x=\left(1+i\sqrt{3}\right)y$$

$$y=\left(\frac{2}{1+i\sqrt{3}}\right)x$$

$$y=\left(\frac{2}{1+i\sqrt{3}}\right)\left(\frac{1-i\sqrt{3}}{1-i\sqrt{3}}\right)x$$

$$y=\left(\frac{1-i\sqrt{3}}{2}\right)x$$

Equating upper elements:

$$4x+y+2z=\left(\frac{7+i\sqrt{3}}{2}\right)x$$

$$4x+\left(\frac{1-i\sqrt{3}}{2}\right)x+2z=\left(\frac{7+i\sqrt{3}}{2}\right)x$$

$$2z=\left(\frac{7+i\sqrt{3}}{2}\right)x-\left(\frac{1-i\sqrt{3}}{2}\right)x-4x$$

$$2z=\left(-1+i\sqrt{3}\right)x$$

$$z=\left(\frac{-1+i\sqrt{3}}{2}\right)x$$

10 d (continued)

So choosing $x = 1$, you obtain the eigenvector $\begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{pmatrix}$

Note: Any scalar multiple of this is also an eigenvector, so the following would also be valid:

$$\left(-1-i\sqrt{3}\right)\begin{pmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \\ \frac{-1+i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -1-i\sqrt{3} \\ -2 \\ 2 \end{pmatrix}$$

Taking $\lambda = \frac{7-i\sqrt{3}}{2}$

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 0 \\ -1 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\frac{7-i\sqrt{3}}{2}\right)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:

$$x + 3y = \left(\frac{7-i\sqrt{3}}{2}\right)y$$

$$2x + 6y = \left(7-i\sqrt{3}\right)y$$

$$2x = \left(1-i\sqrt{3}\right)y$$

$$y = \left(\frac{2}{1-i\sqrt{3}}\right)x$$

$$y = \left(\frac{2}{1-i\sqrt{3}}\right)\left(\frac{1+i\sqrt{3}}{1+i\sqrt{3}}\right)x$$

$$y = \left(\frac{1+i\sqrt{3}}{2}\right)x$$

10 d (continued)

Equating upper elements:

$$4x + y + 2z = \left(\frac{7 - i\sqrt{3}}{2}\right)x$$

$$4x + \left(\frac{1 + i\sqrt{3}}{2}\right)x + 2z = \left(\frac{7 - i\sqrt{3}}{2}\right)x$$

$$2z = \left(\frac{7 - i\sqrt{3}}{2}\right)x - \left(\frac{1 + i\sqrt{3}}{2}\right)x - 4x$$

$$2z = \left(-1 - i\sqrt{3}\right)x$$

$$z = \left(\frac{-1 - i\sqrt{3}}{2}\right)x$$

So choosing $x = 1$, you obtain the eigenvector $\begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{-1 - i\sqrt{3}}{2} \end{pmatrix}$

Note: Any scalar multiple of this is also an eigenvector, so the following would also be valid:

$$\left(-1 + i\sqrt{3}\right)\begin{pmatrix} 1 \\ \frac{1 + i\sqrt{3}}{2} \\ \frac{-1 - i\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} -1 + i\sqrt{3} \\ -2 \\ 2 \end{pmatrix}$$

11 a $\mathbf{A}=\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}$

To find the eigenvalues:

$$\begin{vmatrix}3-\lambda & 0 & 0\\ 1 & 1-\lambda & 1\\ 4 & -1 & 3-\lambda\end{vmatrix}=0$$

$$(3-\lambda)\left[(1-\lambda)(3-\lambda)+1\right]=0$$

$$(3-\lambda)\left(\lambda^2-4\lambda+4\right)=0$$

$$3\lambda^2-12\lambda+12-\lambda^3+4\lambda^2-4\lambda=0$$

$$\lambda^3-7\lambda^2+16\lambda-12=0$$

Let $f(\lambda)=\lambda^3-7\lambda^2+16\lambda-12$

$f(2)=8-28+32-12=0\Rightarrow(\lambda-2)$ is a factor

So $f(\lambda)=(\lambda-2)\left(\lambda^2+k\lambda+6\right)$

Equating coefficients of λ^2 gives

$-2+k=-7$, so $k=-5$

$$(\lambda-2)\left(\lambda^2-5\lambda+6\right)=0$$

$$(\lambda-2)(\lambda-2)(\lambda-3)=0$$

$$(\lambda-2)^2(\lambda-3)=0$$

Therefore 2 is a repeated eigenvalue and the other eigenvalue is 3

Taking $\lambda=2$:

$$\begin{pmatrix}3 & 0 & 0\\ 1 & 1 & 1\\ 4 & -1 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

Equating upper elements:

$3x=2x\Rightarrow x=0$

Equating middle elements:

$x+y+z=2y$

$x-y+z=0$

$y=z$

So choosing $y=1$ gives an eigenvector of $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$

11 a (continued)

Taking $\lambda = 3$:

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & -1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating middle elements:
$x + y + z = 3y \Rightarrow x - 2y + z = 0$

Equating lower elements:
$4x - y + 3z = 3z \Rightarrow y = 4x$

So choosing $x = 1$ leads to $y = 4$ and $z = 7$, giving an eigenvector of $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

b Since the eigenvectors are $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$, and each invariant line goes through the origin,

the required lines are: $L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$

12 Every linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ must have one real eigenvalue, hence at least one real eigenvector. Therefore every such transformation must have at least one invariant line.

Challenge

$$\mathbf{M}=\begin{pmatrix}\frac{1}{9} & \frac{8}{9} & -\frac{4}{9}\\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9}\\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9}\end{pmatrix}$$

To find the eigenvalues:

$$\begin{vmatrix}\frac{1}{9}-\lambda & \frac{8}{9} & -\frac{4}{9}\\ \frac{8}{9} & \frac{1}{9}-\lambda & \frac{4}{9}\\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9}-\lambda\end{vmatrix}=0$$

$$\left(\frac{1}{9}-\lambda\right)\left[\left(\frac{1}{9}-\lambda\right)\left(\frac{7}{9}-\lambda\right)-\frac{16}{81}\right]-\left(\frac{8}{9}\right)\left[\frac{8}{9}\left(\frac{7}{9}-\lambda\right)+\frac{16}{81}\right]-\left(\frac{4}{9}\right)\left[\frac{32}{81}+\frac{4}{9}\left(\frac{1}{9}-\lambda\right)\right]=0$$

$$\left(\frac{1}{9}-\lambda\right)\left(\frac{7}{81}-\frac{8\lambda}{9}+\lambda^2-\frac{16}{81}\right)-\left(\frac{8}{9}\right)\left(\frac{56}{81}-\frac{8\lambda}{9}+\frac{16}{81}\right)-\left(\frac{4}{9}\right)\left(\frac{32}{81}+\frac{4}{81}-\frac{4\lambda}{9}\right)=0$$

$$\left(\frac{1}{9}-\lambda\right)\left(\lambda^2-\frac{8\lambda}{9}-\frac{1}{9}\right)-\left(\frac{8}{9}\right)\left(\frac{8}{9}-\frac{8\lambda}{9}\right)-\left(\frac{4}{9}\right)\left(\frac{4}{9}-\frac{4\lambda}{9}\right)=0$$

$$\left(\frac{1}{9}\right)\left(\lambda^2-\frac{8\lambda}{9}-\frac{1}{9}\right)-\lambda\left(\lambda^2-\frac{8\lambda}{9}-\frac{1}{9}\right)-\left(\frac{8}{9}\right)\left(\frac{8}{9}-\frac{8\lambda}{9}\right)-\left(\frac{4}{9}\right)\left(\frac{4}{9}-\frac{4\lambda}{9}\right)=0$$

$$\frac{\lambda^2}{9}-\frac{8\lambda}{81}-\frac{1}{81}-\lambda^3+\frac{8\lambda^2}{9}+\frac{\lambda}{9}-\frac{64}{81}+\frac{64\lambda}{81}-\frac{16}{81}+\frac{16\lambda}{81}=0$$

$$-1+\lambda+\lambda^2-\lambda^3=0$$

$$\lambda^3-\lambda^2-\lambda+1=0$$

Let $f(\lambda)=\lambda^3-\lambda^2-\lambda+1$

$f(1)=1-1-1+1=0\Rightarrow(\lambda-1)$ is a factor

So $f(\lambda)=(\lambda-1)(\lambda^2+k\lambda-1)$

Equating coefficients of λ^2 gives

$-1+k=-1$, so $k=0$

$$(\lambda-1)(\lambda^2-1)=0$$

$$(\lambda-1)(\lambda-1)(\lambda+1)=0$$

$$(\lambda-1)^2(\lambda+1)=0$$

Therefore 1 is a repeated eigenvalue and the other eigenvalue is -1.

Challenge (continued)

Taking $\lambda = 1$:

$$\begin{pmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$$\frac{x}{9} + \frac{8y}{9} - \frac{4z}{9} = x$$

$$x + 8y - 4z = 9x$$

$$8x - 8y + 4z = 0$$

$$2x - 2y + z = 0$$

So choosing $x = 1$ and $y = 1$ gives an eigenvector of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Choosing $x = -1$ and $y = 0$ gives an eigenvector of $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Taking $\lambda = -1$:

$$\begin{pmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Equating upper elements:

$$\frac{x}{9} + \frac{8y}{9} - \frac{4z}{9} = -x$$

$$x + 8y - 4z = -9x$$

$$10x + 8y - 4z = 0$$

$$5x + 4y - 2z = 0$$

So choosing $x = 2$ and $z = 1$ gives an eigenvector of $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Eigenvectors relating to the eigenvalue $\lambda = 1$ are invariant under the transformation and so both eigenvectors corresponding to this eigenvalue lie in the plane Π. It can be verified by the dot (scalar) product that the both of these vectors are perpendicular to the vector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

Hence it follows that this vector is perpendicular to the plane Π. Hence the Cartesian equation of the plane is given by $2x - 2y + z = 0$.