Recurrence relations Mixed exercise

1 $u_n = 2u_{n-1} - 1$

The associated homogeneous recurrence relation is $u_n = 2u_{n-1}$, so the complementary function is $u_n = c(2^n)$.

Particular solution: $u_n = \lambda$

$\lambda = 2\lambda - 1 \Rightarrow \lambda = 1$

So a particular solution to the recurrence relation is $u_n = 1$

The general solution is $u_n = c(2^n) + 1$

Since $u_0 = 4$, $4 = c + 1$ so $c = 3$

The solution is $u_n = 3(2^n) + 1$

2 a The recurrence relation is of the form $u_n = u_{n-1} + \text{g}(n)$ where $\text{g}(n) = n$ so use the general form of the solution:

$$u_n = u_0 - \sum_{r=1}^{n} \text{g}(r) \text{ so } u_n = 2000 - \sum_{r=1}^{n} r = 2000 - \frac{1}{2}n(n+1)$$

b Solving $2000 - \frac{1}{2}n(n+1) < 0$:

$n(n+1) > 4000 \Rightarrow n^2 + n - 4000 > 0$

Positive solution is $n > 62.74\ldots$ so first integer value is 63

$u_{63} = -16$

3 a The associated homogeneous recurrence relation is $u_n = 3u_{n-1}$, so the complementary function is $u_n = c(3^n)$.

Particular solution: $u_n = \lambda$

$$\lambda = 3\lambda + 5 \Rightarrow \lambda = -\frac{5}{2}$$

So a particular solution to the recurrence relation is $u_n = -\frac{5}{2}$

The general solution is $u_n = c(3^n) - \frac{5}{2}$

Since $u_0 = 0$, $0 = c - \frac{5}{2}$ so $c = \frac{5}{2}$

The solution is $u_n = \frac{5}{2}(3^n) - \frac{5}{2} = \frac{5}{2}(3^n - 1)$

b When $n = 10$, $u_{10} = \frac{5}{2}(3^{10} - 1) = 147620$

c $\frac{5}{2}(3^n - 1) > 10\,000\,000 \Rightarrow 3^n > 4\,000\,001$

$$n\log 3 > \log 4\,000\,001 \Rightarrow n > \frac{\log 4\,000\,001}{\log 3} \Rightarrow n > 13.83...$$

Hence $n = 14$. $u_{14} = \frac{5}{2}(3^{14} - 1) = 11\,957\,420$

4 a $T_0 = 12000$ represents the number of trees planted at the start of the first year.

Removing 20% of the trees compared to year $n - 1$ leaves 80% of the trees, i.e. $0.8T_{n-1}$, then planting 1000 trees gives $T_n = 0.8T_{n-1} + 1000$.

b The associated homogeneous recurrence relation is $T_n = 0.8T_{n-1}$, so the complementary function is $T_n = c(0.8^n)$.

Particular solution: $T_n = \lambda$

$\lambda = 0.8\lambda + 1000 \Rightarrow \lambda = 5000$

So a particular solution to the recurrence relation is $T_n = 5000$

The general solution is $T_n = c(0.8^n) + 5000$

Since $T_0 = 12000$, $12000 = c + 5000 \Rightarrow c = 7000$

The solution is $T_n = 7000(0.8)^n + 5000$

c As $n \to \infty$, $T_n \to 5000$

5 a $S_0 = 2000$ represents the number of salmon at the beginning of the first month.

An increase of 25% means that their number is multiplied by $\frac{5}{4}$ and as X salmon are removed, you subtract X, i.e. $S_n = \frac{5}{4}S_{n-1} - X = \frac{5S_{n-1} - 4X}{4}$

b Basis step:

$$S_0 = \left(\frac{5}{4}\right)^0 (2000 - 4X) + 4X = 2000$$

Assumption step:

Assume $S_k = \left(\frac{5}{4}\right)^k (2000 - 4X) + 4X$

Inductive step:

$$S_{k+1} = \frac{5}{4}S_k - X = \frac{5}{4}\left[\left(\frac{5}{4}\right)^k (2000 - 4X) + 4X\right] - X$$

$$= \frac{5}{4}\left(\frac{5}{4}\right)^k (2000 - 4X) + \frac{5}{4}(4X) - X$$

$$= \left(\frac{5}{4}\right)^{k+1} (2000 - 4X) + 4X$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$. Since the closed form is true for $n = 0$, by induction the closed form is true for all $n \geqslant 0$.

c If $x < 500$ the population tends to infinity.
If $x = 500$ the population is constant at 2000.
If $x > 500$ the population dies out.

6 a $b_0 = 200000 - 25000 = 175000$

Interest is added at 0.25% per month so multiplier of 1.0025 and pay off £1200 so

$b_n = 1.0025b_{n-1} - 1200$

b The associated homogeneous recurrence relation is $b_n = 1.0025b_{n-1}$, so the complementary function is $b_n = c(1.0025^n)$.

Particular solution: $b_n = \lambda$

$\lambda = 1.0025\lambda - 1200 \Rightarrow \lambda = 480\,000$

So a particular solution to the recurrence relation is $b_n = 480\,000$

The general solution is $b_n = c(1.0025^n) + 480\,000$

Since $b_0 = 175\,000$, $175\,000 = c + 480\,000 \Rightarrow c = -305\,000$

The solution is $b_n = -305\,000(1.0025^n) + 480\,000$

Need n when $b_n = 0$:

$$305\,000(1.0025^n) = 480\,000 \Rightarrow 1.0025^n = \frac{96}{61}$$

$$n\log(1.0025) = \log\left(\frac{96}{61}\right) \Rightarrow n = 181.616...$$

So n = 182. In years, this is 15 years and 2 months, so they will pay off their mortgage in 2033

7 a $P_4 = 6$

b Since the nth line intersects each of the n – 1 lines once, the recurrence relation is $P_n = P_{n-1} + n - 1$.

The recurrence relation is of the form $P_n = P_{n-1} + \mathrm{g}(n)$ where g(n) = n – 1, so use the general form of the solution:

$$P_n = P_1 + \sum_{r=2}^{n} \mathrm{g}(r) \text{ so } P_n = 0 + \sum_{r=2}^{n}(r-1) = \sum_{r=1}^{n} r - (n-1) = \frac{1}{2}n(n+1) - 1 - n + 1 = \frac{1}{2}n(n-1)$$

c When n = 100, $P_{100} = \frac{1}{2}(100)(99) = 4950$

8 a $t_5 = 25$, $t_6 = 36$ and $t_7 = 49$

b The sequence of square numbers is formed by taking the (n – 1)th square number and adding $2n$ – 1 so $t_n = t_{n-1} + 2n - 1$.

c The recurrence relation is of the form $t_n = t_{n-1} + \mathrm{g}(n)$ where g(n) = 2n – 1, so use the general form of the solution:

$$t_n = t_1 + \sum_{r=2}^{n} \mathrm{g}(r) \text{ so } t_n = 1 + \sum_{r=2}^{n}(2r-1) = 1 + \sum_{r=1}^{n}(2r-1) - 1 = 2\sum_{r=1}^{n} r - n = n(n+1) - n = n^2$$

When n = 100, $t_{100} = 100^2 = 10\,000$

9 a $$\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 4+3p \\ 0 & 3q \end{pmatrix}$$

9 b $a_n = 3a_{n-1} + 4$

$b_n = 3b_{n-1}$

c For a_n:

The associated homogeneous recurrence relation is $a_n = 3a_{n-1}$, so the complementary function is $a_n = c(3^n)$.

Particular solution: $a_n = \lambda$

$\lambda = 3\lambda + 4 \Rightarrow \lambda = -2$

So a particular solution to the recurrence relation is $a_n = -2$

The general solution is $a_n = c(3^n) - 2$

Since $a_1 = 4$, $4 = 3c - 2 \Rightarrow c = 2$

The solution is $a_n = 2(3^n) - 2 = 2(3^n - 1)$

For b_n:

The recurrence relation is of the form $b_n = ab_{n-1}$ with $a = 3$ so use the general form of the solution:

$b_n = b_1 a^{n-1}$ so $b_n = 3(3)^{n-1} = 3^n$

Hence $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 2(3^n - 1) \\ 0 & 3^n \end{pmatrix}$

10 a $S_5 = 55$, $S_6 = 91$ and $S_7 = 140$

b The square pyramidal numbers are formed by adding on the next square number so $S_n = S_{n-1} + n^2$.

c The recurrence relation is of the form $S_n = S_{n-1} + \text{g}(n)$ where $\text{g}(n) = n^2$ so use the general form of the solution:

$$S_n = S_1 + \sum_{r=2}^{n} \text{g}(r) \text{ so } S_n = 1 + \sum_{r=2}^{n} r^2 = 1 + \frac{1}{6}n(n+1)(2n+1) - 1 = \frac{1}{6}n(n+1)(2n+1)$$

11 Basis step:

$$u_1 = \frac{1}{2}(1!)(2!) = 1$$

Assumption step:

Assume $u_k = \frac{1}{2}k!(k+1)!$

Inductive step:

$$u_{k+1} = ((k+1)^2 + (k+1))\frac{1}{2}k!(k+1)!$$

$$= (k+2)(k+1)\frac{1}{2}k!(k+1)!$$

$$= \frac{1}{2}(k+1)!(k+2)!$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$. Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{N}$.

12 Basis step:

$u_1 = \frac{3!}{6} = 1$

Assumption step:

Assume $u_k = \frac{(k+2)!}{6}$

Inductive step:

$u_{k+1} = (k+1+2)\frac{(k+2)!}{6} = \frac{(k+3)!}{6}$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$. Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{N}$

13 a An increase of 20% per hour represents a multiplier of 1.2 so $u_n = 1.2u_{n-1} - k(2^n)$

$u_0 = 100$

b The associated homogeneous recurrence relation is $u_n = 1.2u_{n-1}$, so the complementary function is $u_n = c(1.2^n)$.

Particular solution: $u_n = \lambda k 2^n$

$$u_n = 1.2u_{n-1} - k(2^n)$$
$$\lambda k(2^n) = 1.2\lambda k 2^{n-1} - k(2^n)$$
$$2\lambda k(2^{n-1}) = 1.2\lambda k(2^{n-1}) - k(2^n)$$
$$0.8\lambda k(2^{n-1}) = -k(2^n)$$
$$0.8\lambda k = -2k \Rightarrow \lambda = -\frac{5}{2}$$

So a particular solution to the recurrence relation is $u_n = -\frac{5}{2}k(2^n)$

The general solution is $u_n = c(1.2^n) - \frac{5}{2}k(2^n)$

Since $u_0 = 100$, $100 = c - \frac{5}{2}k$ so $c = 100 + \frac{5}{2}k$

The solution is $u_n = \left(100 + \frac{5}{2}k\right)(1.2^n) - \frac{5}{2}k(2^n)$

14 a

b There are f_{n-1} paths of length n ending in a small flagstone and f_{n-2} paths of length n ending in a long flagstone. This gives a total of $f_{n-1} + f_{n-2}$ paths of length n.

There is only one path of length 1 m and there are two paths of length 2 m so $f_1 = 1$ and $f_2 = 2$.

14 c Auxiliary equation: $r^2 - r - 1 = 0$

Solving gives $r = \dfrac{1 \pm \sqrt{5}}{2}$

General solution: $f_n = A\left(\dfrac{1+\sqrt{5}}{2}\right)^n + B\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

Substituting initial conditions:

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

$$2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right)$$

Solving simultaneously gives $A = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)$ and $B = -\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)$

Hence closed form of the recurrence relation is:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

So when $n = 200$, $f_{200} = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^{201} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{201}\right]$

15 a $t_2 = 8$

$t_3 = 22$

b If the final digit of the string is not 0, then there are t_{n-1} possibilities for the rest of the string for each of final digits 1 and 2. If the final digit is zero, then the penultimate digit must *not* be zero, i.e. can be either 1 or 2, and there are t_{n-2} possibilities for the rest of the string for each of these two cases.

Thus $t_n = 2t_{n-1} + 2t_{n-2}$

c $t_4 = 2 \times 22 + 2 \times 8 = 60$

$t_5 = 2 \times 60 + 2 \times 22 = 164$

$t_6 = 2 \times 164 + 2 \times 60 = 448$

15 d i Auxiliary equation: $r^2 - 2r - 2 = 0$

Solving gives $r = 1 \pm \sqrt{3}$

General solution is $t_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$

Substituting initial conditions:

$8 = A(1+\sqrt{3})^2 + B(1-\sqrt{3})^2$

$22 = A(1+\sqrt{3})^3 + B(1-\sqrt{3})^3$

Solving simultaneously gives $A = \frac{2+\sqrt{3}}{2\sqrt{3}}$ and $B = -\frac{2-\sqrt{3}}{2\sqrt{3}}$

Hence closed form of the recurrence relation is

$$t_n = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})(1+\sqrt{3})^n + (\sqrt{3}-2)(1-\sqrt{3})^n\right]$$

ii $t_{15} = \frac{1}{2\sqrt{3}}\left[(2+\sqrt{3})(1+\sqrt{3})^{15} + (\sqrt{3}-2)(1-\sqrt{3})^{15}\right] = 3\,799\,168$

16 a Auxiliary equation: $r^2 - r - 2 = 0$

Solving gives $r = 2$ or $r = -1$

Complementary function is $u_n = A(2^n) + B(-1)^n$

b Substituting initial conditions:

$1 = 2A - B$

$2 = 4A + B$

Solving simultaneously gives $A = \frac{1}{2}$ and $B = 0$

Hence closed form of the recurrence relation is $u_n = \frac{1}{2}(2^n) = 2^{n-1}$

17 a Auxiliary equation: $r^2 - 7r + 10 = 0$

Solving gives $r = 2$ or $r = 5$

Complementary function is $x_n = A(2^n) + B(5^n)$

Try particular solution λ:

$\lambda = 7\lambda - 10\lambda + 3 \Rightarrow \lambda = \frac{3}{4}$

Hence general solution is $x_n = A(2^n) + B(5^n) + \frac{3}{4}$

b Substituting initial conditions:

$1 = 2A + 5B + \frac{3}{4}$

$2 = 4A + 25B + \frac{3}{4}$

Solving simultaneously gives $A = 0$ and $B = \frac{1}{20}$

Hence closed form of the recurrence relation is $x_n = \frac{1}{20}(5^n) + \frac{3}{4} = \frac{1}{4}\left(\frac{1}{5}(5^n) + 3\right) = \frac{1}{4}(5^{n-1} + 3)$

18 Auxiliary equation: $r^2 - 2r - 15 = 0$
Solving gives $r = -3$ or $r = 5$
Complementary function is $a_n = A(5)^n + B(-3)^n$
Try particular solution $\lambda(2^n)$:
$$\lambda(2^n) = 2\lambda(2^{n-1}) + 15\lambda(2^{n-2}) + 2^n$$
$$4\lambda = 4\lambda + 15\lambda + 4 \Rightarrow \lambda = -\frac{4}{15}$$
Hence general solution is $a_n = A(5)^n + B(-3)^n - \frac{4}{15}(2^n)$
Substituting initial conditions:
$$2 = 5A - 3B - \frac{8}{15}$$
$$4 = 25A + 9B - \frac{16}{15}$$
Solving simultaneously gives $A = \frac{19}{60}$ and $B = -\frac{19}{60}$
Hence closed form of the recurrence relation is:
$$a_n = \frac{19}{60}(5^n) - \frac{19}{60}(-3)^n - \frac{4}{15}(2^n) = \frac{1}{60}(19(5^n) - 19(-3)^n - 2^{n+4})$$

19 Basis step:
When $n = 0$, $u_0 = \frac{0!}{4}(5 - (-3)^0) = 1$; when $n = 1$, $u_1 = \frac{1!}{4}(5 - (-3)^1) = 2$
Assumption step:
Assume the closed form is true for $n = k$ and $n = k + 1$, so $u_k = \frac{k!}{4}(5 - (-3)^k)$ and
$$u_{k+1} = \frac{(k+1)!}{4}(5 - (-3)^{k+1})$$
Inductive step:
$$u_{k+2} = -2(k+2)\frac{(k+1)!}{4}(5 - (-3)^{k+1}) + 3(k+2)(k+1)\frac{k!}{4}(5 - (-3)^k)$$
$$= -2\frac{(k+2)!}{4}(5 - (-3)^{k+1}) + 3\frac{(k+2)!}{4}(5 - (-3)^k)$$
$$= \frac{(k+2)!}{4}(5 - (-3)^{k+2})$$
So if the closed form is valid for $n = k$ and $n = k + 1$ it is valid for $n = k + 2$. Since the closed form is true for $n = 0$ and $n = 1$, by induction the closed form is true for all $n \geqslant 0$.

20 a There are S_{n+1} messages of length $n + 2$ ending in a type A signal packet and S_n messages of length $n + 2$ ending in a type B signal packet. This gives a total of $S_{n+2} = S_{n+1} + S_n$ messages of length $n + 2$.
There is only one possible message of length 1 and there are two possible messages of length 2 so $S_1 = 1$ and $S_2 = 2$.

b Auxiliary equation: $r^2 - r - 1 = 0$

Solving gives $r = \dfrac{1 \pm \sqrt{5}}{2}$

General solution: $S_n = A\left(\dfrac{1+\sqrt{5}}{2}\right)^n + B\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

Substituting initial conditions:

$$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$$

$$2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right)$$

Solving simultaneously gives $A = \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)$ and $B = -\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)$

Hence closed form of the recurrence relation is:

$$S_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

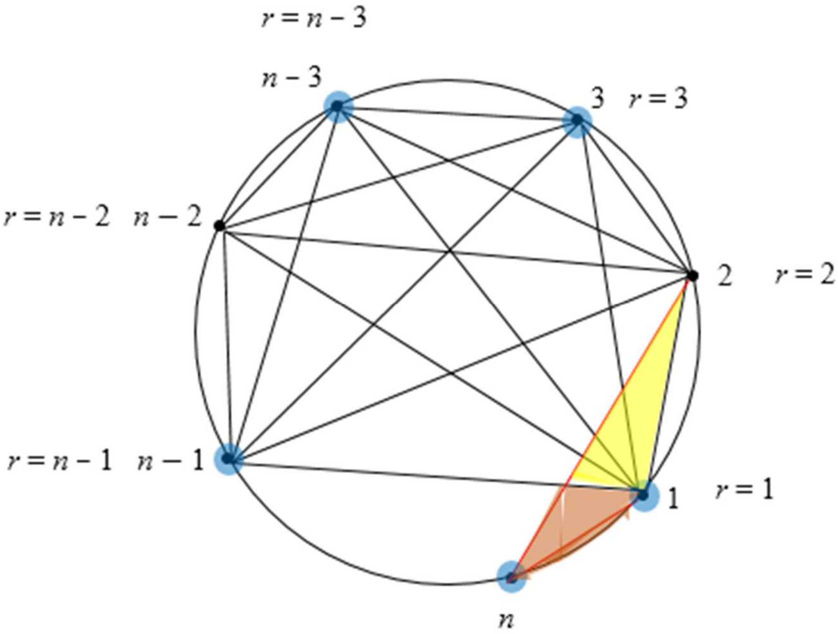
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Challenge

1 a By drawing it out, $C_6 = 31$

b Consider one of the points already on the circumference of the circle (A), and one of the existing points of intersection of two diagonals (B). The line through these two points will meet the circle at two points, A and C. Since there are finitely many pairs $\{A, B\}$, there will be finitely many points C on the circumference of the circle such that the line AC goes through an existing intersection point. However, since there are infinitely many points on the circumference of the circle, it is possible to choose one, D, which doesn't coincide with any of the points C, and thus the chords AD do not go through any of the existing intersection points.

c



Each new line from the point n to point r divides existing points into two sets.

$A = \{1, \ldots r-1\}$ and $B = \{r+1, \ldots n-1\}$

$r - 1$ $\quad$ $n - r - 1$

The new line crosses a line between each pair of points, where one is from A and one is from B forming a new region each time. This accounts for $(r - 1)(n - r - 1)$ new regions.

The new line also crosses one more (the first) region.

So line r creates $(r - 1)(n - r - 1) + 1$ new regions.

The total number of new regions is

$$\sum_{r=1}^{n-1}\left(1+(r-1)(n-1-r)\right)$$

Which simplifies to

$$\frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 2$$

$$C_n = C_{n-1} + \frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 2$$

Challenge

1 d The recurrence relation is of the form $C_n = C_{n-1} + \mathrm{g}(n)$ where $\mathrm{g}(n) = \frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 2$ so use the general form of the solution:

$C_n = C_1 + \sum_{r=2}^{n} \mathrm{g}(r)$ so

$$C_n = 1 + \sum_{r=2}^{n}\left(\frac{1}{6}r^3 - r^2 + \frac{17}{6}r - 2\right) = 1 + \frac{1}{6}\left(\frac{1}{4}n^2(n+1)^2\right) - \frac{1}{6}n(n+1)(2n+1) + \frac{17}{6}\left(\frac{1}{2}n(n+1)\right) - 2n$$

which simplifies to $C_n = \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$

$$C_{100} = \frac{1}{24}(100^4) - \frac{1}{4}(100^3) + \frac{23}{24}(100^2) - \frac{3}{4}(100) + 1 = 3\,926\,176$$

2 a Walks of length 1 start at A and end at any of the other points; the spider cannot return to A.

b ABCA, ABDA, ACBA, ACDA, ADBA and ADCA so six walks of length 3

c $u_n = 2u_{n-1} + 3u_{n-2}$ with $u_1 = 0$ and $u_2 = 3$

Auxiliary equation: $r^2 - 2r - 3 = 0$

Solving gives $r = 3$ or $r = -1$

General solution is $u_n = A(3^n) + B(-1)^n$

Substituting initial conditions:

$0 = 3A - B$

$3 = 9A + B$

Solving simultaneously gives $A = \frac{1}{4}$ and $B = \frac{3}{4}$

Hence closed form of the recurrence relation is $u_n = \frac{1}{4}(3^n) + \frac{3}{4}(-1)^n = \frac{1}{4}(3^n + 3(-1)^n)$