Recurrence relations 4D

1 Basis step:

When $n = 1$, $u_1 = 5^1 - 1 = 4$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 5^k - 1$

Inductive step:

$u_{k+1} = 5u_k + 4 = 5(5^k - 1) + 4 = 5^{k+1} - 1$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{N}$.

2 Basis step:

When $n = 1$, $u_1 = 2^3 - 5 = 3$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 2^{k+2} - 5$

Induction step:

$u_{k+1} = 2u_k + 5 = 2(2^{k+2} - 5) + 5 = 2^{k+3} - 5$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{N}$.

3 Basis step:

When $n = 1$, $u_1 = 5^0 + 2 = 3$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 5^{k-1} + 2$

Inductive step:

$u_{k+1} = 5u_k - 8 = 5(5^{k-1} + 2) - 8 = 5^k + 2$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{N}$.

4 Basis step:

When $n = 1$, $u_1 = \dfrac{3^1 - 1}{2} = 1$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = \dfrac{3^k - 1}{2}$

Inductive step:

$$u_{k+1} = 3u_k + 1 = 3\left(\frac{3^k - 1}{2}\right) + 1 = \frac{3^{k+1} - 3}{2} + 1 = \frac{3^{k+1} - 1}{2}$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{N}$.

5 a $u_1 = 2$

$$u_2 = \frac{3\times 2 - 1}{4} = \frac{5}{4}$$

$$u_3 = \frac{3\left(\frac{5}{4}\right) - 1}{4} = \frac{11}{16}$$

$$u_4 = \frac{3\left(\frac{11}{16}\right) - 1}{4} = \frac{17}{64}$$

b Basis step:

When $n = 1$, $u_1 = 4\left(\frac{3}{4}\right)^1 - 1 = 2$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = \frac{3u_k - 1}{4}$

Inductive step:

$$u_{k+1} = \frac{3\left(4\left(\frac{3}{4}\right)^k - 1\right) - 1}{4} = 4\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)^k - \frac{4}{4} = 4\left(\frac{3}{4}\right)^{k+1} - 1$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$. Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

6 Basis step:

When $n = 1$, $u_1 = 4^1 + 3(1) + 1 = 8$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 4^k + 3k + 1$

Inductive step:

$$u_{k+1} = 4\left(4^k + 3k + 1\right) - 9k = 4^{k+1} + 3k + 4 = 4^{k+1} + 3(k+1) + 1$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

7 Basis step:

When $n = 1$, $2u_1 = 2(1) - 1 + (-1)^1 = 0$

Assumption step:

Assume the closed form is true for $n = k$, so $2u_k = 2k - 1 + (-1)^k$

Inductive step:

$$2u_{k+1} = 4k - 2u_k = 4k - (2k - 1 + (-1)^k) = 2k + 1 - (-1)^k = 2(k+1) - 1 + (-1)^{k+1}$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

8 Basis step:

When $n = 1$, $u_1 = 2 - \left(-\dfrac{1}{2}\right)^{-1} = 2 - (-2) = 4$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 2 - \left(-\frac{1}{2}\right)^{k-2}$

Inductive step:

$$u_{k+1} = 3 - \frac{1}{2}u_k = 3 - \frac{1}{2}\left(2 - \left(-\frac{1}{2}\right)^{k-2}\right) = 3 - 1 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^{k-2} = 2 - \left(-\frac{1}{2}\right)^{(k+1)-2}$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

9 Basis step:

When $n = 1$, $u_1 = 3^0(1)! = 1$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 3^{k-1}k!$

Inductive step:

$$u_{k+1} = 3(k+1)u_k = 3(k+1)3^{k-1}k! = 3(3^{k-1})(k+1)k! = 3^{(k+1)-1}(k+1)!$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$.

Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

10 a With $2n$ people, any person can pair with any of the $2n - 1$ other people. Having made this pairing, the remaining $2n - 2$ people can be paired in P_{n-1} ways, so multiplying gives $P_n = (2n-1)P_{n-1}$.

b Basis step:

When $n = 1$, $P_1 = \dfrac{2!}{2^1(1)!} = 1$

Assumption step:

Assume the closed form is true for $n = k$, so $P_k = \dfrac{(2k)!}{2^k k!}$

Inductive step:

$$P_{k+1} = (2(k+1)-1)P_k = (2k+1)\frac{(2k)!}{2^k k!} = \frac{(2k+2)!}{(2k+2)2^k k!} = \frac{(2(k+1))!}{2^{k+1}(k+1)!}$$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$. Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

11 Basis step:

When $n = 1$, $u_1 = 3^1 - 2^1 = 1$; when $n = 2$, $u_2 = 3^2 - 2^2 = 5$

Assumption step:

Assume the closed form is true for $n = k$ and $n = k + 1$, so $u_k = 3^k - 2^k$ and $u_{k+1} = 3^{k+1} - 2^{k+1}$

Inductive step:

$$u_{k+2} = 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k) = 15(3^k) - 10(2^k) - 6(3^k) + 6(2^k) = 9(3^k) - 4(2^k) = 3^{k+2} - 2^{k+2}$$

So if the closed form is valid for $n = k$ and $n = k + 1$ it is valid for $n = k + 2$. Since the closed form is true for $n = 1$ and $n = 2$, by induction the closed form is true for all $n \in \mathbb{N}$.

12 Basis step:

When $n = 1$, $u_1 = (-1)3^0 = -1$; when $n = 2$, $u_2 = (0)3^1 = 0$

Assumption step:

Assume the closed form is true for $n = k$ and $n = k + 1$, so $u_k = (k-2)3^{k-1}$ and $u_{k+1} = (k-1)3^k$

Inductive step:

$u_{k+2} = 6(k-1)3^k - 9(k-2)3^{k-1} = 2(k-1)3^{k+1} - (k-2)3^{k+1} = k3^{k+1}$

So if the closed form is valid for $n = k$ and $n = k + 1$ it is valid for $n = k + 2$. Since the closed form is true for $n = 1$ and $n = 2$, by induction the closed form is true for all $n \in \mathbb{N}$.

13 Basis step:

When $n = 1$, $u_1 = 2(5^0) - 2^0 = 1$; when $n = 2$, $u_2 = 2(5^1) - 2^1 = 8$

Assumption step:

Assume the closed form is true for $n = k$ and $n = k + 1$, so $u_k = 2(5^{k-1}) - 2^{k-1}$ and $u_{k+1} = 2(5^k) - 2^k$

Inductive step:

$u_{k+2} = 7(2(5^k) - 2^k) - 10(2(5^{k-1}) - 2^{k-1})$

$= 14(5^k) - 7(2^k) - 4(5^k) + 5(2^k) = 10(5^k) - 2(2^k) = 2(5^{k+1}) - 2^{k+1}$

So if the closed form is valid for $n = k$ and $n = k + 1$ it is valid for $n = k + 2$. Since the closed form is true for $n = 1$ and $n = 2$, by induction the closed form is true for all $n \in \mathbb{N}$.

14 Basis step:

When $n = 1$, $u_1 = 1 \times 3^1 = 3$; when $n = 2$, $u_2 = 4 \times 3^2 = 36$

Assumption step:

Assume the closed form is true for $n = k$ and $n = k + 1$, so $u_k = (3k-2)3^k$ and

$u_{k+1} = (3(k+1)-2)3^{k+1} = (3k+1)3^{k+1}$

Inductive step:

$u_{k+2} = 6(3k+1)3^{k+1} - 9(3k-2)3^k = 2(3k+1)3^{k+2} - (3k-2)3^{k+2} = (3k+4)3^{k+2} = (3(k+2)-2)3^{k+2}$

So if the closed form is valid for $n = k$ and $n = k + 1$ it is valid for $n = k + 2$. Since the closed form is true for $n = 1$ and $n = 2$, by induction the closed form is true for all $n \in \mathbb{N}$.

15 a $u_1 = 7$

$u_2 = 5 \times 7 - 3(2^1) = 29$

$u_3 = 5 \times 29 - 3(2^2) = 133$

$u_4 = 5 \times 133 - 3(2^3) = 641$

b Basis step:

When $n = 1$, $u_1 = 5^1 + 2^1 = 7$

Assumption step:

Assume the closed form is true for $n = k$, so $u_k = 5^k + 2^k$

Inductive step:

$u_{k+1} = 5(5^k + 2^k) - 3(2^k) = 5(5^k) + 2(2^k) = 5^{k+1} + 2^{k+1}$

So if the closed form is valid for $n = k$ it is valid for $n = k + 1$. Since the closed form is true for $n = 1$, by induction the closed form is true for all $n \in \mathbb{Z}^+$.

16 Basis step:

When $n = 1$, $L_1 = \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2} = 1$; when $n = 2$, $L_2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 + \left(\frac{1-\sqrt{5}}{2}\right)^2 = 3$

Assumption step:

Assume the closed form is true for $n = k$ and $n = k + 1$, so $L_k = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k$ and

$$L_{k+1} = \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}$$

Inductive step:

$$L_{k+2} = \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1-\sqrt{5}}{2}\right)^{k}$$

$$= \left(\frac{1+\sqrt{5}}{2}+1\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1-\sqrt{5}}{2}+1\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}$$

$$= \left(\frac{3+\sqrt{5}}{2}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{3-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{k}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{2}\left(\frac{1+\sqrt{5}}{2}\right)^{k} + \left(\frac{1-\sqrt{5}}{2}\right)^{2}\left(\frac{1-\sqrt{5}}{2}\right)^{k}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}$$

So if the closed form is valid for $n = k$ and $n = k + 1$ it is valid for $n = k + 2$. Since the closed form is true for $n = 1$ and $n = 2$, by induction the closed form is true for all $n \in \mathbb{N}$.