Recurrence relations 4B

1 a The recurrence relation is of the form $u_n = au_{n-1}$ with $a = 2$, so use the general form of the solution:

$u_n = u_0 a^n$ so $u_n = 5\left(2^n\right)$

b The recurrence relation is of the form $b_n = ab_{n-1}$ with $a = \frac{5}{2}$, so use the general form of the solution:

$b_n = b_1 a^{n-1}$ so $b_n = 4\left(\frac{5}{2}\right)^{n-1}$

c The recurrence relation is of the form $d_n = ad_{n-1}$ with $a = -\frac{11}{10}$, so use the general form of the solution:

$d_n = d_1 a^{n-1}$ so $d_n = 10\left(-\frac{11}{10}\right)^{n-1}$

d The recurrence relation is of the form $x_{n+1} = ax_n$, so $x_n = ax_{n-1}$ with $a = -3$ so use the general form of the solution:

$x_n = x_0 a^n$ so $x_n = 2\left(-3\right)^n$

2 a The recurrence relation is of the form $u_n = u_{n-1} + \mathrm{g}(n)$ where $\mathrm{g}(n) = 3$, so use the general form of the solution:

$u_n = u_0 + \sum_{r=1}^{n} \mathrm{g}(r)$ so $u_n = 5 + \sum_{r=1}^{n} 3 = 5 + 3n$

b The recurrence relation is of the form $x_n = x_{n-1} + \mathrm{g}(n)$ where $\mathrm{g}(n) = n$, so use the general form of the solution:

$x_n = x_0 + \sum_{r=1}^{n} \mathrm{g}(r)$ so $x_n = 2 + \sum_{r=1}^{n} r = 2 + \frac{n(n+1)}{2} = 2 + \tfrac{1}{2}n + \tfrac{1}{2}n^2$

c The recurrence relation is of the form $y_n = y_{n-1} + \mathrm{g}(n)$ where $\mathrm{g}(n) = n^2 - 2$, so use the general form of the solution:

$y_n = y_0 + \sum_{r=1}^{n} \mathrm{g}(r)$ so $y_n = 3 + \sum_{r=1}^{n} (r^2 - 2) = 3 + \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 2 = 3 - 2n + \frac{1}{6}n(n+1)(2n+1)$

d Substitute n for $n - 1$ throughout the recurrence relation:

$s_n = s_{n-1} + 2(n-1) - 1 = s_{n-1} + 2n - 3$

The recurrence relation is now of the form $s_n = s_{n-1} + \mathrm{g}(n)$ where $\mathrm{g}(n) = 2n - 3$ so use the general form of the solution:

$s_n = s_0 + \sum_{r=1}^{n} \mathrm{g}(r)$ so $s_n = 1 + \sum_{r=1}^{n} (2r - 3) = 1 + 2\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3 = 1 + 2\frac{n(n+1)}{2} - 3n = 1 - 2n + n^2$

3 **a** The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$, so the complementary function is $a_n = c(2^n)$

Particular solution: $a_n = \lambda$

$a_n = 2a_{n-1} + 1$

$\lambda = 2\lambda + 1$

$\Rightarrow \lambda = -1$

So a particular solution to the recurrence relation is $a_n = -1$.

The general solution is $a_n = c(2^n) - 1$

Since $a_1 = 1$, $1 = 2c - 1$ so $c = 1$

The solution is $a_n = 2^n - 1$

b The associated homogeneous recurrence relation is $u_n = -u_{n-1}$, so the complementary function is $u_n = c(-1)^n$.

Particular solution: $u_n = \lambda$

$u_n = -u_{n-1} + 2$

$\lambda = -\lambda + 2$

$\Rightarrow \lambda = 1$

So a particular solution to the recurrence relation is $u_n = 1$.

The general solution is $u_n = c(-1)^n + 1$

Since $u_1 = 3$, $3 = -c + 1$ so $c = -2$

The solution is $u_n = -2(-1)^n + 1$

c The associated homogeneous recurrence relation is $h_n = 3h_{n-1}$, so the complementary function is $h_n = c(3^n)$.

Particular solution: $h_n = \lambda$

$h_n = 3h_{n-1} + 5$

$\lambda = 3\lambda + 5$

$\Rightarrow \lambda = -\frac{5}{2}$

So a particular solution to the recurrence relation is $h_n = -\frac{5}{2}$

The general solution is $h_n = c(3^n) - \frac{5}{2}$

Since $h_0 = 1$, $1 = c - \frac{5}{2}$ so $c = \frac{7}{2}$

The solution is $h_n = \frac{7}{2}(3^n) - \frac{5}{2} = \frac{1}{2}(7 \times 3^n - 5)$

3 d The associated homogeneous recurrence relation is $b_n = -2b_{n-1}$, so the complementary function is $b_n = c(-2)^n$.

Particular solution: $b_n = \lambda$

$b_n = -2b_{n-1} + 6$

$\lambda = -2\lambda + 6$

$\Rightarrow \lambda = 2$

So a particular solution to the recurrence relation is $b_n = 2$.

The general solution is $b_n = c(-2)^n + 2$

Since $b_1 = 3$, $3 = -2c + 2$ so $c = -\frac{1}{2}$

The solution is $b_n = -\frac{1}{2}(-2)^n + 2 = 2 + (-2)^{n-1}$

4 a $n - 1$ teams play each other g_{n-1} times. When an nth team is added, this team has to play each of the other $n - 1$ teams once so there are $g_{n-1} + n - 1$ games in total.

Hence $g_n = g_{n-1} + n - 1$ and $g_1 = 0$

b Using the standard form of the solution when g_1 is given:

$$g_n = g_1 + \sum_{r=2}^{n}(r-1) = 0 + \sum_{r=2}^{n} r - \sum_{r=2}^{n} 1$$

Hence $g_n = \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n-1)}{2}$

5 a The associated homogeneous recurrence relation is $u_n = 4u_{n-1}$, so the complementary function is $u_n = c(4^n)$.

Particular solution: $u_n = \lambda$

$u_n = 4u_{n-1} - 1$

$\lambda = 4\lambda - 1$

$\Rightarrow \lambda = \frac{1}{3}$

So a particular solution to the recurrence relation is $u_n = \frac{1}{3}$

The general solution is $u_n = c(4^n) + \frac{1}{3}$

b i Since $u_1 = 3$, $3 = 4c + \frac{1}{3}$ so $c = \frac{2}{3}$

The solution is $u_n = \frac{2}{3}(4^n) + \frac{1}{3} = \frac{1}{3}(2 \times 4^n + 1)$

ii Since $u_1 = 0$, $0 = 4c + \frac{1}{3}$ so $c = -\frac{1}{12}$

The solution is $u_n = -\frac{1}{12}(4^n) + \frac{1}{3} = \frac{1}{3}(1 - 4^{n-1})$

5 b iii Since $u_1 = 200$, $200 = 4c + \frac{1}{3}$ so $c = \frac{599}{12}$

The solution is $u_n = \frac{599}{12}(4^n) + \frac{1}{3} = \frac{1}{3}(599 \times 4^{n-1} + 1)$

6 a The associated homogeneous recurrence relation is $u_n = 3u_{n-1}$, so the complementary function is $u_n = c(3^n)$.

Particular solution: $u_n = \lambda n + \mu$

$u_n = 3u_{n-1} + n$

$\lambda n + \mu = 3(\lambda(n-1) + \mu) + n$

$\lambda n + \mu = 3\lambda n - 3\lambda + 3\mu + n$

Equating coefficients:

$\lambda = 3\lambda + 1 \Rightarrow \lambda = -\frac{1}{2}$

$\mu = -3\lambda + 3\mu \Rightarrow \mu = -\frac{3}{4}$

So a particular solution to the recurrence relation is $u_n = -\frac{1}{2}n - \frac{3}{4}$

The general solution is $u_n = c(3^n) - \frac{1}{2}n - \frac{3}{4}$

b Since $u_1 = 5$, $5 = 3c - \frac{1}{2} - \frac{3}{4}$ so $c = \frac{25}{12}$

The solution is $u_n = \frac{25}{12}(3^n) - \frac{1}{2}n - \frac{3}{4} = \frac{1}{4}(25 \times 3^{n-1} - 2n - 3)$

7 a $u_1 = 0.6 \times 7 + 4 = 8.2$

$u_2 = 0.6 \times 8.2 + 4 = 8.92$

$u_3 = 0.6 \times 8.92 + 4 = 9.352$

b The associated homogeneous recurrence relation is $u_n = 0.6u_{n-1}$, so the complementary function is $u_n = c(0.6^n)$.

Particular solution: $u_n = \lambda$

$u_n = 0.6u_{n-1} + 4$

$\lambda = 0.6\lambda + 4$

$\Rightarrow \lambda = 10$

So a particular solution to the recurrence relation is $u_n = 10$

The general solution is $u_n = c(0.6^n) + 10$

Since $u_0 = 7$, $7 = c + 10$ so $c = -3$

The solution is $u_n = -3(0.6^n) + 10 = 10 - 3(0.6^n)$

c $10 - 3(0.6^n) > 9.9 \Rightarrow 0.6^n < \frac{1}{30}$

$n \log 0.6 < \log\left(\frac{1}{30}\right) \Rightarrow n > 6.658... \Rightarrow n = 7$

8 a Since the population falls by 5% each year, and 20 deer are added, a recurrence relation is $D_n = 0.95D_{n-1} + 20$ with $D_0 = 200$.

b The associated homogeneous recurrence relation is $D_n = 0.95D_{n-1}$, so the complementary function is $D_n = c(0.95^n)$.

Particular solution: $D_n = \lambda$

$$D_n = 0.95D_{n-1} + 20$$
$$\lambda = 0.95\lambda + 20$$
$$\Rightarrow \lambda = 400$$

So a particular solution to the recurrence relation is $D_n = 400$

The general solution is $D_n = c(0.95^n) + 400$

Since $D_0 = 200$, $200 = c + 400$ so $c = -200$

The solution is $D_n = -200(0.95^n) + 400 = 200(2 - 0.95^n)$

c As $n \to \infty$, $0.95^n \to 0$ so the deer population approaches $200 \times 2 = 400$.

9 $u_n = 4u_{n-1} - 3$

The associated homogeneous recurrence relation is $u_n = 4u_{n-1}$, so the complementary function is $u_n = c(4^n)$.

Particular solution: $u_n = \lambda$

$$u_n = 4u_{n-1} - 3$$
$$\lambda = 4\lambda - 3$$
$$\Rightarrow \lambda = 1$$

So a particular solution to the recurrence relation is $u_n = 1$

The general solution is $u_n = c(4^n) + 1$

Since $u_0 = 7$, $7 = c + 1$ so $c = 6$

The solution is $u_n = 6(4^n) + 1$

10 $u_0 = u_1 - 2^1 = 5 - 2 = 3$

Now use the formula $u_n = u_0 + \sum_{r=1}^{n} \mathrm{g}(r)$:

$$u_n = 3 + \sum_{r=1}^{n} 2^r = 3 + \frac{2(1-2^n)}{1-2} = 3 - 2 + 2 \times 2^n = 2^{n+1} + 1$$

11 The associated homogeneous recurrence relation is $u_n = 4u_{n-1}$, so the complementary function is $u_n = c(4^n)$.

Particular solution: $u_n = \lambda n + \mu$

$u_n = 4u_{n-1} + 2n$

$\lambda n + \mu = 4(\lambda(n-1) + \mu) + 2n$

$\lambda n + \mu = 4\lambda n - 4\lambda + 4\mu + 2n$

Equating coefficients:

$\lambda = 4\lambda + 2 \Rightarrow \lambda = -\frac{2}{3}$

$\mu = -4\lambda + 4\mu \Rightarrow \mu = -\frac{8}{9}$

So a particular solution to the recurrence relation is $u_n = -\frac{2}{3}n - \frac{8}{9}$

The general solution is $u_n = c(4^n) - \frac{2}{3}n - \frac{8}{9}$

Since $u_0 = 7$, $7 = c - \frac{8}{9}$ so $c = \frac{71}{9}$

The solution is $u_n = \frac{71}{9}(4^n) - \frac{2}{3}n - \frac{8}{9} = \frac{1}{9}(71 \times 4^n - 6n - 8)$

12 a The associated homogeneous recurrence relation is $u_n = 2u_{n-1}$, so the complementary function is $u_n = c(2^n)$.

Particular solution: $u_n = \lambda$

$u_n = 2u_{n-1} - 1005$

$\lambda = 2\lambda - 1005$

$\Rightarrow \lambda = 1005$

So a particular solution to the recurrence relation is $u_n = 1005$

The general solution is $u_n = c(2^n) + 1005$

Since $u_0 = 1000$, $1000 = c + 1005$ so $c = -5$

The solution is $u_n = -5(2^n) + 1005 = -5(2^n - 201)$

b $1005 - 5(2^n) < 0 \Rightarrow 2^n > 201$

$n \log 2 > \log(201) \Rightarrow n > 7.65 \Rightarrow n = 8$

Hence $u_8 = -5(2^8 - 201) = -275$

13 a The associated homogeneous recurrence relation is $u_n = 2u_{n-1}$, so the complementary function is $u_n = c(2^n)$.

Particular solution: $u_n = \lambda n 2^n$

$$u_n = 2u_{n-1} - 2^n$$
$$\lambda n(2^n) = 2\lambda(n-1)2^{n-1} - 2^n$$
$$\lambda n(2^n) = 2\lambda n(2^{n-1}) - 2\lambda(2^{n-1}) - 2^n$$
$$\lambda n(2^n) = \lambda n(2^n) - \lambda(2^n) - 2^n$$
$$\Rightarrow \lambda = -1$$

So a particular solution to the recurrence relation is $u_n = -n(2^n)$

The general solution is $u_n = c(2^n) - n(2^n) = 2^n(c-n)$

b Since $u_1 = 3$, $3 = 2(c - 1)$ so $c = \frac{5}{2}$

The solution is $u_n = 2^n\left(\frac{5}{2} - n\right)$

14 a $u_1 = k \times 0 + 1 = 1$

$u_2 = k \times 1 + 1 = k + 1$

$u_3 = k \times (k+1) + 1 = k^2 + k + 1$

b The associated homogeneous recurrence relation is $u_n = ku_{n-1}$, so the complementary function is $u_n = c(k^n)$.

Particular solution: $u_n = \lambda$

$$u_n = ku_{n-1} + 1$$
$$\lambda = k\lambda + 1$$
$$\Rightarrow \lambda = \frac{1}{1-k}$$

So a particular solution to the recurrence relation is $u_n = \frac{1}{1-k}$

The general solution is $u_n = c(k^n) + \frac{1}{1-k}$

Since $u_0 = 0$, $0 = c + \frac{1}{1-k}$ so $c = -\frac{1}{1-k}$

The solution is $u_n = -\frac{1}{1-k}(k^n) + \frac{1}{1-k} = \frac{k^n - 1}{k-1}$

c i When $k > 1$, $u_n \to \infty$

ii When $-1 < k < 1$, $u_n \to -\frac{1}{k-1} = \frac{1}{1-k}$

iii When $k = -1$, u_n alternately takes the values 0 and 1

iv When $k < -1$, u_n is divergent to $\pm\infty$ and the sign alternates.

15 a $\sum_{r=1}^{n}(6r+1)=6\sum_{r=1}^{n}r+\sum_{r=1}^{n}1=3n(n+1)+n=3n^2+4n$

b $u_n=u_0+\sum_{r=1}^{n}g(r)$ so using part **a**:

$u_n=2+3n^2+4n$

c $561=2+3n^2+4n\Rightarrow 3n^2+4n-559=0$

Hence $n=-\frac{43}{3}$ or 13 and since $n\in\mathbb{Z}^+$, $n=13$

16 a The recurrence relation is of the form $u_n=u_{n-1}+g(n)$ where $g(n)=-6n^2$ so use the general form of the solution:

$u_n=u_0+\sum_{r=1}^{n}g(r)$ so $u_n=89-\sum_{r=1}^{n}6r^2=89-6\sum_{r=1}^{n}r^2=89-n(n+1)(2n+1)$

b $89-n(n+1)(2n+1)<0\Rightarrow 2n^3+3n^2+n-89>0$

The first integer value of n for which this is true is $n=4$ therefore $u_4=89-4(5)(9)=-91$

c Adding an odd number (89) to an even number ($n(n+1)(2n+1)$) always gives an odd number.

17 a The recurrence relation is of the form $u_n=u_{n-1}+g(n)$ where $g(n)=-2n$ so use the general form of the solution:

$u_n=u_0+\sum_{r=1}^{n}g(r)$ so $u_n=3-2\sum_{r=1}^{n}r=3-n(n+1)$

b Solving $3-n(n+1)=-103\Rightarrow n^2+n-106=0$ which has no integer solutions

c $3-k(k+1)=-459\Rightarrow k^2+k-462=0$

Hence $k=21$ or -22 and since k has to be positive, $k=21$

18 a Since the interest added is 1.5% each month, and Alison pays off P per month, a recurrence relation is:

$u_n=1.015u_{n-1}-P$ with $u_0=2000$

18 b The associated homogeneous recurrence relation is $u_n = 1.015u_{n-1}$, so the complementary function is $u_n = c(1.015^n)$.

Particular solution: $u_n = \lambda$

$u_n = 1.015u_{n-1} - P$

$\lambda = 1.015\lambda - P$

$\Rightarrow \lambda = \frac{200}{3}P$

So a particular solution to the recurrence relation is $u_n = \frac{200}{3}P$

The general solution is $u_n = c(1.015^n) + \frac{200}{3}P$

Since $u_0 = 2000$, $2000 = c + \frac{200}{3}P$ so $c = 2000 - \frac{200}{3}P$

The solution is $u_n = \left(2000 - \frac{200}{3}P\right)(1.015^n) + \frac{200}{3}P = \frac{200}{3}(1.015^n(30 - P) + P)$

c When $n = 18$, $u_n = 0$:

$\frac{200}{3}(1.015^{18}(30 - P) + P) = 0 \Rightarrow 30 \times 1.015^{18} - P \times 1.015^{18} + P = 0$

Hence $P = \frac{30 \times 1.015^{18}}{1.015^{18} - 1} = 127.611...$

So $P = £127.61$

Challenge

a The first disk cannot be moved from *A* to *C* in one jump, so must move from *A* to *B*, then *B* to *C*.

b The sequence of moves is as follows:
A to *B*, *B* to *C*, *A* to *B*, *C* to *B*, *B* to *A*, *B* to *C*, *A* to *B* and *B* to *C*

c Transfer $n - 1$ disks from *A* to *C* (H_{n-1} moves), then move *n*th disk from *A* to *B* (1 move), then transfer $n - 1$ disks from *C* to *A* (H_{n-1} moves), then move *n*th disk from *B* to *C* (1 move), then transfer $n - 1$ disks from *A* to *C* (H_{n-1} moves). In total, therefore, $H_n = 3H_{n-1} + 2$.

d i The associated homogeneous recurrence relation is $H_n = 3H_{n-1}$ so the complementary function is $H_n = c(3^n)$

Particular solution: $u_n = \lambda$

$H_n = 3H_{n-1} + 2$

$\lambda = 3\lambda + 2$

$\Rightarrow \lambda = -1$

So a particular solution to the recurrence relation is $H_n = -1$

The general solution is $H_n = c(3^n) - 1$

Since $H_1 = 2$, $2 = 3c - 1$ so $c = 1$

The solution is $H_n = 3^n - 1$

ii When $n = 10$, $H_{10} = 3^{10} - 1 = 59048$