Complex numbers Mixed exercise

1 a i Write $z = x + \mathrm{i}y$ and square both sides:

$$|x+\mathrm{i}y|^2 = |x-4+\mathrm{i}y|^2$$
$$x^2+y^2 = (x-4)^2+y^2$$
$$x^2 = x^2-8x+16$$
$$x = 2$$

ii This equation describes points on the perpendicular bisector of the line segment connecting $(0,0)$ and $(4,0)$

b i Write $z = x + \mathrm{i}y$ and square both sides:

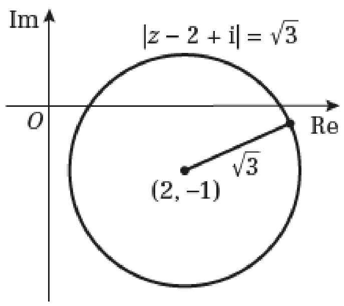
$$|x+\mathrm{i}y|^2 = 4|x-4+\mathrm{i}y|^2$$
$$x^2+y^2 = 4(x-4)^2+4y^2$$
$$x^2+y^2 = 4(x^2-8x+16)+4y^2$$
$$x^2+y^2 = 4x^2-32x+64+4y^2$$
$$3y^2+3x^2-32x+64=0$$
$$x^2-\frac{32}{3}x+y^2+\frac{64}{3}=0$$

Complete the square:

$$\left(x-\frac{16}{3}\right)^2+y^2=\frac{64}{9}$$

ii The equation describes a circle centred at $\left(\frac{16}{3},0\right)$, radius $r=\frac{8}{3}$

2 a The equation $|z-2+\mathrm{i}|=\sqrt{3}$ describes a circle centred at $(2,-1)$, radius $r=\sqrt{3}$:



2 b The half-line L $y = mx - 1,\ x \geqslant 0,\ m > 0$ is tangent to the circle from part **a**. This means that it has to lie on the circle and thus satisfy the equation $(x-2)^2 + (y+1)^2 = 3$. Substituting the expression for y into this equation gives:

$(x-2)^2 + (mx)^2 = 3$

$x^2 - 4x + 4 + m^2x^2 - 3 = 0$

$x^2(1+m^2) - 4x + 1 = 0$

$x^2 - \frac{4}{1+m^2} + \frac{1}{1+m^2} = 0$

This equation must have exactly one solution, as the line and the circle only touch at one point. Therefore, it needs to be of the form:

$\left(x - \frac{2}{1+m^2}\right)^2 = 0$ which means that $\left(\frac{2}{1+m^2}\right)^2 = \frac{1}{1+m^2}$

Solving this gives:

$\frac{4}{(1+m^2)^2} = \frac{1}{1+m^2}$

$4 = 1 + m^2$

$m^2 = 3$

$m = \sqrt{3}$

since we know that $m > 0$.

So L is given by $y = x\sqrt{3} - 1$.

c For $x = 0$, $y = -1$ so the line goes through $(0,-1)$. The gradient is equal to $\sqrt{3}$, so $\tan\theta = \sqrt{3}$.

Therefore $\theta = \frac{\pi}{3}$. So the equation for L can be written as $\arg(z+i) = \frac{\pi}{3}$.

d Using $\left(x - \frac{2}{1+m^2}\right)^2 = 0$ from part **b** and substituting $m = \sqrt{3}$ we obtain $\left(x - \frac{1}{2}\right)^2 = 0$, so $x = \frac{1}{2}$.

Now substituting this value into $y = x\sqrt{3} - 1$ we see $y = \frac{\sqrt{3}-2}{2}$. So $a = \frac{1}{2} + \left(\frac{\sqrt{3}-2}{2}\right)\text{i}$.

3 a $|z+2|=|2z-1|$

$\Rightarrow |x+\mathrm{i}y+2| = |2(x+\mathrm{i}y)-1|$

$\Rightarrow |x+\mathrm{i}y+2| = |2x+2\mathrm{i}y-1|$

$\Rightarrow |(x+2)+\mathrm{i}y| = |(2x-1)+\mathrm{i}(2y)|$

$\Rightarrow |(x+2)+\mathrm{i}y|^2 = |(2x-1)+\mathrm{i}(2y)|^2$

$\Rightarrow (x+2)^2+y^2=(2x-1)^2+(2y)^2$

$\Rightarrow x^2+4x+4+y^2=4x^2-4x+1+4y^2$

$\Rightarrow 0=3x^2-8x+3y^2+1-4$

$\Rightarrow 3x^2-8x+3y^2-3=0$

$\Rightarrow x^2-\frac{8}{3}x+y^2-1=0$

$\Rightarrow \left(x-\frac{4}{3}\right)^2-\frac{16}{9}+y^2-1=0$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{16}{9}+1$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$

$\Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\left(\frac{5}{3}\right)^2$

This is a circle, centre $\left(\frac{4}{3},0\right)$, radius $\frac{5}{3}$.

The Cartesian equation of the locus of points representing $|z+2|=|2z-1|$ is $\left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$.

3 b $|z+2|=|2z-1| \Rightarrow \left(x-\frac{4}{3}\right)^2+y^2=\frac{25}{9}$ **(1)**

$\arg z=\frac{\pi}{4} \Rightarrow \arg(x+\text{i}y)=\frac{\pi}{4}$

$\Rightarrow \frac{y}{x}=\tan\frac{\pi}{4}$

$\Rightarrow \frac{y}{x}=1$

$\Rightarrow y=x \quad \text{where } x>0, y>0$ **(2)**

Solving simultaneously:

$\left(x-\frac{4}{3}\right)^2+x^2=\frac{25}{9}$

$\Rightarrow x^2-\frac{4}{3}x-\frac{4}{3}x+\frac{16}{9}+x^2=\frac{25}{9}$

$\Rightarrow 2x^2-\frac{8}{3}x=\frac{25}{9}-\frac{16}{9}$

$\Rightarrow 2x^2-\frac{8}{3}x=\frac{9}{9}$

$\Rightarrow 2x^2-\frac{8}{3}x=1 \qquad (\times 3)$

$\Rightarrow 6x^2-8x=3$

$\Rightarrow 6x^2-8x-3=0$

$\Rightarrow x=\frac{8\pm\sqrt{64-4(6)(-3)}}{2(6)}$

$\Rightarrow x=\frac{8\pm\sqrt{136}}{12}$

$\Rightarrow x=\frac{8\pm 2\sqrt{34}}{12}$

$\Rightarrow x=\frac{4\pm\sqrt{34}}{6}$

As $x>0$ then we reject $x=\frac{4-\sqrt{34}}{6}$

and accept $x=\frac{4+\sqrt{34}}{6}$

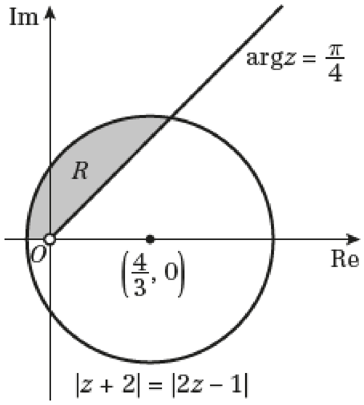
as $y=x$, then $y=\frac{4+\sqrt{34}}{6}$

So $z=\left(\frac{4+\sqrt{34}}{6}\right)+\left(\frac{4+\sqrt{34}}{6}\right)\text{i}$

The value of z satisfying $|z+2|=|2z-1|$ and $\arg z=\frac{\pi}{4}$

is $z=\left(\frac{4+\sqrt{34}}{6}\right)+\left(\frac{4+\sqrt{34}}{6}\right)\text{i}$ OR $z=1.64+1.64\text{i}$ (2 d.p.)

3 c



The region R (shaded) satisfies both $|z+2| \geqslant |2z-1|$ and $\frac{\pi}{4} \leqslant \arg z \leqslant \pi$.

Note that $|z+2| \geqslant |2z-1|$

$\Rightarrow (x+2)^2 + y^2 \geqslant (2x-1)^2 + (2y)^2$

$\Rightarrow 0 \geqslant 3x^2 - 8x + 3y^2 - 3$

$\Rightarrow 0 \geqslant \left(x - \frac{4}{3}\right)^2 - \frac{16}{9} + y^2 - 1$

$\Rightarrow \frac{25}{9} \geqslant \left(x - \frac{4}{3}\right)^2 + y^2$

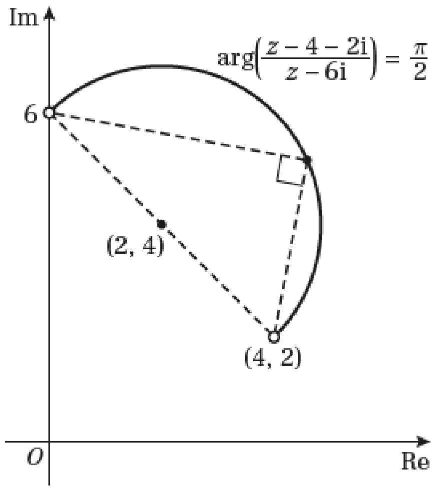
$\Rightarrow \left(x - \frac{4}{3}\right)^2 + y^2 \leqslant \frac{25}{9}$

represents region inside and bounded by the circle, centre $\left(\frac{4}{3}, 0\right)$, radius $\frac{5}{3}$.

4 a $\arg\left(\dfrac{z-4-2\text{i}}{z-6\text{i}}\right)=\dfrac{\pi}{2}$

$\Rightarrow \arg(z-4-2\text{i})-\arg(z-6\text{i})=\dfrac{\pi}{2}$

$\Rightarrow \theta-\phi=\dfrac{\pi}{2}$, where $\arg(z-4-2\text{i})=\theta$ and $\arg(z-6\text{i})=\phi$.



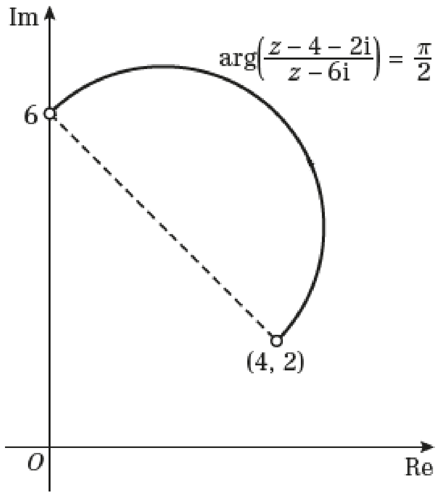
Using geometry,

$\Rightarrow A\hat{P}B=-\phi+\theta$

$\Rightarrow A\hat{P}B=\theta-\phi$

$\Rightarrow A\hat{P}B=\dfrac{\pi}{2}$

The locus of z is the arc of a circle (in this case, a semi-circle) cut off at (4, 2) and (0, 6) as shown below.



4 b $|z-2-4\text{i}|$ is the distance from the point (2, 4) to the locus of points P.

Note, as the locus is a semi-circle, its centre is $\left(\frac{4+0}{2},\frac{2+6}{2}\right)=(2,4)$.

Therefore $|z-2-4\text{i}|$ is the distance from the centre of the semi-circle to points on the locus of points P.

Hence $|z-2-4\text{i}|$ = radius of semi-circle

$$= \sqrt{(0-2)^2+(6-4)^2}$$
$$= \sqrt{4+4}$$
$$= \sqrt{8}$$
$$= 2\sqrt{2}$$

The exact value of $|z-2-4\text{i}|$ is $2\sqrt{2}$

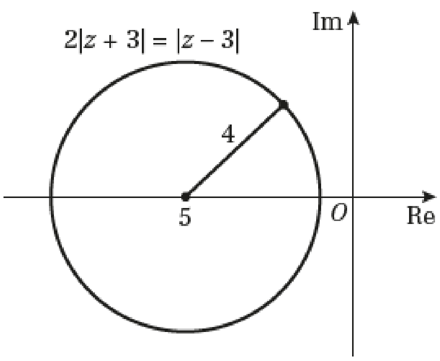
5 We have $2|z+3|=|z-3|$

a To show that this describes a circle, write $z=x+\text{i}y$ and square both sides:

$$2|x+3+iy|=|x-3+iy|$$
$$4|x+3+\text{i}y|^2=|x-3+\text{i}y|^2$$
$$4(x+3)^2+4y^2=(x-3)^2+y^2$$
$$4x^2+24x+36+3y^2=x^2-6x+9$$
$$3x^2+30x+27+3y^2=0$$
$$x^2+y^2+10x+9=0$$

as required.

b



5 c L is given by $b^*z + bz^* = 0$ where $b, z \in \mathbb{C}$.
We know that L is tangent to the circle and that $\arg b = \theta$.
We want to find possible values of $\tan\theta$.
Write $z = x + \text{i}y$, $b = u + \text{i}v$.
Then the equation for L becomes:

$$b(x - \text{i}y) + b^*(x + \text{i}y) = 0$$

$$(u + \text{i}v)(x - \text{i}y) + (u - \text{i}v)(x + \text{i}y) = 0$$

$$ux + vy = 0$$

$$y = -\frac{ux}{v},\ v \neq 0$$

If $v = 0$ then $ux = 0$, so either $u = 0$ i.e. $b = 0$, which means that the line does not exist, or $x = 0$ but this is not tangent to the circle.
So we can assume $v \neq 0$.

Now, since $b = u + \text{i}v$, $\tan\theta = \dfrac{v}{u}$.

So L can be written as $y = -\dfrac{x}{\tan\theta}$

We want to find $\tan\theta$ such that the line is tangent to the circle.
Therefore, it has to satisfy the equation for the circle $x^2 + y^2 + 10x + 9 = 0$:

$$x^2 + \frac{x^2}{\tan^2\theta} + 10x + 9 = 0$$

$$x^2\tan^2\theta + x^2 + 10x\tan^2\theta + 9\tan^2\theta = 0$$

$$x^2(\tan^2\theta + 1) + 10x\tan^2\theta + 9\tan^2\theta = 0$$

Let $a = \tan^2\theta$. Then $x^2(a+1) + 10xa + 9a = 0$.
Since the line is tangent to the circle, this equation can only have one solution.
Therefore we need $\Delta = 0$.
Therefore

$$100a^2 - 36a(a+1) = 0$$

$$100a^2 - 36a^2 - 36a = 0$$

$$a(64a - 36) = 0$$

$$a = 0 \quad \text{or} \quad a = \frac{36}{64} = \frac{9}{16}$$

Recall that $a = \tan^2\theta = \dfrac{v^2}{u^2}$

Since $v \neq 0$, we have $a \neq 0$

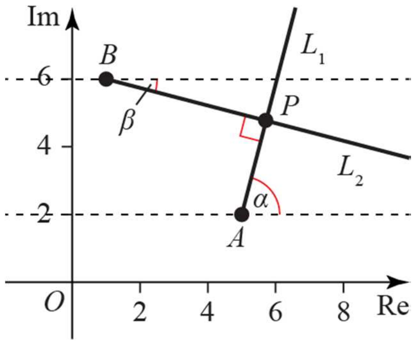
So $a = \dfrac{9}{16}$ and we can solve for $\tan\theta$:

$$\tan^2\theta = \frac{9}{16}$$

$$\tan\theta = \pm\frac{3}{4}$$

6 a Let $\arg(z-5-2\text{i})=\alpha$ and $\arg(z-1-6\text{i})=\beta$.

Then we have $\arg\left(\frac{z-5-2\text{i}}{z-1-6\text{i}}\right)=\arg(z-5-2\text{i})-\arg(z-1-6\text{i})=\alpha-\beta=\frac{\pi}{2}$



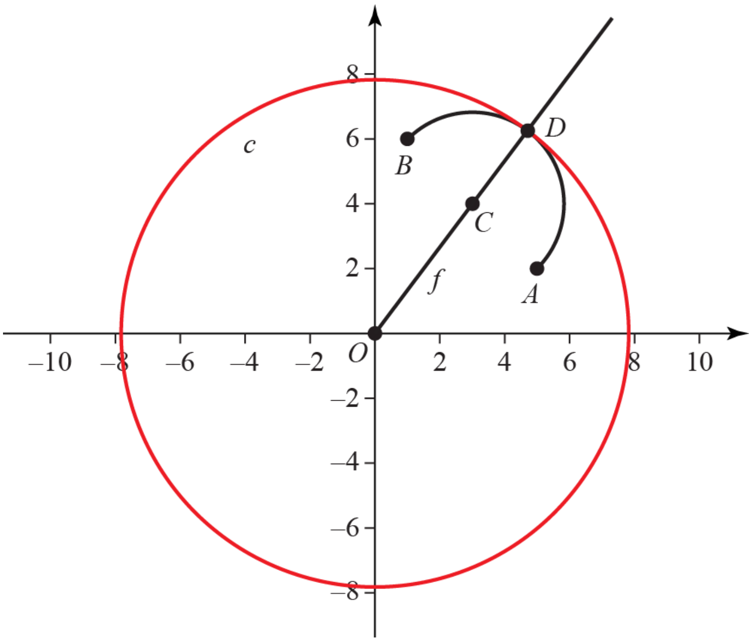
As α,β vary, P i.e. the intersection of L_1 and L_2 creates an arc.

Since $A\hat{P}B=\frac{\pi}{2}$, this arc will be a semicircle and the line segment AB is its diameter.

The centre of the circle lies in the middle of the line segment AB. $A=(5,2)$ and $B=(1,6)$ so the midpoint is $C=(3,4)$.

The radius is the distance CA. $CA=\sqrt{(5-3)^2+(4-2)^2}=2\sqrt{2}$, so $r=2\sqrt{2}$.

6 b The maximum value will lie on the line connecting the centre of this circle with the origin. This is represented by point D on the diagram below:



Thus D satisfies both the equation of the semicircle described in part **a**, $(x-3)^2+(y-4)^2=8$, and the equation of the line going through the origin and the centre of that semicircle, $y=\frac{4x}{3}$.

Substituting this into the equation for the circle we obtain:

$$(x-3)^2+\left(\frac{4x}{3}-4\right)^2=8$$

$$x^2-6x+9+\frac{16x^2}{9}-\frac{32x}{3}+16-8=0$$

$$\frac{25x^2}{9}-\frac{50x}{3}+17=0$$

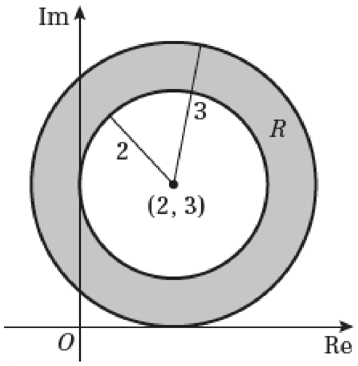
$$x^2-6x+\frac{153}{25}=0$$

$$x=3\pm\frac{6\sqrt{2}}{5}$$

We're looking for the larger value, so $x=3+\frac{6\sqrt{2}}{5}$, $y=4+\frac{8\sqrt{2}}{5}$ and so:

$$|z|=\sqrt{x^2+y^2}=\sqrt{\left(5+2\sqrt{2}\right)^2}=5+2\sqrt{2}.$$

7 a First note that both $2 = |z-2-3\text{i}|$ and $3 = |z-2-3\text{i}|$ represent circles centred at $(2,3)$ with radius $r=2$ and $r=3$ respectively.
Thus $2 \leqslant |z-2-3\text{i}| \leqslant 3$ represents the region between these two circles, including the circles since the inequalities are not strict.



b The area of this region can be found by subtracting the area of the smaller circle from the area of the larger circle $P_{\text{region}} = P_{\text{large}} - P_{\text{small}} = 9\pi - 4\pi = 5\pi$.

c We want to determine whether $z = 4+\text{i}$ lies within the region.
We have $|4+\text{i}-2-3\text{i}| = |2-2\text{i}| = 2|1-\text{i}| = 2\sqrt{2}$.
Since $2 \leqslant 2\sqrt{2} \leqslant 3$, the point lies in the region.

8 $T: w = \frac{1}{z}$

a line $x = \frac{1}{2}$ in the z-plane

$$w = \frac{1}{z}$$
$$\Rightarrow wz = 1$$
$$\Rightarrow z = \frac{1}{w}$$
$$\Rightarrow z = \frac{1}{u + \mathrm{i}v}$$
$$\Rightarrow z = \frac{1}{(u + \mathrm{i}v)} \times \frac{(u - \mathrm{i}v)}{(u - \mathrm{i}v)}$$
$$\Rightarrow z = \frac{u - \mathrm{i}v}{u^2 + v^2}$$
$$\Rightarrow z = \frac{u}{u^2 + v^2} + \mathrm{i}\left(\frac{-v}{u^2 + v^2}\right)$$

So, $x + \mathrm{i}y = \frac{u}{u^2 + v^2} + \mathrm{i}\left(\frac{-v}{u^2 + v^2}\right)$

$\Rightarrow x = \frac{u}{u^2 + v^2}$ and $y \frac{-v}{u^2 + v^2}$

As $x = \frac{1}{2}$, then $\frac{1}{2} = \frac{u}{u^2 + v^2}$

$$\Rightarrow u^2 + v^2 = 2u$$
$$\Rightarrow u^2 - 2u + v^2 = 0$$
$$\Rightarrow (u - 1)^2 - 1 + v^2 = 0$$
$$\Rightarrow (u - 1)^2 + v^2 = 1$$

Therefore the transformation T maps the line $x = \frac{1}{2}$ in the z-plane to a circle C, with centre (1, 0), radius 1. The equation of C is $(u - 1)^2 + v^2 = 1$.

8 b $x \geqslant \frac{1}{2}$

$\frac{u}{u^2+v^2} \geqslant \frac{1}{2}$

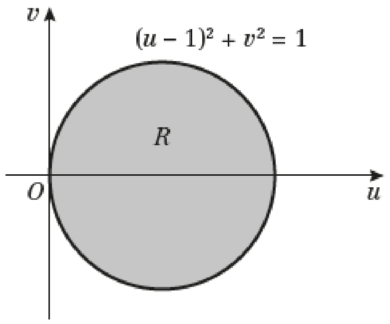
$\Rightarrow 2u \geqslant u^2+v^2$

$\Rightarrow 0 \geqslant u^2+v^2-2u$

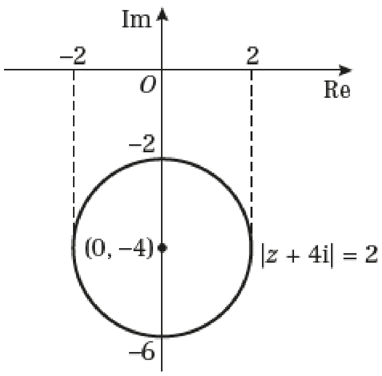
$\Rightarrow 0 \geqslant (u-1)^2+v^2-1$

$\Rightarrow 1 \geqslant (u-1)^2+v^2$

$\Rightarrow (u-1)^2+v^2 \leqslant 1$



9 a $|z+4\mathrm{i}| = 2$ is represented by a circle centre (0, –4), radius 2.



b $|z|$ represents the distance from (0, 0) to points on the locus of P.

Hence $|z|_{\text{max}}$ occurs when $z = -6\mathrm{i}$

Therefore $|z|_{\text{max}} = |-6\mathrm{i}| = 6.$

9 c i $T_1 : w = 2z$

METHOD **(1)** z lies on circle with equation $|z+4\text{i}| = 2$

$$\Rightarrow w = 2z$$

$$\Rightarrow \frac{w}{2} = z$$

$$\Rightarrow \frac{w}{2} + 4\text{i} = z + 4\text{i}$$

$$\Rightarrow \frac{w+8\text{i}}{2} = z + 4\text{i}$$

$$\Rightarrow \left|\frac{w+8\text{i}}{2}\right| = |z+4\text{i}|$$

$$\Rightarrow \frac{|w+8\text{i}|}{|2|} = |z+4\text{i}|$$

$$\Rightarrow \frac{|w+8\text{i}|}{2} = 2$$

$$\Rightarrow |w+8\text{i}| = 4$$

So the locus of the image of P under T_1 is a circle centre (0, –8), radius 4, with equation $u^2 + (v+8)^2 = 16$.

METHOD **(2)** z lies on circle centre (0, –4), radius 2

↓ enlargement scale factor 2, centre 0.

$w = 2z$ lies on circle centre (0,–8), radius 4.

So the locus of the image of P under T_1 is a circle centre (0, –8), radius 4, with equation $u^2 + (v+8)^2 = 16$.

9 c ii $T_2 : w = \text{i}z$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = \text{i}z$

$\Rightarrow \frac{w}{\text{i}} = z$

$\Rightarrow \frac{w}{\text{i}}\left(\frac{\text{i}}{\text{i}}\right) = z$

$\Rightarrow \frac{w\text{i}}{(-1)} = z$

$\Rightarrow -w\text{i} = z$

$\Rightarrow z = -w\text{i}$

Hence $|z + 4\text{i}| = 2 \Rightarrow |-w\text{i} + 4\text{i}| = 2$

$\Rightarrow |(-\text{i})(w - 4)| = 2$

$\Rightarrow |(-\text{i})||w - 4| = 2$

$\Rightarrow |w - 4| = 2$

So the locus of the image of P under T_2 is a circle centre (4, 0), radius 2, with equation $(u-4)^2 + v^2 = 4$.

iii $T_3 : w = -\text{i}z$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = -\text{i}z$

$\Rightarrow \text{i}w = \text{i}(-\text{i}z)$

$\Rightarrow \text{i}w = z$

$\Rightarrow z = \text{i}w$

Hence $|z + 4\text{i}| = 2 \Rightarrow |\text{i}w + 4\text{i}| = 2$

$\Rightarrow |\text{i}(w + 4)| = 2$

$\Rightarrow |\text{i}||w + 4| = 2$

$\Rightarrow |w + 4| = 2$ ← $|\text{i}| = 1$

So the locus of the image of P under T_3 is a circle centre (–4, 0), radius 2, with equation $(u+4)^2 + v^2 = 4$.

9 c iv $T_4 : w = z^*$

z lies on a circle with equation $|z + 4\text{i}| = 2$

$w = z^* \Rightarrow u + \text{i}v = x - \text{i}y$

$z = x + \text{i}y$

$\Rightarrow z^* = x - \text{i}y$

So $u = x$, $v = -y$ and $x = u$ and $y = -v$

$$|z + 4\text{i}| = 2 \Rightarrow |x + \text{i}y + 4\text{i}| = 2$$
$$\Rightarrow |x + \text{i}(y + 4)| = 2$$
$$\Rightarrow |u + \text{i}(-v + 4)| = 2$$
$$\Rightarrow |u + \text{i}(4 - v)| = 2$$
$$\Rightarrow |u + \text{i}(4 - v)|^2 = 2^2$$
$$\Rightarrow u^2 + (4 - v)^2 = 4$$
$$\Rightarrow u^2 + (v - 4)^2 = 4$$

So the locus of the image of P under T_4 is a circle centre (0, 4), radius 2, with equation $u^2 + (v - 4)^2 = 4$.

10 $T: w = \dfrac{z+2}{z+\text{i}}, z \neq -\text{i}$

a the imaginary axis in z-plane $\Rightarrow x = 0$

$$w = \frac{z+2}{z+\text{i}}$$
$$\Rightarrow w(z+\text{i}) = z+2$$
$$\Rightarrow wz + \text{i}w = z+2$$
$$\Rightarrow wz - z = 2 - \text{i}w$$
$$\Rightarrow z(w-1) = 2 - \text{i}w$$
$$\Rightarrow z = \frac{2-\text{i}w}{w-1}$$
$$\Rightarrow z = \frac{2-\text{i}(u+\text{i}v)}{u+\text{i}v-1}$$
$$\Rightarrow z = \frac{2-\text{i}u+v}{(u-1)+\text{i}v}$$
$$\Rightarrow z = \left[\frac{(2+v)-\text{i}u}{(u-1)+\text{i}v}\right] \times \left[\frac{(u-1)-\text{i}v}{(u-1)-\text{i}v}\right]$$
$$\Rightarrow z = \frac{(2+v)(u-1)-uv-\text{i}v(2+v)-\text{i}u(u-1)}{(u-1)^2+v^2}$$
$$\Rightarrow z = \frac{(2+v)(u-1)-uv}{(u-1)^2+v^2} - \text{i}\left(\frac{v(2+v)+u(u-1)}{(u-1)^2+v^2}\right)$$

So $x + \text{i}y = \dfrac{(2+v)(u-1)-uv}{(u-1)^2+v^2} - \text{i}\left(\dfrac{v(2+v)+u(u-1)}{(u-1)^2+v^2}\right)$

$\Rightarrow x = \dfrac{(2+v)(u-1)-uv}{(u-1)^2+v^2}$ and $y = \dfrac{-v(2+v)-u(u-1)}{(u-1)^2+v^2}$

As $x = 0$, then

$$\frac{(2+v)(u-1)-uv}{(u-1)^2+v^2} = 0$$
$$\Rightarrow (2+v)(u-1)-uv = 0$$
$$\Rightarrow 2u - 2 + vu - v - uv = 0$$
$$\Rightarrow 2u - 2 - v = 0$$
$$\Rightarrow v = 2u - 2$$

The transformation T maps the imaginary axis in the z-plane to the line l with equation $v = 2u - 2$ in the w-plane.

10 b As $y = x$, then

$$\frac{-v(2+v)-u(u-1)}{(u-1)^2+v^2} = \frac{(2+v)(u-1)-uv}{(u-1)^2+v^2}$$

$$\Rightarrow -v(2+v)-u(u-1) = (2+v)(u-1)-uv$$

$$\Rightarrow -2v-v^2-u^2+u = 2u-2+vu-v-uv$$

$$\Rightarrow -2v-v^2-u^2+u = 2u-2-v$$

$$\Rightarrow 0 = u^2+v^2+u+v-2$$

$$\Rightarrow \left(u+\frac{1}{2}\right)^2-\frac{1}{4}+\left(v+\frac{1}{2}\right)^2-\frac{1}{4}-2=0$$

$$\Rightarrow \left(u+\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\frac{5}{2}$$

$$\Rightarrow \left(u+\frac{1}{2}\right)^2+\left(v+\frac{1}{2}\right)^2=\left(\frac{\sqrt{10}}{2}\right)^2$$

$$\sqrt{\frac{5}{2}} = \frac{\sqrt{5}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{10}}{2} = \frac{1}{2}\sqrt{10}$$

The transformation T maps the line $y = x$ in the z-plane to the circle C with centre $\left(-\frac{1}{2},-\frac{1}{2}\right)$, radius $\frac{\sqrt{10}}{2}$ in the w-plane.

11 $T: \ w = \frac{4-z}{z+\mathrm{i}} \quad z \neq -\mathrm{i}$

circle with equation $|z| = 1$ in the z-plane.

$w = \frac{4-z}{z+\mathrm{i}}$

$\Rightarrow w(z+\mathrm{i}) = 4-z$

$\Rightarrow wz + \mathrm{i}w = 4-z$

$\Rightarrow wz + z = 4 - \mathrm{i}w$

$\Rightarrow z(w+1) = 4 - \mathrm{i}w$

$\Rightarrow z = \frac{4-\mathrm{i}w}{w+1}$

$\Rightarrow |z| = \left|\frac{4-\mathrm{i}w}{w+1}\right|$

$\Rightarrow |z| = \frac{|4-\mathrm{i}w|}{|w+1|}$

Applying $|z| = 1$ gives $1 = \frac{|4-\mathrm{i}w|}{|w+1|}$

$\Rightarrow |w+1| = |4-\mathrm{i}w|$

$\Rightarrow |w+1| = |-\mathrm{i}(w+4\mathrm{i})|$

$\Rightarrow |w+1| = |-\mathrm{i}||w+4\mathrm{i}|$

$\Rightarrow |w+1| = |w+4\mathrm{i}|$

$\Rightarrow |u+\mathrm{i}v+1| = |u+\mathrm{i}v+4\mathrm{i}|$

$\Rightarrow |(u+1)+\mathrm{i}v| = |u+\mathrm{i}(v+4)|$

$\Rightarrow |(u+1)+\mathrm{i}v|^2 = |u+\mathrm{i}(v+4)|^2$

$\Rightarrow (u+1)^2 + v^2 = u^2 + (v+4)^2$

$\Rightarrow u^2 + 2u + 1 + v^2 = u^2 + v^2 + 8v + 16$

$\Rightarrow 2u + 1 = 8v + 16$

$\Rightarrow 2u - 8v - 15 = 0$

The circle $|z| = 1$ is mapped by T onto the line l: $2u - 8v - 15 = 0$
(i.e. $a = 2,\ b = -8,\ c = -15$).

12 $T: \ w = \dfrac{3iz+6}{1-z}; \quad z \neq 1$

circle with equation $|z| = 2$

$$w = \frac{3iz+6}{1-z}$$
$$\Rightarrow w(1-z) = 3iz+6$$
$$\Rightarrow w - wz = 3iz + 6$$
$$\Rightarrow w - 6 = 3iz + wz$$
$$\Rightarrow w - 6 = z(3i + w)$$
$$\Rightarrow \frac{w-6}{w+3i} = z$$
$$\Rightarrow \left|\frac{w-6}{w+3i}\right| = |z|$$
$$\Rightarrow \frac{|w-6|}{|w+3i|} = |z|$$

Applying $|z| = 2 \Rightarrow \dfrac{|w-6|}{|w+3i|} = 2$

$$\Rightarrow |w-6| = 2|w+3i|$$
$$\Rightarrow |u + iv - 6| = 2|u + iv + 3i|$$
$$\Rightarrow |(u-6) + iv| = 2|u + i(v+3)|$$
$$\Rightarrow |(u-6)+iv|^2 = 2^2\,|u + i(v+3)|^2$$
$$\Rightarrow (u-6)^2 + v^2 = 4[u^2 + (v+3)^2]$$
$$\Rightarrow u^2 - 12u + 36 + v^2 = 4[u^2 + v^2 + 6v + 9]$$
$$\Rightarrow u^2 - 12u + 36 + v^2 = 4u^2 + 4v^2 + 24v + 36$$
$$\Rightarrow 0 = 3u^2 + 12u + 3v^2 + 24v$$
$$\Rightarrow 0 = u^2 + 4u + v^2 + 8v$$
$$\Rightarrow 0 = (u+2)^2 - 4 + (v+4)^2 - 16$$
$$\Rightarrow 20 = (u+2)^2 + (v+4)^2$$
$$\Rightarrow (u+2)^2 + (v+4)^2 = (2\sqrt{5})^2$$

$\sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

Therefore the circle with equation $|z| = 2$ is mapped onto a circle C, centre (–2, –4), radius $2\sqrt{5}$. So $k = 2$.

13 a We know that the transformation T given by $w=\frac{az+b}{z+c}$ maps the origin onto itself, so, substituting $w=z=0$ into T we get $0=\frac{b}{c}$.

We have to assume $c\neq 0$ or else it is not possible to map the origin onto itself.
Therefore $b=0$.
We also know that this mapping reflects $z_1=1+2\text{i}$ in the real axis, i.e. $w_1=1-2\text{i}$. Substituting these values into T we obtain:

$$1-2\text{i}=\frac{a+2a\text{i}}{1+2\text{i}+c}$$

$$(1-2\text{i})(1+2\text{i}+c)=a+2a\text{i}$$

$$1+2\text{i}+c-2\text{i}+4-2c\text{i}=a+2a\text{i}$$

$$1+4+c-2ci=a+2ai$$

$$5+c-2ci=a+2ai$$

We now equate the real and complex parts:

$$5+c=a$$

$$-c=a$$

Solving simultaneously gives:

$$2c=-5$$

$$c=-\frac{5}{2}$$

$$a=\frac{5}{2}$$

So $a=\frac{5}{2}$, $b=0$ and $c=-\frac{5}{2}$.

b We know that another complex number, ω, is mapped onto itself. i.e. we have:

$$\omega=\frac{\frac{5}{2}\omega}{\omega-\frac{5}{2}}$$

$$\omega^2-\frac{5}{2}\omega=\frac{5}{2}\omega$$

$$\omega^2=5\omega$$

$$\omega^2-5\omega=0$$

$$\omega(\omega-5)=0$$

$$\omega=0 \text{ or } \omega=5$$

$\omega=0$ is the origin, so the other number mapped onto itself is $\omega=5$.

14 a $w = \dfrac{az+b}{z+c} \quad a,\ b,\ c \in \mathbb{R}.$

$w = 1$ when $z = 0$ **(1)**

$w = 3 - 2\text{i}$ when $z = 2 + 3\text{i}$ **(2)**

(1) $\Rightarrow 1 = \dfrac{a(0)+b}{0+c} \Rightarrow 1 = \dfrac{b}{c} \Rightarrow c = b$ **(3)**

(3) $\Rightarrow w = \dfrac{az+b}{z+b}$

(2) $\Rightarrow 3 - 2\text{i} = \dfrac{a(2+3\text{i})+b}{2+3\text{i}+b}$

$3 - 2\text{i} = \dfrac{(2a+b)+3a\text{i}}{(2+b)+3\text{i}}$

$(3-2\text{i})[(2+b)+3\text{i}] = 2a + b + 3a\text{i}$

$6 + 3b + 9\text{i} - 4\text{i} - 2b\text{i} + 6 = 2a + b + 3a\text{i}$

$(12+3b) + (5-2b)\text{i} = (2a+b) + 3a\text{i}$

Equate real parts: $12 + 3b = 2a + b$

$\Rightarrow \quad 12 = 2a - 2b$ **(4)**

Equate imaginary parts: $5 - 2b = 3a$

$\Rightarrow \quad 5 = 3a + 2b$ **(5)**

(4) + **(5)**: $17 = 5a$

$\Rightarrow \dfrac{17}{5} = a$

(5) $\Rightarrow 5 = \dfrac{51}{5} + 2b$

$\Rightarrow -\dfrac{26}{5} = 2b$

$\Rightarrow -\dfrac{13}{5} = b$

As $b = c$ then $c = -\dfrac{13}{5}$

The values are $a = \dfrac{17}{5},\ b = -\dfrac{13}{5},\ c = -\dfrac{13}{5}$

14 b $w = \dfrac{\frac{17}{5}z - \frac{13}{5}}{z - \frac{13}{5}}$

$w = \dfrac{17z - 13}{5z - 13}$

invariant points $\Rightarrow z = \dfrac{17z - 13}{5z - 13}$

$z(5z - 13) = 17z - 13$

$5z^2 - 13z = 17z - 13$

$5z^2 - 30z + 13 = 0$

$z = \dfrac{30 \pm \sqrt{900 - 4(5)(13)}}{10}$

$z = \dfrac{30 \pm \sqrt{900 - 260}}{10}$

$z = \dfrac{30 \pm \sqrt{640}}{10}$

$z = \dfrac{30 \pm \sqrt{64}\sqrt{10}}{10}$

$z = \dfrac{30 \pm 8\sqrt{10}}{10} = 3 \pm \dfrac{4\sqrt{10}}{5}$

The exact values of the two points which remain invariant are

$z = 3 + \dfrac{4\sqrt{10}}{5}$ and $z = 3 - \dfrac{4\sqrt{10}}{5}$

15 $T: \quad w = \dfrac{z+\mathrm{i}}{z}, \quad z \neq 0.$

a the line $y = x$ in the z-plane other than (0, 0)

$$w = \frac{z+\mathrm{i}}{z}$$
$$\Rightarrow wz = z + \mathrm{i}$$
$$\Rightarrow wz - z = \mathrm{i}$$
$$\Rightarrow z(w-1) = \mathrm{i}$$
$$\Rightarrow z = \frac{i}{w-1}$$
$$\Rightarrow z = \frac{\mathrm{i}}{(u+\mathrm{i}v)-1} = \frac{\mathrm{i}}{(u-1)+\mathrm{i}v}$$
$$\Rightarrow z = \left[\frac{\mathrm{i}}{(u-1)+\mathrm{i}v}\right]\left[\frac{(u-1)-\mathrm{i}v}{(u-1)-\mathrm{i}v}\right]$$
$$\Rightarrow z = \frac{\mathrm{i}(u-1)+v}{(u-1)^2+v^2}$$
$$\Rightarrow z = \frac{v}{(u-1)^2+v^2} + \mathrm{i}\frac{(u-1)}{(u-1)^2+v^2}$$

So $x + \mathrm{i}y = \dfrac{v}{(u-1)^2+v^2} + \mathrm{i}\dfrac{(u-1)}{(u-1)^2+v^2}$

$\Rightarrow x = \dfrac{v}{(u-1)^2+v^2}$ and $y = \dfrac{u-1}{(u-1)^2+v^2}$

Applying $y = x$, gives $\dfrac{u-1}{(u-1)^2+v^2} = \dfrac{v}{(u-1)^2+v^2}$

$$\Rightarrow u - 1 = v$$
$$\Rightarrow v = u - 1$$

Therefore the line l has equation $v = u - 1$.

15 b the line with equation $x + y + 1 = 0$ in the z-plane

$$x + y + 1 = 0 \Rightarrow \frac{v}{(u-1)^2 + v^2} + \frac{u-1}{(u-1)^2 + v^2} + 1 = 0 \; [\times (u-1)^2 + v^2]$$

$$\Rightarrow v + (u-1) + (u-1)^2 + v^2 = 0$$

$$\Rightarrow v + u - 1 + u^2 - 2u + 1 + v^2 = 0$$

$$\Rightarrow u^2 + v^2 - u + v = 0$$

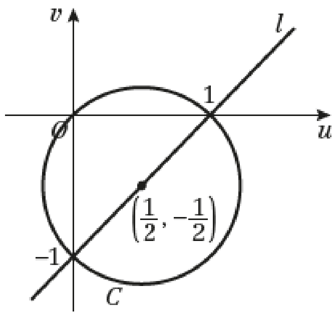
$$\Rightarrow \left(u - \frac{1}{2}\right)^2 - \frac{1}{4} + \left(v + \frac{1}{2}\right)^2 - \frac{1}{4} = 0$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \frac{1}{2}$$

$$\Rightarrow \left(u - \frac{1}{2}\right)^2 + \left(v + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{2}}{2}\right)^2$$

The image of $x + y + 1 = 0$ under T is a circle C, centre $\left(\frac{1}{2}, -\frac{1}{2}\right)$, radius $\frac{\sqrt{2}}{2}$ with equation $u^2 + v^2 - u + v = 0$, as required.

c



Challenge

We want to find a transformation of the form $\mathrm{f}(z) = az^* + b$, which reflects the z-plane in the line $x + y = 1$.

First note that points lying on the line will be mapped onto themselves.

This means that for any $z = x + \mathrm{i}y$ such that $x + y = 1$ we have $\mathrm{f}(z) = z$.

Choose $z_1 = 1$, $z_2 = \mathrm{i}$.

Both these points satisfy $x + y = 1$, so $\mathrm{f}(z_1) = z_1$ and $\mathrm{f}(z_2) = z_2$.

Therefore we have:

$\mathrm{f}(1) = a + b = 1$

and $\mathrm{f}(\mathrm{i}) = -a\mathrm{i} + b = \mathrm{i}$.

Solving simultaneously:

$-(1-b)\mathrm{i} + b = \mathrm{i}$

$-\mathrm{i} + b\mathrm{i} + b = \mathrm{i}$

$b(1+\mathrm{i}) = 2\mathrm{i}$

$$b = \frac{2\mathrm{i}}{1+\mathrm{i}} = \frac{2\mathrm{i}}{1+\mathrm{i}}\left(\frac{1-\mathrm{i}}{1-\mathrm{i}}\right) = \frac{2\mathrm{i}(1-\mathrm{i})}{2} = \mathrm{i}(1-\mathrm{i})$$

$b = 1 + \mathrm{i}$

So $a = 1 - b = -\mathrm{i}$ and $\mathrm{f}(z) = -\mathrm{i}z^* + 1 + \mathrm{i}$