Complex Numbers 3A

1 a $|z+3|=3|z-5|$

$\Rightarrow |x+\text{i}y+3|=3|x+\text{i}y-5|$

$\Rightarrow |(x+3)+\text{i}y|=3|(x-5)+\text{i}y|$

$\Rightarrow |(x+3)+\text{i}y|^2=3^2|(x-5)+\text{i}y|^2$

$\Rightarrow (x+3)^2+y^2=9[(x-5)^2+y^2]$

$\Rightarrow x^2+6x+9+y^2=9[(x^2-10x+25+y^2)]$

$\Rightarrow x^2+6x+9+y^2=9x^2-90x+225+9y^2$

$\Rightarrow 0=8x^2-96x+8y^2+216 \quad (\div 8)$

$\Rightarrow x^2-12x+y^2+27=0$

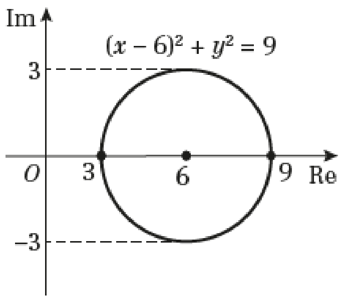
$\Rightarrow (x-6)^2-36+y^2+27=0$

$\Rightarrow (x-6)^2+y^2-9=0$

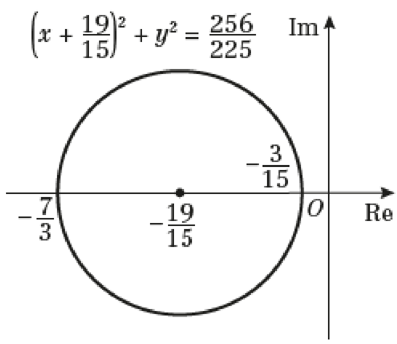
$\Rightarrow (x-6)^2+y^2=9$

The Cartesian equation of the locus of z is $(x-6)^2+y^2=9$.

This is a circle centre (6, 0), radius = 3



b



$|z-3|=4|z+1|$

$|x+\text{i}y-3|=4|x+\text{i}y+1|$

$|x-3+\text{i}y|^2=16|x+1+\text{i}y|^2$

$(x-3)^2+y^2=16((x+1)^2+y^2)$

$x^2-6x+9+y^2=16(x^2+2x+1+y^2)$

$=16x^2+32x+16+16y^2$

$15x^2+38x+15y^2+7=0$

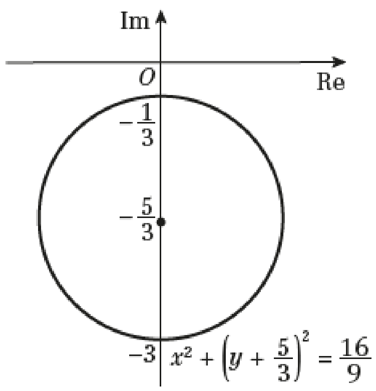
$x^2+\frac{38}{15}x+y^2+\frac{7}{15}=0$

$\left(x+\frac{19}{15}\right)^2-\frac{19^2}{15^2}+y^2+\frac{7}{15}=0$

$\left(x+\frac{19}{15}\right)^2+y^2=\frac{256}{225}$

Circle centre $\left(-\frac{19}{15},0\right)$ radius $\frac{16}{15}$

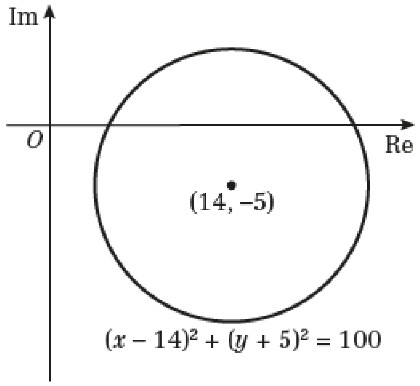
1 c



$$|z - \mathrm{i}| = 2|z + \mathrm{i}|$$
$$|x + \mathrm{i}y - \mathrm{i}| = 2|x + \mathrm{i}y + \mathrm{i}|$$
$$|x + \mathrm{i}(y-1)|^2 = 4|x + \mathrm{i}(y+1)|^2$$
$$x^2 + (y-1)^2 = 4[x^2 + (y+1)^2]$$
$$x^2 + y^2 - 2y + 1 = 4(x^2 + y^2 + 2y + 1)$$
$$= 4x^2 + 4y^2 + 8y + 4$$
$$3x^2 + 3y^2 + 10y + 3 = 0$$
$$x^2 + y^2 + \frac{10}{3}y + 1 = 0$$
$$x^2 + \left(y + \frac{5}{3}\right)^2 - \frac{25}{9} + 1 = 0$$
$$x^2 + \left(y + \frac{5}{3}\right)^2 = \frac{16}{9}$$

Circle centre $\left(0, -\frac{5}{3}\right)$ radius $\frac{4}{3}$

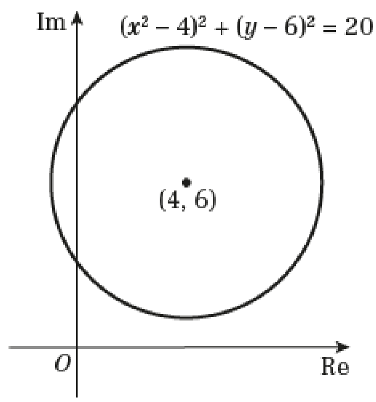
d



$$|z + 2 - 7\mathrm{i}| = 2|z - 10 + 2\mathrm{i}|$$
$$|x + \mathrm{i}y + 2 - 7\mathrm{i}| = 2|x + \mathrm{i}y - 10 + 2\mathrm{i}|$$
$$|(x+2) + \mathrm{i}(y-7)|^2 = 4|(x-10) + \mathrm{i}(y+2)|^2$$
$$(x+2)^2 + (y-7)^2 = 4[(x-10)^2 + (y+2)^2]$$
$$x^2 + 4x + 4 + y^2 - 14y + 49 = 4[x^2 - 20x + 100 + y^2 + 4y + 4$$
$$3x^2 - 84x + 3y^2 + 30y + 363 = 0$$
$$x^2 - 28x + y^2 + 10y + 121 = 0$$
$$(x-14)^2 - 14^2 + (y+5)^2 - 5^2 + 121 = 0$$
$$(x-14)^2 + (y+5)^2 = 100$$

Circle centre $(14, -5)$ radius 10

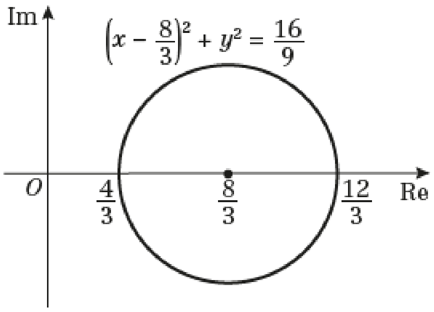
e



$$|z + 4 - 2\mathrm{i}| = 2|z - 2 - 5\mathrm{i}|$$
$$|x + \mathrm{i}y + 4 - 2\mathrm{i}| = 2|x + \mathrm{i}y - 2 - 5\mathrm{i}|$$
$$|(x+4) + \mathrm{i}(y-2)|^2 = 4|(x-2) + \mathrm{i}(y-5)|^2$$
$$(x+4)^2 + (y-2)^2 = 4[(x-2)^2 + (y-5)^2]$$
$$x^2 + 8x + 16 + y^2 - 4y + 4 = 4[x^2 - 4x + 4$$
$$+ y^2 - 10y + 25]$$
$$3x^2 - 24x + 3y^2 + 36y + 96 = 0$$
$$x^2 - 8x + y^2 - 12y + 32 = 0$$
$$(x-4)^2 - 16 + (y-6)^2 - 36 + 32 = 0$$
$$(x-4)^2 + (y-6)^2 = 20$$

Circle centre $(4, 6)$ radius $\sqrt{20} = 2\sqrt{5}$

1 f



$|z| = 2|2-z|$

$= 2|-1||z-2|$

$|x+\mathrm{i}y| = 2\times1\times|x+\mathrm{i}y-2|$

$x^2+y^2 = 4((x-2)^2+y^2)$

$x^2+y^2 = 4(x^2-4x+4+y^2)$

$3x^2-16x+3y^2+16=0$

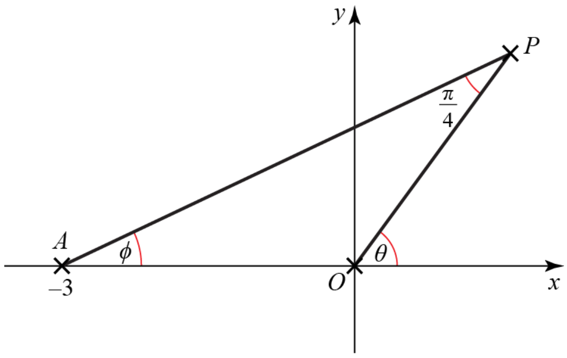
$x^2-\frac{16}{3}x+y^2+\frac{16}{3}=0$

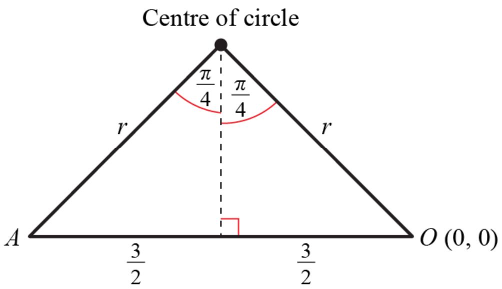
$\left(x-\frac{8}{3}\right)^2-\frac{64}{9}+y^2+\frac{16}{3}=0$

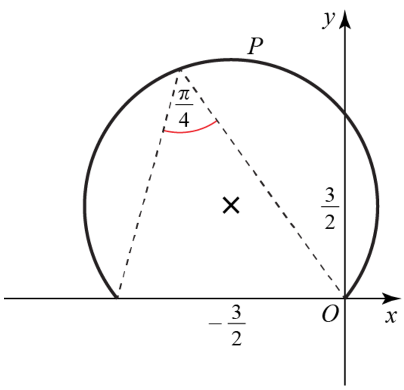
$\left(x-\frac{8}{3}\right)^2+y^2=\frac{16}{9}$

Circle centre $\left(\frac{8}{3},0\right)$ radius $\frac{4}{3}$

2 a







$\arg\left(\frac{z}{z+3}\right)=\frac{\pi}{4}$

$\arg z-\arg(z+3)=\frac{\pi}{4}$

$\arg z-\arg(z-(-3))=\frac{\pi}{4}$

$\arg z=\theta$

$\arg(z-(-3))=\phi$

$\theta-\phi=\frac{\pi}{4}$

$\theta=\phi+\frac{\pi}{4}$

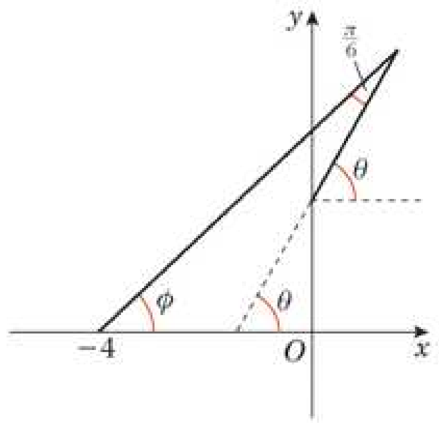
P lies on an arc of a circle cut off at $A(-3,0)$ and $O(0,0)$

Angle at the centre is twice the angle at the circumference $\therefore \frac{\pi}{2}$

It follows that the centre is at $\left(-\frac{3}{2},\frac{3}{2}\right)$

and the radius is $\frac{3}{2}\sqrt{2}$

2 b



$$\arg\left(\frac{z-3\text{i}}{z+4}\right)=\frac{\pi}{6}$$

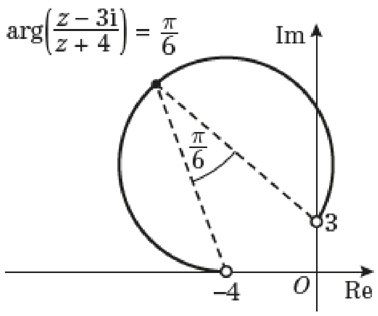
$$\arg(z-3\text{i})-\arg(z-(-4))=\frac{\pi}{6}$$

$$\arg(z-3\text{i})=\theta.$$

$$\arg(z-(-4))=\phi$$

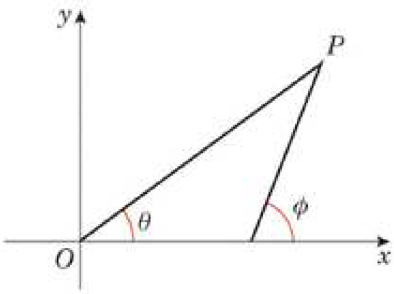
$$\theta-\phi=\frac{\pi}{6}$$

Arc of a circle from $(-4,0)$ to $(0,3)$



$\left(\text{The centre is at } \left(-\frac{4+3\sqrt{3}}{2}, \frac{3+4\sqrt{3}}{2}\right)\text{, though you do not need to calculate this for a sketch.}\right)$

2 c



$\arg\left(\frac{z}{z-2}\right)=\frac{\pi}{3}$

$\arg z=\theta$

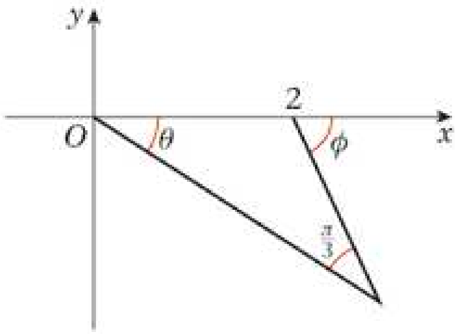
$\arg(z-2)=\phi$

$\theta-\phi=\frac{\pi}{3}$

As our diagram has $\phi>\theta$, we have P on the wrong side of the line joining O or ϕ.

We want the arc below the x-axis.

Redrawing:



$\arg z=-\theta$

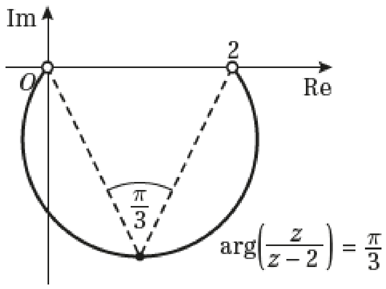
$\arg(z-2)=-\phi$

Hence $\arg z-\arg(z-2)=\frac{\pi}{3}$

becomes $-\theta-(-\phi)=\frac{\pi}{3}$

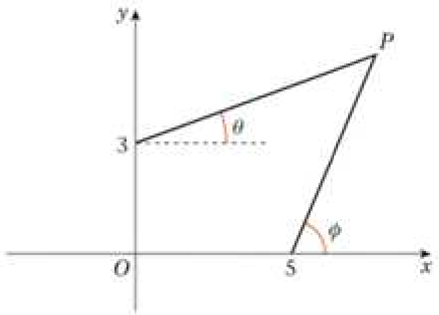
$\phi=\theta+\frac{\pi}{3}$

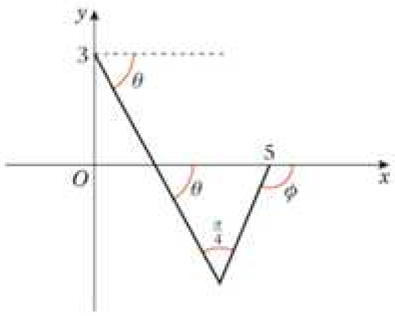
Arc of a circle, ends 0 and 2, subtending angle $\frac{\pi}{3}$



(The centre is at $\left(1,-\frac{1}{\sqrt{3}}\right)$ radius $\frac{2\sqrt{3}}{3}$ not needed to be calculated for a sketch)

2 d





$$\arg\left(\frac{z-3\mathrm{i}}{z-5}\right)=\frac{\pi}{4}$$

$$\arg(z-3\mathrm{i})-\arg(z-5)=\frac{\pi}{4}$$

$$\arg(z-3\mathrm{i})=\theta$$

$$\arg(z-5)=\phi$$

$$\theta-\phi=\frac{\pi}{4}$$

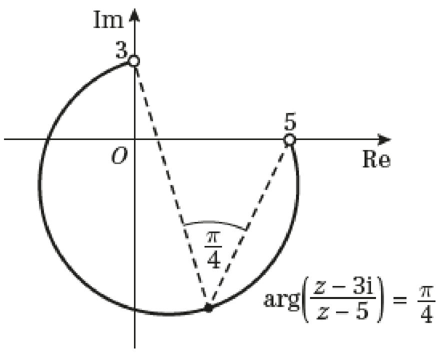
But $\phi>\theta$, we have P on the wrong side of the line joining 3i and 5.

$$\arg(z-3\mathrm{i})=-\theta$$

$$\arg(z-5)=-\phi$$

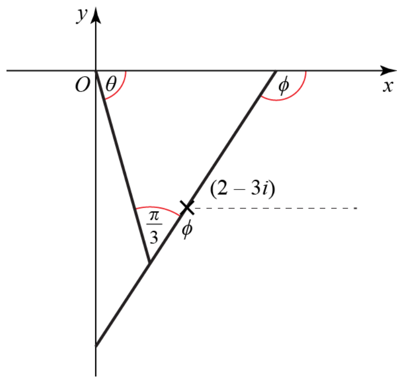
$$-\theta-(-\phi)=\frac{\pi}{4}$$

$$\phi=\theta+\frac{\pi}{4}$$



(Arc of circle centre (1, –1) radius $\sqrt{17}$ not needed for sketch)

2 e



$$\arg z - \arg(z - 2 + 3\text{i}) = \frac{\pi}{3}$$

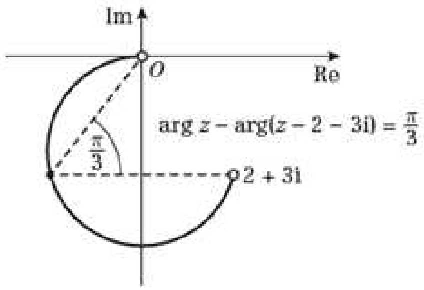
$$\arg z - \arg(z - (2 - 3\text{i})) = \frac{\pi}{3}$$

$$\arg z = -\theta$$

$$\arg(z - (2 - 3\text{i})) = -\phi$$

$$-\theta - (-\phi) = \frac{\pi}{3}$$

$$\phi = \theta + \frac{\pi}{3}$$



Arc of circle, centre at $\left(\frac{2-\sqrt{3}}{2}, -\frac{9+2\sqrt{3}}{6}\right)$, this need not be calculated for your sketch.

f

$$\arg\left(\frac{z - 4\text{i}}{z + 4}\right) = \frac{\pi}{2}$$

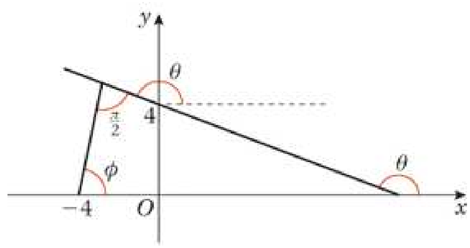
$$\arg(z - 4\text{i}) - \arg(z + 4) = \frac{\pi}{2}$$

$$\arg(z - 4\text{i}) = \theta$$

$$\arg(z + 4) = \phi = \arg(z - (-4\text{i}))$$

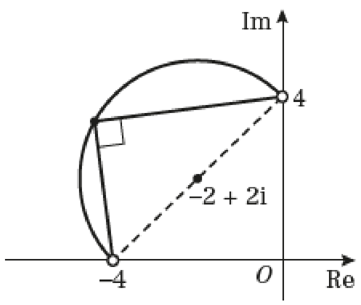
$$\theta - \phi = \frac{\pi}{2}$$

$$\theta = \phi + \frac{\pi}{2}$$



The locus is an arc of a circle, ends at –4 and 4i, angle subtended being $\frac{\pi}{2}$

$\therefore$ It is a semi-circle.



(Circle arc has centre (–2, 2), radius $2\sqrt{2}$)

3 a $|z+1+\mathrm{i}|=2|z+4-2\mathrm{i}|$

$\Rightarrow |x+\mathrm{i}y+1+\mathrm{i}|=2|x+\mathrm{i}y+4-2\mathrm{i}|$

$\Rightarrow |(x+1)+\mathrm{i}(y+1)|=2|(x+4)+\mathrm{i}(y-2)|$

$\Rightarrow |(x+1)+\mathrm{i}(y+1)|^2=2^2|(x+4)+\mathrm{i}(y-2)|^2$

$\Rightarrow (x+1)^2+(y+1)^2=4[(x+4)^2+(y-2)^2]$

$\Rightarrow x^2+2x+1+y^2+2y+1=4[(x^2+8x+16+y^2-4y+4]$

$\Rightarrow x^2+2x+1+y^2+2y+1=4x^2+32x+64+4y^2-16y+16$

$\Rightarrow 0=3x^2+30x+3y^2-18y+64+16-1-1$

$\Rightarrow 3x^2+30x+3y^2-18y+78=0$

$\Rightarrow x^2+10x+y^2-6y+26=0$

$\Rightarrow (x+5)^2-25+(y-3)^2-9+26=0$

$\Rightarrow (x+5)^2+(y-3)^2=25+9-26$

$\Rightarrow (x+5)^2+(y-3)^2=8$

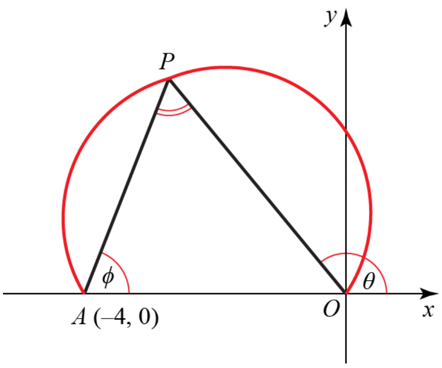
Therefore the locus of P is a circle centre (–5, 3). (as required)

b radius $=\sqrt{8}=\sqrt{4}\sqrt{2}=2\sqrt{2}$

The exact radius is $2\sqrt{2}$.

4 a $\arg(z)-\arg(z+4)=\frac{\pi}{4}$

$\Rightarrow \theta-\phi=\frac{\pi}{4}$, where $\arg(z)=\theta$ and $\arg(z+4)=\phi$



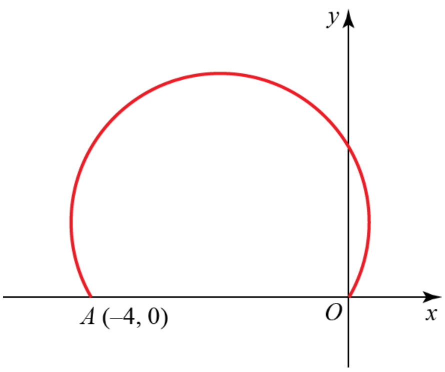
from ΔAOP,

$A\hat{P}O+\phi=\theta$

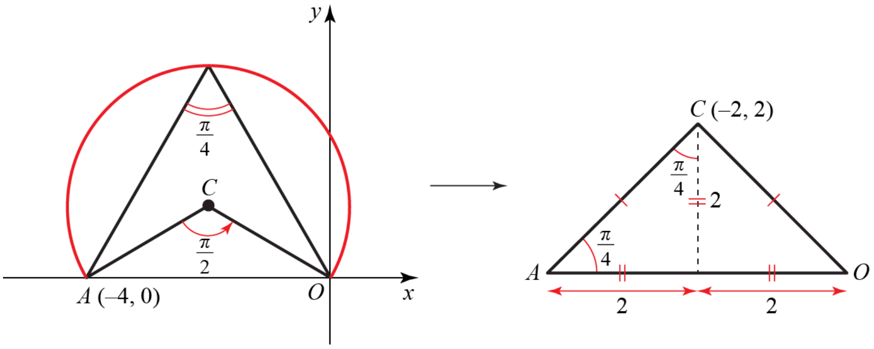
$\Rightarrow A\hat{P}O=\theta-\phi$

$\Rightarrow A\hat{P}O=\frac{\pi}{4}$

The locus of points P is an arc of a circle cut off at (–4, 0) and (0, 0), as shown below.



4 b



Therefore the centre of the circle has coordinates (–2, 2).

c $r = \sqrt{2^2 + 2^2} = \sqrt{8} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$

Therefore, the radius of C is $2\sqrt{2}$.

d The Cartesian equation of C is $(x+2)^2 + (y-2)^2 = 8$.

4 e Finite area = Area of major sector ACO + Area ΔACO

$$= \frac{1}{2}(\sqrt{8})^2\left(2\pi - \frac{\pi}{2}\right) + \frac{1}{2}(4)(2)$$
$$= \frac{1}{2}(8)\left(2\pi - \frac{\pi}{2}\right) + 4$$
$$= 4\left(\frac{3\pi}{2}\right) + 4$$
$$= 6\pi + 4$$

Finite area bounded by the locus of P and the x-axis is $6\pi + 4$.

b, c, d Method **(2)**:

$$\arg z - \arg(z+4) = \arg\left(\frac{z}{z+4}\right)$$
$$= \arg\left(\frac{x+\mathrm{i}y}{x+\mathrm{i}y+4}\right)$$
$$= \arg\left[\frac{x+\mathrm{i}y}{(x+4)+\mathrm{i}y}\right]$$
$$= \arg\left[\frac{x+\mathrm{i}y}{(x+4)+\mathrm{i}y} \times \frac{(x+4)-\mathrm{i}y}{(x+4)-\mathrm{i}y}\right]$$
$$= \arg\left[\frac{x(x+4) - \mathrm{i}yx + \mathrm{i}y(x+4) + y^2}{(x+4)^2 + y^2}\right]$$
$$= \arg\left[\left(\frac{x(x+4)+y^2}{(x+4)^2+y^2}\right) + \mathrm{i}\left(\frac{y(x+4)-yx}{(x+4)^2+y^2}\right)\right]$$
$$= \arg\left[\left(\frac{x^2+4x+y^2}{(x+4)^2+y^2}\right) + \mathrm{i}\left(\frac{xy+4y-xy}{(x+4)^2+y^2}\right)\right]$$
$$= \arg\left[\left(\frac{x^2+4x+y^2}{(x+4)^2+y^2}\right) + \mathrm{i}\left(\frac{4y}{(x+4)^2+y^2}\right)\right]$$

Applying $\arg\left(\frac{z}{z+4}\right) = \frac{\pi}{4} \Rightarrow \frac{\left(\frac{4y}{(x+4)^2+y^2}\right)}{\left(\frac{x^2+4x+y^2}{(x+4)^2+y^2}\right)} = \tan\left(\frac{\pi}{4}\right) = 1$

$$\Rightarrow \frac{4y}{x^2+4x+y^2} = 1$$
$$\Rightarrow 4y = x^2 + 4x + y^2$$
$$\Rightarrow 0 = x^2 + 4x + y^2 - 4y$$
$$\Rightarrow (x+2)^2 - 4 + (y-2)^2 - 4 = 0$$
$$\Rightarrow (x+2)^2 + (y-2)^2 = 8$$
$$\Rightarrow (x+2)^2 + (y-2)^2 = (2\sqrt{2})^2$$

C is a circle with centre (–2, 2), radius $2\sqrt{2}$ and has Cartesian equation $(x+2)^2 + (y-2)^2 = 8$.

5 a Curve F is described by $|z| = 2|z+4|$. First, note that z can be written as $z = x + iy$:

$|x+yi| = |2x+2yi+8|$. Next, group the real and imaginary parts

$|x+yi| = 2|(x+4)+yi|$. Square both sides

$|x+yi|^2 = 2^2|(x+4)+yi|^2$

$x^2+y^2 = 4(x+4)^2+4y^2$

$x^2+y^2 = 4(x^2+8x+16)+4y^2$

$x^2+y^2 = 4x^2+32x+64+4y^2$

$4x^2+32x+64-x^2+4y^2-y^2 = 0$

$3x^2+32x+3y^2+64 = 0$

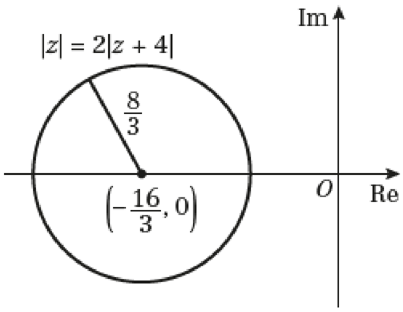
$x^2+\frac{32}{3}x+y^2+\frac{64}{3} = 0$

Completing the square for x

$\left(x+\frac{16}{3}\right)^2+y^2 = \frac{64}{9} = \left(\frac{8}{3}\right)^2$

Thus we see that F is a circle centred at $\left(-\frac{16}{3},0\right)$ with radius $r = \frac{8}{3}$

b



c The circle is centred at $\left(-\frac{16}{3},0\right)$ and its radius is $r = \frac{8}{3}$. This means that it stretches out from $-\frac{8}{3}$ to $\frac{8}{3}$ along the imaginary axis. Thus $-\frac{8}{3} \leqslant \text{Im}(z) \leqslant \frac{8}{3}$

6 We are given curve defined by $|z-8|=2|z-2-6\text{i}|$. To visualise this, express z as real and imaginary parts and square both sides

$$|x-8+y\text{i}|=2|x-2+y\text{i}-6\text{i}|$$

$$|x-8+y\text{i}|^2=2^2|x-2+y\text{i}-6\text{i}|^2$$

$$(x-8)^2+y^2=4(x-2)^2+4(y-6)^2$$

$$x^2-16x+64+y^2=4x^2-16x+16+4y^2-48y+144$$

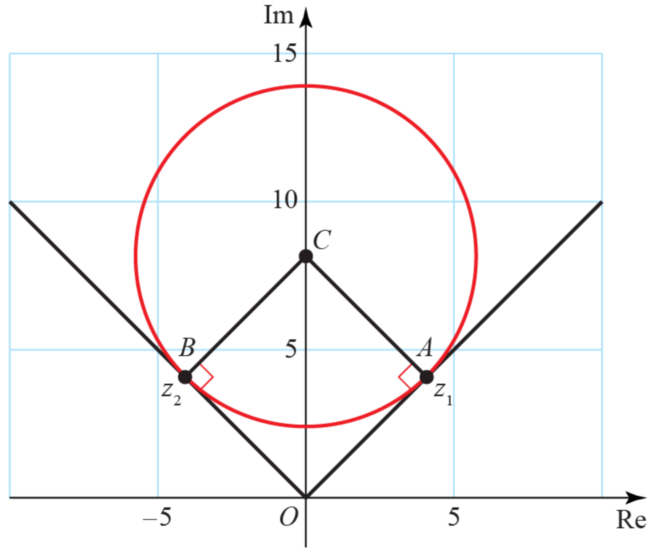
$$3x^2+3y^2-48y+96=0$$

$$x^2+y^2-16y+32=0$$

$$x^2+(y-8)^2-64+32=0$$

$$x^2+(y-8)^2=32=\left(4\sqrt{2}\right)^2$$

So this curve is a circle centred at $(0,8)$ with radius $r=4\sqrt{2}$. Now the largest and smallest values of $\arg(z)$ will be found at the points of tangency of the circle to the lines going through the origin. These are shown below as z_1 and z_2.



We can calculate the distance x from the origin to A using Pythagoras Theorem:

$$x^2+r^2=8^2$$

$$x^2=64-32$$

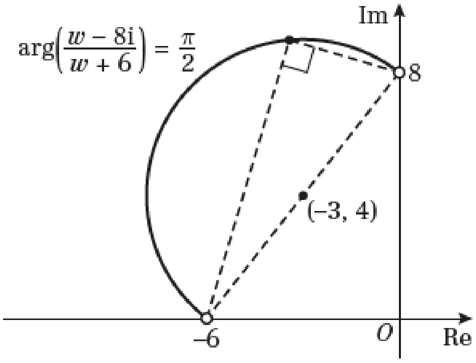
$$x=4\sqrt{2}=r$$

So the triangle created by the origin, z_1 and the centre of the circle is a right-angled isosceles triangle, so the angle $\angle COA=\frac{\pi}{4}$. Similarly, $\angle COB=\frac{\pi}{4}$. Thus we conclude that $\arg(z_1)=\frac{\pi}{4}$ and $\arg(z_2)=\frac{3\pi}{4}$. So for any z lying on this circle we have $\frac{\pi}{4}\leqslant\arg(z)\leqslant\frac{3\pi}{4}$

7 a We want to sketch the curve S satisfying $\arg\left(\frac{w-8\text{i}}{w+6}\right)=\frac{\pi}{2}$. We have

$\arg\left(\frac{w-8\text{i}}{w+6}\right)=\arg(w-8\text{i})-\arg(w+6)=\alpha-\beta=\frac{\pi}{2}$, where $\arg(w-8\text{i})=\alpha$ and $\arg(w+6)=\beta$.

Since the constant angle is $\frac{\pi}{2}$, S is a semicircle from $(0,8)$ anticlockwise to $(-6,0)$ but not including these two points.



b The centre of this semicircle lies in the middle of the line connecting $(-6,0)$ and $(0,8)$, i.e. at $(-3,4)$. The radius can be found be using Pythagoras Theorem:

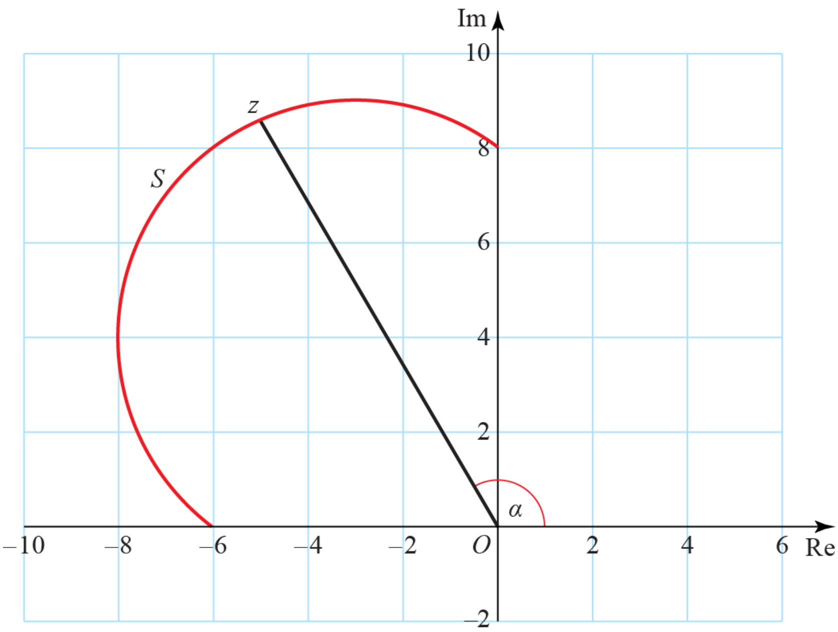
$r^2=3^2+4^2=25$

$r=5$

Thus the Cartesian equation for S can be written as $(x+3)^2+(y-4)^2=25,\ x<0, y>0$.

Remember to specify the range of x and y. Here the inequalities are strict since $(-6,0)$ and $(0,8)$ are not included in the curve.

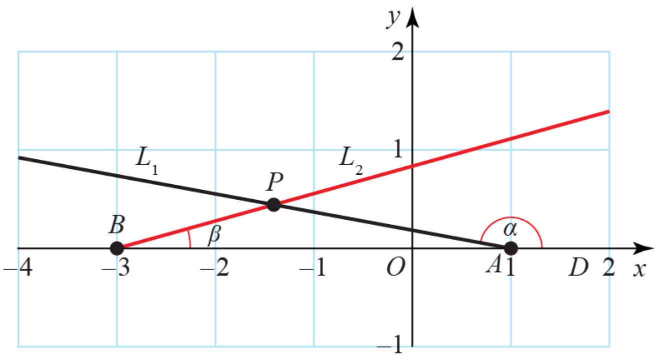
c The argument of an imaginary number z is the angle between the line connecting z to the origin and the real axis.



For curve S the smallest such angle is for $z=8i$ and the largest for $z=-6$. Remember that the endpoints are not included in the curve, so we have $\frac{\pi}{2}<\arg(z)<\pi$

7 d The point furthest to the left is $-8+4i$, so the smallest possible value of $\text{Re}(z)$ is -8. The endpoints of the semicircle are not included in the curve, so we need to use a strict inequality for the largest value of $\text{Re}(z)$. Thus $-8 \leqslant \text{Re}(z) < 0$.

8 We have $\arg(z-1)-\arg(z+3)=\frac{3\pi}{4}$, $z \neq -3$. Let L_1 be the half-line satisfying $\arg(z-1)=\alpha$ and L_2 be the half-line satisfying $\arg(z+3)=\beta$. From the initial equation we have $\alpha-\beta=\frac{3\pi}{4}$



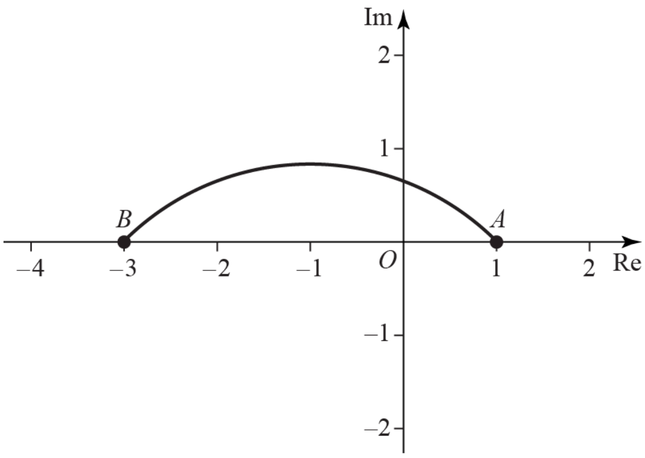
Now considering the triangle APB we see that

$P\hat{B}A + A\hat{P}B = D\hat{A}P$

$A\hat{P}B = D\hat{A}P - P\hat{B}A = \alpha - \beta = \frac{3\pi}{4}$

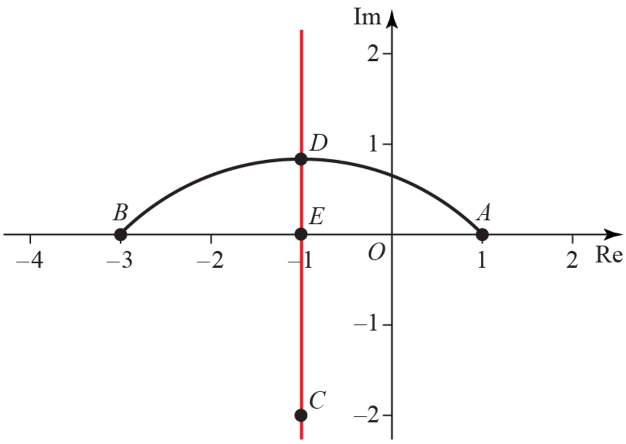
So, as α and β vary, the angle APB remains constant at $\frac{3\pi}{4}$

So the locus will be an arc going anticlockwise from A to B:



8 (continued)

Now we know that the centre of this circle lies on the perpendicular bisector of the line segment connecting A and B, which has equation $x = -1$. Let C be the centre of this circle.



We know that $A\hat{D}B = \frac{3\pi}{4}$, so $B\hat{C}A = 2\pi - 2A\hat{D}B = \frac{\pi}{2}$.
So ACB is an isosceles, right-angled triangle.
So we have:

$r^2 + r^2 = 4^2$

$r^2 = 8$

$r = 2\sqrt{2}$

Now, using Pythagoras Theorem again, on triangle BEC we have that

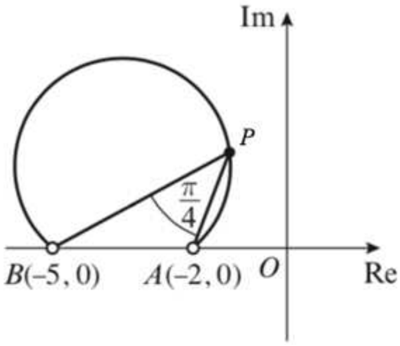
$CE^2 + 2^2 = r^2$

$CE^2 = 4$

$CE = 2$

So the centre has coordinates $C = (-1, -2)$ and the Cartesian equation of this locus can be written as $(x+1)^2 + (y+2)^2 = 8,\ y > 0$.

9 a



By considering the triangle APB, we have that

$P\hat{B}A + A\hat{P}B = O\hat{A}P$

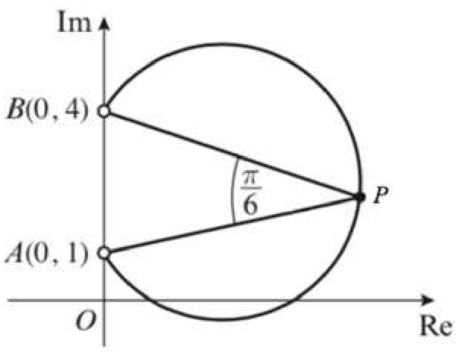
$A\hat{P}B = O\hat{A}P - P\hat{B}A$

$O\hat{A}P - P\hat{B}A = \frac{\pi}{4}$

Moreover, we know that angles in the same segment of a circle are equal, so we're looking for all numbers z for which $\arg(z+2) - \arg(z+5) = \frac{\pi}{4}$

Thus the equation describing this locus is $\arg\left(\frac{z+2}{z+5}\right) = \frac{\pi}{4}$

b

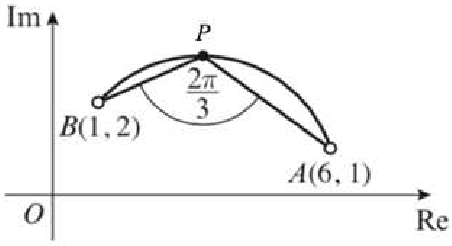


Similar to example **a**, we have

$\arg(z-\mathrm{i}) - \arg(z-4\mathrm{i}) = \frac{\pi}{6}$

So $\arg\left(\frac{z-\mathrm{i}}{z-4\mathrm{i}}\right) = \frac{\pi}{6}$

9 c



Using the same the same techniques as for part **a** and **b** we have that the locus can be described as

$$\arg\left(z-(6+\mathrm{i})\right)-\arg\left(z-(1+2\mathrm{i})\right)=\frac{2\pi}{3}$$

$$\arg\left(z-6-\mathrm{i}\right)-\arg\left(z-1-2\mathrm{i}\right)=\frac{2\pi}{3}$$

$$\arg\left(\frac{z-6-\mathrm{i}}{z-1-2\mathrm{i}}\right)=\frac{2\pi}{3}$$

10 a We have $|z+3|=3|z-5|$. By representing z as real and imaginary parts and squaring both sides of the equation we see that:

$$|x+3+y\mathrm{i}|=3|x-5+y\mathrm{i}|$$

$$|x+3+y\mathrm{i}|^2=9|x-5+y\mathrm{i}|^2$$

$$(x+3)^2+y^2=9(x-5)^2+9y^2$$

$$x^2+6x+9+y^2=9x^2-90x+225+9y^2$$

$$8x^2-96x+216+8y^2=0$$

$$x^2+y^2-12x+27=0$$

as required.

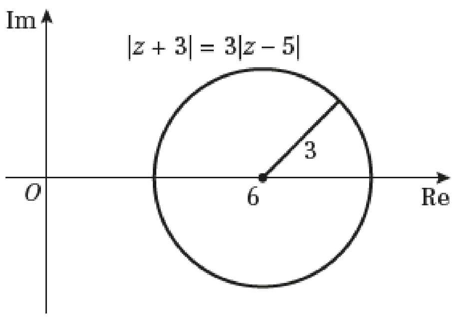
b The above equation can be rewritten as follows:

$$(x-6)^2-36+y^2+27=0$$

$$(x-6)^2+y^2=9$$

$$(x-6)^2+y^2=3^2$$

So the equation describes a circle centred at $(6,0)$ with radius $r=3$



10 c We have that $\arg(z_1) = \frac{\pi}{6}$ and that $z_1 \in C$. If we write $z_1 = r(\cos\theta + i\sin\theta)$ where $\theta = \frac{\pi}{6}$, we see that $z_1 = \frac{\sqrt{3}}{2}r + \frac{1}{2}r\text{i}$. Moreover, we know that z_1 lies on the circle, so if we write $z_1 = x + y\text{i}$, x and y must satisfy $(x-6)^2 + y^2 = 3^2$. Comparing the two expressions for z_1, we obtain $x = \frac{\sqrt{3}}{2}r,\ y = \frac{1}{2}r$. Substituting these values into the circle equation we have:

$$\left(\frac{\sqrt{3}}{2}r - 6\right)^2 + \left(\frac{1}{2}r\right)^2 = 9$$

$$\frac{3}{4}r^2 - 6r\sqrt{3} + 36 + \frac{1}{4}r^2 = 9$$

$$r^2 - 6r\sqrt{3} + 27 = 0$$

$$\left(r - 3\sqrt{3}\right)^2 = 0$$

$$r = 3\sqrt{3}$$

Thus we can write $z_1 = 3\sqrt{3}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right)$

11 a We have the locus of points P satisfying $|z - z_1| = k|z - z_2|$. Moreover, we know that $AP = 2BP$, $A = (0,6)$, $B = (3,0)$. Thus we can write $|z - 6\text{i}| = 2|z - 3|$.

b Write $z = x + y\text{i}$ and square both sides of equation derived in part **a**:

$$|x + y\text{i} - 6\text{i}| = 2|x - 3 + y\text{i}|$$

$$|x + y\text{i} - 6\text{i}|^2 = 4|x - 3 + y\text{i}|^2$$

$$x^2 + (y-6)^2 = 4(x-3)^2 + 4y^2$$

$$x^2 + y^2 - 12y + 36 = 4x^2 - 24x + 36 + 4y^2$$

$$3x^2 - 24x + 3y^2 + 12y = 0$$

$$x^2 + y^2 - 8x + 4y = 0$$

as required.

c The equation for circle C derived in part **b** can be written as $(x-4)^2 + (y+2)^2 = 20 = \left(2\sqrt{5}\right)^2$. This means the circle is centred at $(4,-2)$ and has radius $r = 2\sqrt{5}$. We are given the locus of points w satisfying $\arg(w-6) = \alpha$ and α passes through the centre of the circle. The centre is at point $c = 4 - 2\text{i}$ and we know that, since the centre lies in the 4th quadrant, $\frac{\text{Im}(c)}{\text{Re}(c)} = \tan(2\pi - \alpha)$.

Thus we can write $\tan(2\pi - \alpha) = -\frac{1}{2}$ and so, since $\alpha \in (0, 2\pi)$, we have that

$$2\pi - \alpha = \tan^{-1}\left(-\frac{1}{2}\right) \approx -0.46$$

$$\alpha \approx 5.82$$

11 d We know that Q satisfies both $\arg(w-6)=\alpha$ and $m+n=b$, since it lies on the intersection of the line and the circle. Thus, writing $q=x_1+y_1\mathrm{i}$ we have $\frac{y_1}{x_1}=-\frac{1}{2}\Rightarrow x_1=-2y_1$.

Substituting this into the circle equation, we obtain:

$4y_1^2+y_1^2+16y_1+4y_1=0$

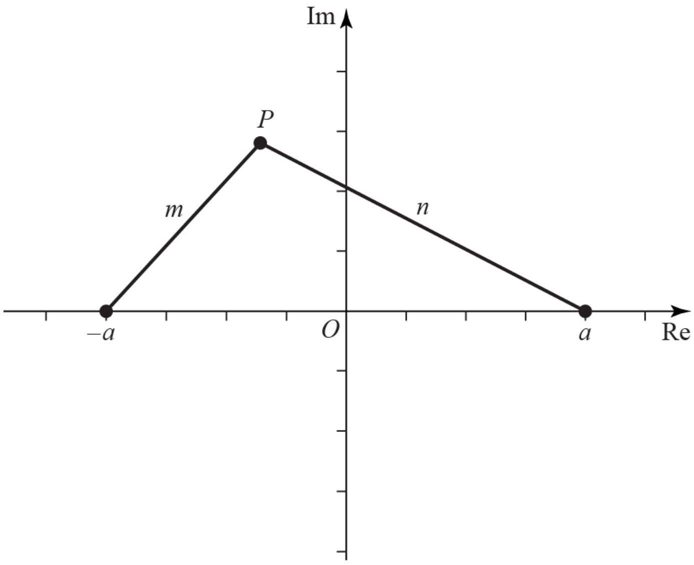
$5y_1^2+20y_1=0$

$y_1(y_1+4)=0$

$y_1=0$ or $y_1=-4$

$y_1=0$ leads to $x_1=0$, so the origin. Thus we take $y_1=-4$ and $x_1=8$. So $Q=(8,-4)$.

Challenge



The equation $|z-a|+|z+a|=b$ describes all points P for which the sum of distances from a and $-a$ is equal to b. According to the graph above, we have $m+n=b$. This is exactly the definition of an ellipse with foci at a and $-a$ and the major axis of length b.