Groups Mixed exercise

1 a Suppose $ab^2 = a^2b$, then $a^{-1}ab^2 = a^{-1}a^2b \Rightarrow b^2 = ab \Rightarrow b^2b^{-1} = abb^{-1} \Rightarrow b = a$
But this contradicts the assumption that a and b are distinct, so $ab^2 \neq a^2b$.

b Suppose $ab = ba$, then $ab^2 = ba \Rightarrow ab^2 = ab \Rightarrow ab^2b^{-1} = abb^{-1} \Rightarrow ab = a \Rightarrow b = e$
But this contradicts the assumption that b and e are distinct, so $ab \neq ba$.

2 The complete Cayley table is:

$\times_{14}$	**1**	**3**	**5**	**9**	**11**	**13**
1	1	3	5	9	11	13
3	3	9	1	13	5	11
5	5	1	11	3	13	9
9	9	13	3	11	1	5
11	11	5	13	1	9	3
13	13	11	9	5	3	1

3 a 1 has order 1.
For 3: $3^2 \equiv_{16} 9$, $3^3 \equiv_{16} 11$, $3^4 \equiv_{16} 1$, so 3 has order 4 and 9 has order 2.
For 5: $5^2 \equiv_{16} 9$, so 5 has order 4.
For 7: $7^2 \equiv_{16} 1$, so 7 has order 2.
For 11: $11^2 \equiv_{16} 9$, so 11 has order 4.
For 13: $13^2 \equiv_{16} 9$, so 13 has order 4.
For 15: $15^2 \equiv_{16} 1$, so 15 has order 2.

b The group has order 8 and none of its elements has order 8, so it is not cyclic.

c The group has order 8, so by Lagrange's theorem the order of any of its subgroups must divide 8.
As 3 does not divide 8, there is no subgroup of order 3.

d For example, {1, 3, 9, 11} is a cyclic subgroup generated by 3.
Other cyclic subgroups of order 4 are generated by 9, 11 and 13.

4 a This matrix represents a rotation of $\left(\frac{\pi}{4}\right)$ anticlockwise.

b G is the group of rotations of $\frac{\pi k}{4}$, for $k = 0, 1, \ldots, 7$.
So its elements can be written as $\{\mathbf{M}, \mathbf{M}^2, \mathbf{M}^3, \ldots, \mathbf{M}^7, \mathbf{M}^8\}$.

c i The inverse is the rotation $\frac{7\pi}{4}$ anticlockwise, which is $\mathbf{M}^7 = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$

4 c ii The group is cyclic, so besides **M** and its inverse, the other generators are $\mathbf{M}^3$ and $\mathbf{M}^7$. (Elements $\mathbf{M}^2$ and $\mathbf{M}^6$ have order 4, and $\mathbf{M}^2$ has order 2.)

$$\mathbf{M}^3 = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} \qquad \mathbf{M}^5 = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

d Clearly a subgroup of order 4 is the group of rotations of $\frac{\pi k}{2}$ for $k = 0, 1, 2, 3$.

In terms of *M*, it is the set $\{\mathbf{M}^2, \mathbf{M}^4, \mathbf{M}^6, \mathbf{M}^8\}$ under matrix multiplication.

5 a

$\circ$	***A***	***B***	***C***	***D***
A	*D*	*C*	*B*	*A*
B	*C*	*D*	*A*	*B*
C	*B*	*A*	*D*	*C*
D	*A*	*B*	*C*	*D*

b The Cayley table shows closure (all entries are in the set of operations), the identity element is *D* and each element is a self-inverse. As associativity is assumed, this is a group.

c The group has order 4 and no element has order 4, so this is not a cyclic group. (In fact, it is isomorphic to the Klein four-group).

6 a i

$\circ$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	1	2	3	4	5	6	7
1	1	0	7	6	5	4	3	2
2	2	7	4	1	6	3	0	5
3	3	6	1	4	7	2	5	0
4	4	5	6	7	0	1	2	3
5	5	4	3	2	1	0	7	6
6	6	3	0	5	2	7	4	1
7	7	2	5	0	3	6	1	4

ii The Cayley table shows closure (all entries are in the set *G*), the identity element is 0, and each element has an inverse – 0, 1, 4, and 5 are self-inverse while $2^{-1} = 6$ and $3^{-1} = 7$. As associativity is assumed, this is a group.

b i As noted in part **aii**, 1, 4 and 5 are non-identity elements which are self-inverses, so $1 = 1^{-1}$ etc.

ii Look for an element of order 4.
For 2: $2 \circ 2 = 4$, $2 \circ 2 \circ 2 = 6$, $2 \circ 2 \circ 2 \circ 2 = 0$, so 2 has order 4.
So the set {0, 2, 4, 6} under operation $\circ$ is a group of order 4. It is cyclic, generated by 2.

6 c Elements 2, 3, 6 and 7 all have order 4. This can be deduced from the fact that in each case the square of the elements is 4, which has order 2. Elements 1, 4 and 5 have order 2, and the identity element 0 has order 1. Therefore there is no element in *G* that has order 8, so the group cannot be cyclic.

7 a There are elements that do not have a multiplicative inverse. Finding the inverse of a 2 × 2 matrix involves dividing by the determinant, so any matrix which has a zero determinant (singular matrices) does not have an inverse. So the inverse axiom fails.

b Closure: The set is closed under multiplication because the determinant of the product of two matrices is the product of the determinants, so if two matrices are non-singular then so is their product.

Identity: the identity element is $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is a non-singular real-valued 2 × 2 matrix.

Inverse: for any non-singular matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc \neq 0$ by definition, and

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Associativity is assumed, so the set of non-singular real-valued 2 × 2 matrices under matrix multiplication is a group.

8 a i

∘	**2**	**4**	**8**	**10**	**14**	**16**
2	4	8	16	2	10	14
4	8	16	14	4	2	10
8	16	14	10	8	4	2
10	2	4	8	10	14	16
14	10	2	4	14	16	8
16	14	10	2	16	8	4

ii The Cayley table shows closure (all entries are in the set *G*), the identity element is 10, and each element has an inverse – 8 is self-inverse while $2^{-1} = 14$ and $4^{-1} = 16$. As associativity is assumed, this is a group.

b As $4^2 \equiv_{18} 16,\ 4^3 \equiv_{18} 10,$ so 4 has order 3.

c These elements cannot be generators: 10 (order 1), 8 (order 2), 4 (order 3) and 16 (order 3).

For 2: $2^2 = 4,\ 2^3 = 8,\ 2^4 = 8,\ 2^3 = 16,\ 2^5 = 14,\ 2^6 = 10$. So 2 is order 6 and a generator.

Writing each element in terms of the generator: $2 = 2^1,\ 4 = 2^2, 8 = 2^3, 10 = 2^6,\ 14 = 2^5,\ 16 = 2^4$

The element14 is also a generator: $2 = 14^5,\ 4 = 14^4, 8 = 14^3, 10 = 14^6,\ 14 = 14^1,\ 16 = 14^2$

d *H* is the set of elements on the diagonal of the Cayley table, so *H* = {4, 10, 16}.
From part **b**, this is a cyclic group of order 3 generated by 4, so it is a subgroup of *G*.

9 a The set contains all integers modulo 6 so the operation must be closed. The identity element is 0 as it is an identity for normal addition. Each element has an inverse: $1+5 = 2+4 = 3+3 = 0$
Associativity follows from associativity of normal addition. So all axioms hold.

9 b The element 1 has order 6 (1 + 1 = 2, 1 + 1 +1 = 2, 1 + 1 + 1 + 1 = 4, 1 + 1 + 1 + 1 + 1 = 5, and 1 + 1 + 1 + 1 + 1 + 1 = 0) so the group is cyclic.
The 5 is also a generator: (5 + 5 = 4, 5 + 5 +5 = 3, 5 + 5 + 5 + 5 = 2, 5 + 5 + 5 + 5 + 5 = 1, and 5 + 5 + 5 + 5 + 5 + 5 = 0).
The element 3 has order 2, 2 and 4 have order 3, and 0 has order 1, so these are not generators.

c As 4 does not divide 6, which is the order of the group, by Lagrange's theorem it cannot be the order of any subgroup.

d The set {0, 2, 4} under addition modulo 6 is a subgroup (as it is closed, contains 0 and $2+4=0$). Thus it is a subgroup, and it has three elements.

10 a Closure: $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} z & w \\ -w & z \end{pmatrix} = \begin{pmatrix} xz - wy & -yz - xw \\ xw + yz & -wy + xz \end{pmatrix}$, which is an element of S.

Identity: The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is clearly an element of S.

Inverse: Given $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, its inverse is $\begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$

This inverse exists because x and y are not both zero, and it is clearly an element of S.
Associativity: Matrix multiplication is associative.
So S forms a group under matrix multiplication.

b R is clearly closed under multiplication, as $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} xy & 0 \\ 0 & xy \end{pmatrix}$

The identity matrix is in R (for $x = 1$)

The inverse of $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ is $\begin{pmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{pmatrix}$, which is in R.

Matrix multiplication is associative.
So R is a subgroup of S.

c T is not a subgroup of S, since it does not contain the identity matrix.

11 a 1 has order 1.

All other elements have order 2 as $5^2 = 7^2 = 11^2 = 13^2 = 17^2 = 19^2 = 23^2 \equiv_{24} 1$

b There is no element of order 4, so no element can generate a cyclic group of order 4.

c The element $e^{\frac{\pi i}{4}}$ has order 8 in H, as $(e^{\frac{\pi i}{4}})^8 = (e^{2\pi i}) = (e^{\pi i})^2 = (-1)^2 = 1$ and $e^{\frac{k\pi i}{4}} \neq 1$ for k = 1, 2, 3, 4, 5, 6, 7.

So $e^{\frac{\pi i}{4}}$ is a generator and H is cyclic.

G doesn't have any elements of order 8, so it is not cyclic, and therefore G and H are not isomorphic.

12 a For groups A and B, the identity element is 1 (the fact that 1 is an identity for normal multiplication implies that it is an identity for multiplication modulo 10 and 15).

For group C, the identity element is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

This is the identity matrix for normal matrix multiplication.

12 b i Groups A and B are both cyclic groups of order 4 (A is generated by 3, B by 2).
So they are isomorphic.

ii All the non-identity elements of C are self-inverse, so they have order 2.
Therefore C is isomorphic to the Klein four-group, and as it has no element of order 4 it is not isomorphic to B.

iii Similarly, C is not isomorphic to A.

13 a This is the group of anticlockwise rotations of $\frac{k\pi}{3}$ for $k = 0, 1, 2, 3, 4, 5$.

So it has order 6 and it is cyclic, generated by the matrix $\mathbf{A} = \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$

$\mathbf{A}$ has order 6, as does the matrix $\mathbf{A}^5$, its inverse, while $\mathbf{A}^3$ has order 2 and $\mathbf{A}^2$ and $\mathbf{A}^4$ have order 3, and the identity element $\mathbf{A}^6$ has order 1.

b The group of permutations S_3 is not cyclic, and it has three elements of order 2.
The group G is cyclic and only has one element of order 2.
So S_3 is not isomorphic to G.

Challenge

a There are $4! = 24$ possible permutations of 4 objects, so $|S_4| = 24$

b i A cyclic group of order 4 generated by $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ is:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\}$$

ii A cyclic group of order 3 is:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \right\}$$

iii A group of order 6 is the copy of S_3 (based on permutations of the first three objects that is contained in this group:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \right\}$$

c i A group isomorphic to the Klein four-group is:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \right\}$$

Challenge

c ii A group isomorphic to D_8 is:

$$\left\{\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix},\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix},\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix},\begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix},\begin{pmatrix}1&2&3&4\\4&2&1&3\end{pmatrix}\right\}$$

d i The elements of S_4 are permutations of 4 objects, therefore they can be categorised as:
one element of order 1 (the identity);
six elements of order 2 that fix two objects and swap the other two;
six elements of order 2 that swap two disjoint pairs of objects;
eight elements of order 3 that fix one object and rotate the other three;
three elements of order 4 that rotate all objects.
Therefore there is no element of order 6, so there is not a cyclic subgroup of order 6.

ii G has four elements of order 4 (2, 7, 8 and 13) while S_4 only has three (from part **di**), so there cannot be any subgroup of S_4 isomorphic to G.