Groups 2B

1

$\times$	**1**	**–1**
1	1	–1
–1	–1	1

2 a

$+$	**1**	**0**	**–1**
1	2	1	0
0	1	0	–1
–1	0	–1	–2

This isn't a group because the set is not closed under the operation.
For example, $1+1=2\notin\{-1, 0, 1\}$.

b

$\times_{12}$	**1**	**5**	**7**	**11**
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

The elements of the table are all members of the set and so it is closed.
The identity element is 1.
The identity element is in each row and each column; each element is a self-inverse.
Multiplication on integers is associative, so multiplication modulo 12 is associative.
So the set $\{1, 5, 7, 11\}$ under multiplication modulo 12 is a group.

3

$\times_7$	**1**	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

4 a i

$\times_{15}$	**1**	**2**	**4**	**8**
1	1	2	4	8
2	2	4	8	1
4	4	8	1	2
8	8	1	2	4

ii The elements of the table are all members of the set and so it is closed.
The identity element is 1.
The identity element is in each row and each column; each element has an inverse.
Multiplication on integers is associative, so multiplication modulo 15 is associative.
So the set {1, 2, 4, 8} under multiplication modulo 15 is a group.

b The identity element would be 1, as $1\times_{12}1=1, 1\times_{12}2=2,\ 1\times_{12}4=4,\ 1\times_{12}8=8$
But $2\times_{12}1=2, 2\times_{12}2=4,\ 2\times_{12}4=8,\ 2\times_{12}8=4$
So 2 doesn't have an inverse, and so S can't be a group under multiplication modulo 12.

5 a As $2+_8 2=4$ and $4+_8 6=2$, neither 2, 4 or 6 is the identity of this group.
So a must be the identity of this group; and this must be $a = 0$.

b

$+_8$	**0**	**2**	**4**	**6**
0	0	2	4	6
2	2	4	6	0
4	4	6	0	2
6	6	0	2	4

c The generator is 2: $2^1=2, 2^2=2+_8 2=4, 2^3=2+_8 2+_8 2=6, 2^4=2+_8 2+_8 2+_8 2=0$

6 a

$\circ$	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	3	0	2
2	2	0	3	1
3	3	2	1	0

b The elements of the table are all members of the set and so it is closed.
The identity element is 0.
Each element has an inverse; 0 and 3 are self-inverses, and 2 is the inverse of 1 and 1 the inverse of 2.
Associativity is assumed.
So the set {0, 1, 2, 3} under $\circ$ is a group.

7 a

$\circ$	q	r	s	t
q	q	q	q	q
r	q	r	s	t
s	t	s	r	q
t	t	t	t	t

b From the Cayley table that r is the identity.

c From the Cayley table, q and t don't have inverses
So $(M, \circ)$ is not a group.

8 a From the Cayley table it is clear that for every $x, y \in A$, $x * y \in A$, so A is closed under $*$.

b An element of a set is an identity if and only if the corresponding row and column entries in the Cayley table contain the same elements as the headings of the table. Since this doesn't happen for any element of the set, there is no identity.

c The operation is not associative.
For example, $10*(30*30) = 10*30 = 20$ while $(10*30)*30 = 20*30 = 10$

d From parts **b** and **c**, two axioms do not hold, so this can't be a group.

9 a

$*$	0	1	2	3
0	0	2	0	2
1	1	0	3	2
2	2	2	2	2
3	3	0	1	2

b The operation is not associative
For example, $1*(2*3) = 1*2 = 3$, while $(1*2)*3 = 3*3 = 2$.

c Since $2*x = 2+2x+2x = 2+4x = 2$ for all x, this is equivalent to finding solutions to $x*1 = 2$.
From the second column of the table, $x*1 = 2$ when $x = 0$ and $x = 2$.

10 a Check whether the closure axiom holds.

$9 \times 16 = 144 \equiv 53 \pmod{91}$

$9 \times 22 = 198 \equiv 16 \pmod{91}$

$9 \times 53 = 477 \equiv 22 \pmod{91}$

$9 \times 74 = 666 \equiv 29 \pmod{91}$

As $29 \notin S$, the closure axiom does not hold.

So the set S under multiplication modulo 91 is not a group.

10 b Add 29 to S. The Cayley table is:

$\times_{91}$	**1**	**9**	**16**	**22**	**29**	**53**	**74**	**79**	**81**
1	1	9	16	22	29	53	74	79	81
9	9	81	53	16	79	22	29	74	1
16	16	53	74	79	9	29	1	81	22
22	22	16	79	29	1	74	81	9	53
29	29	79	9	1	22	81	53	16	74
53	53	22	29	74	81	79	9	1	16
74	74	29	1	81	53	9	16	22	74
79	79	74	81	9	16	1	22	53	29
81	81	1	22	53	74	16	74	29	9

The elements of the table are all members of the set and so it is closed.
The identity element is 1.
Each element has an inverse (the identity element is in each row and each column).
Associativity is assumed.
So the augmented set is a group under multiplication modulo 91.

11 Assume $n > 0$
Given a such that $a \mid n$, there must be $d < n$ such that $n = ad$
If S forms a group under multiplication modulo n, a has an inverse.
So there is b such that $a \times_n b = 1$
Then $(d \times_n a) \times_n b = 0$ and $d \times_n (a \times_n b) = d$, so by associativity $d \equiv 0 \mod n$.
But as $d < n$, $d = 0$, and as $n = ad$ this gives $n = 0$, so there is a contradiction.
Hence S does not form a group under multiplication modulo n

12 a

$\circ$	***e***	***p***	***q***	***r***	***s***	***t***
e	*e*	*p*	*q*	*r*	*s*	*t*
p	*p*	*e*	*t*	*s*	*r*	*q*
q	*q*	*s*	*e*	*t*	*p*	*r*
r	*r*	*t*	*s*	*e*	*q*	*p*
s	*s*	*q*	*r*	*p*	*t*	*e*
t	*t*	*r*	*p*	*q*	*e*	*s*

b The elements of the table are all members of the set and so it is closed.
The identity element is e.
Each element has an inverse (the identity element is in each row and each column): p, q and r are self-inverses while s and t are inverses.

13 a $b \circ a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

b $a \circ b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}$

13 c $a^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} = a$

d $b^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

e $b^{-1} \circ a^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$

f $a^{-1} \circ b^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix}$

g This is the same as part **f**.

h This is the same as part **e**.

14 a Working out in multiplication modulo 16
$3 \times (9 \times 11) = 3 \times 3 = 9$ and $(3 \times 9) \times 11 = 11 \times 11 = 9$
So $3 \times (9 \times 11) = (3 \times 9) \times 11$

b The Cayley table is:

$\times_{16}$	**1**	**3**	**9**	**11**
1	1	3	9	11
3	3	9	11	1
9	9	11	1	3
11	11	1	3	9

The elements of the table are all members of the set and so it is closed.
The identity element is 1.
Each element has an inverse: 1 and 9 are self-inverses while 3 and 11 are inverses.
Associativity follows from associativity of normal multiplication.
So the set {1, 3, 9, 11} under multiplication modulo 16 is a group.

c The group is cyclic.
3 and 11 are both generators:
$3 \times 3 = 9,\ 3 \times 3 \times 3 = 11,\ 3 \times 3 \times 3 \times 3 = 1$
$11 \times 11 = 9,\ 11 \times 11 \times 11 = 3,\ 11 \times 11 \times 11 \times 11 = 1$

15 a This is generated by 3: $3 \times_{10} 3 = 9,\ 3 \times_{10} 3 \times_{10} 3 = 7,\ 3 \times_{10} 3 \times_{10} 3 \times_{10} 3 = 1$
The element 7 is also a generator.

b This is generated by 8: $8 \times_{20} 8 = 4,\ 8 \times_{20} 8 \times_{20} 8 = 12,\ 8 \times_{20} 8 \times_{20} 8 \times_{20} 8 = 16$
The element 12 is also a generator.

c This is generated by 2: $2 \times_9 2 = 4,\ 2 \times_9 2 \times_9 2 = 8,\ 2 \times_9 2 \times_9 2 \times_9 2 = 7,\ 2 \times_9 2 \times_9 2 \times_9 2 \times_9 2 = 5$
and $2 \times_9 2 \times_9 2 \times_9 2 \times_9 2 \times_9 2 = 1$
The element 5 is also a generator.

16 If 6 generates a group under multiplication modulo 8, then it has an inverse x, i.e. $6x \equiv 1 \pmod 8$. But this is impossible because $6x$ must be an even number, so there is a contradiction.

17 $5^2 = 25 = 4 \pmod{21}$, $5^3 = 125 = 20 \pmod{21}$, $5^4 = 625 = 16 \pmod{21}$

$5^5 = 3125 = 17 \pmod{21}$, $5^6 = 15\,625 = 1 \pmod{21}$

So $G = \{1, 4, 5, 16, 17, 20\}$

18 a Multiplying out, $\omega^2 = \mathrm{i}$; therefore, $\omega^8 = \mathrm{i}^4 = 1$.

So ω^k generates the set $G = \{\omega^n : n = 0, 1, 2 \ldots 7\}$

This is closed; for any $j, k \in \mathbb{Z}$ and $0 \leqslant j, k \leqslant 7$ then $\omega^j \omega^k = \omega^{j+k}$

If $j + k \leqslant 7$, then $\omega^{j+k} \in G$, if $j + k > 7$, then $\omega^{j+k} = \omega^8 \omega^{j+k-8} = \omega^{j+k-8} \in G$.

The identity element is $\omega^0 = 1$.

Each element has an inverse: $\omega^j \omega^{8-j} = \omega^8 = 1$, so $(\omega^j)^{-1} = \omega^{8-j}$.

Associativity holds as complex multiplication is always associative.

So the set G under complex multiplication is a group.

b The other generators are of the form ω^n with gcd(n, 8) = 1, i.e. ω^3, ω^5, ω^7.

$$\omega^3 = \omega^2\omega = \mathrm{i}\left(\frac{\sqrt{2}}{2}(1+\mathrm{i})\right) = \frac{\sqrt{2}}{2}(-1+\mathrm{i})$$

$$\omega^5 = \omega^4\omega = -\left(\frac{\sqrt{2}}{2}(1+\mathrm{i})\right) = \frac{\sqrt{2}}{2}(-1-\mathrm{i})$$

$$\omega^7 = \omega^4\omega = -\left(\frac{\sqrt{2}}{2}(-1+\mathrm{i})\right) = \frac{\sqrt{2}}{2}(1-\mathrm{i})$$

19 a $p_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$, $p_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}$

b

$\circ$	p_1	p_2	p_3	p_4	p_5
p_1	p_1	p_2	p_3	p_4	p_5
p_2	p_2	p_3	p_4	p_5	p_1
p_3	p_3	p_4	p_5	p_1	p_2
p_4	p_4	p_5	p_1	p_2	p_3
p_5	p_5	p_1	p_2	p_3	p_4

c From the Cayley table it is clear that the closure axiom is satisfied

The identity element is p_1

Each element has an inverse ($p_1^{-1} = p_1$, $p_5^{-1} = p_2$ and $p_3^{-1} = p_4$).

For associativity, notice from the table that $p_i p_j = p_{i+j-1}$ for all i, j.

Therefore, $p_i(p_j p_k) = p_i p_{j+k-1} = p_{i+j+k-2} = p_{i+j-1} p_k = (p_i p_j) p_k$.

Therefore all the group axioms are satisfied.

19 d Again, use the fact that $p_i p_j = p_{i+j-1}$:

This means that p_2 is a generator for this cyclic group, as $p_2^2 = p_3$, $p_2^3 = p_4$, $p_2^4 = p_5$ and $p_2^5 = p_1$.

20 a $h_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$ (identity)

$h_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ (horizontal reflection)

$h_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$ (rotation of 180°)

$h_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ (vertical reflection)

b The Cayley table is:

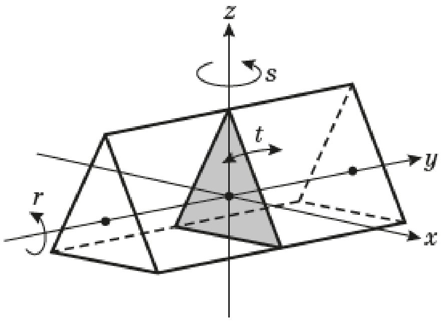
$\circ$	h_1	h_2	h_3	h_4
h_1	h_1	h_2	h_3	h_4
h_2	h_2	h_1	h_4	h_3
h_3	h_3	h_4	h_1	h_2
h_4	h_4	h_3	h_2	h_1

From the Cayley table it is clear that the closure axiom is satisfied.
The identity element is h_1.
Each element has an inverse (all elements are self-inverse).
Associativity is satisfied, $h_i(h_j h_k) = h_i h_j h_k = (h_i h_j)h_k$
Therefore all the group axioms are satisfied.

c The group cannot be cyclic because each element is self-inverse, and therefore no element can be a generator.

Challenge

Denote by r the 120° rotation about the y-axis, s the 180° rotation about the z-axis, t the reflection in the xz-plane and e the identity:



Then the Cayley table is:

$\circ$	e	r	r^2	s	sr	sr^2	t	tr	tr^2	ts	tsr	tsr^2
e	e	r	r^2	s	sr	sr^2	t	tr	tr^2	ts	tsr	tsr^2
r	r	r^2	e	sr^2	s	sr	tr	tr^2	t	tsr^2	ts	tsr
r^2	r^2	e	r	sr	sr^2	s	tr^2	t	tr	tsr	tsr^2	ts
s	s	sr	sr^2	e	r	r^2	ts	tsr	tsr^2	t	tr	tr^2
sr	sr	sr^2	s	r^2	e	r	tsr	tsr^2	ts	tr^2	t	tr
sr^2	sr^2	s	sr	r	r^2	e	tsr^2	ts	tsr	tr	tr^2	t
t	t	tr	tr^2	ts	tsr	tsr^2	e	r	r^2	s	sr	sr^2
tr	tr	tr^2	t	tsr^2	ts	tsr	r	r^2	e	sr^2	s	sr
tr^2	tr^2	t	tr	tsr	tsr^2	ts	r^2	e	r	sr	sr^2	s
ts	ts	tsr	tsr^2	t	tr	tr^2	s	sr	sr^2	e	r	r^2
tsr	tsr	tsr^2	ts	tr^2	t	tr	sr	sr^2	s	r^2	e	r
tsr^2	tsr^2	ts	tsr	tr	tr^2	t	sr^2	s	sr	r	r^2	e