Number theory 1G

1 **a** $^5P_5 = \dfrac{5!}{(5-5)!} = 5! = 120$

b $^5C_2 = \dfrac{5!}{2!(5-2)!} = \dfrac{5!}{2!3!} = \dfrac{120}{2\times 6} = 10$

c $^{20}P_1 = \dfrac{20!}{(20-1)!} = \dfrac{20!}{19!} = 20$

d $^8P_3 = \dfrac{8!}{(8-3)!} = \dfrac{8!}{5!} = 336$

e $^{20}C_7 = \dfrac{20!}{7!(20-7)!} = \dfrac{20!}{7!13!} = 77\,520$

f $^{100}C_3 = \dfrac{100!}{3!(100-3)!} = \dfrac{100!}{3!97!} = 161\,700$

2 The total number of possible configurations will be $3\times 2\times 4 = 24$

3 The total number of possible combinations will be $3\times 4\times 3 = 36$

4 The total number of possible ways of answering 12 questions is $4^{12} = 16\,777\,216$

5 **a** In the first position there are 26 options, and there are 10 options in both the second and third positions. Hence in total there are $26\times 10\times 10 = 2600$ possible settings.

b In the first two positions there are 5 options, and there are 4 options in the last two positions. So there are $5\times 5\times 4\times 4 = 400$ possible settings.

6 There are 5 distinct odd digits. After the first 3 odd digits are chosen, there are 7 distinct digits remaining. Hence there are $5\times 4\times 3\times 7\times 6\times 5 = 12\,600$ options.

7 If a number is odd, the last digit is be one of the 5 odd digits 1,3,5,7,9.
If 0 is not allowed, the other digits can be one of 9 numbers.
Hence there are $5\times 9^4 = 32\,805$ possible such numbers.

8 **a** If 7 is the last digit, then there are 9×8 possibilities (the first digit cannot be 0).
If 7 is the middle digit, then again there are 9×8 possibilities (the first digit cannot be 0).
If 7 is the first digit, then there are 9×9 possibilities.
In total, there are $1\times 9\times 8 + 9\times 1\times 8 + 9\times 9\times 1 = 225$ combinations.

b The total number of three-digit numbers is $9\times 10\times 10 = 900$ (the first digit cannot be 0).
The total number of three-digit numbers that do not contain 7 is $8\times 9\times 9 = 648$
So the number of three-digit numbers that contain at least one 7 is $900 - 648 = 252$

9 **a** The number of ways of shelving all 7 textbooks is $7! = 5040$

9 b There are 7 ways of choosing the first textbook and 6 ways of choosing the second one.
Hence there are $7\times6=42$ ways of shelving 2 textbooks.

c Similarly, there are $7\times6\times5\times4\times3=2520$ ways of shelving 5 textbooks.

10 The first member can be assigned 11 numbers, the second $11-1=10$ numbers, the third 9 numbers, and so on until the last player will have only one number left. Multiplying these options together, the total number of possible assignments is $11\times10\times\ldots\times2\times1=11!=39\,916\,800$.

11 a i The first card can be chosen in 5 ways, the second in 4, and so on until only one option left.
Hence Jonjo can make $5\times4\times3\times2\times1=5!=120$ different 5-digit numbers.

ii The first card can be chosen in 5 ways, the second in 4 ways and the third in 3 ways.
Hence there $5\times4\times3=60$ ways of making a 3-digit number out of the cards.

b With an extra card, there are now 6 numbers but 2 are identical.

Hence Jonjo can make $\frac{6!}{2!}=360$ different 6-digit numbers from the cards. 6

12 There are 8 letters in total, of which 4 letters are identical (E) and another 2 letters are identical (D).

Hence the total number of permutations is $\frac{8!}{4!2!}=840$

13 a If there are no restrictions who sits where, there are $8!=40\,320$ possible seating arrangements.

b If each couple must want sit next to each other, there are $4!=24$ ways of arranging the couples.
For each of these arrangements, each couple can be arranged in two ways, one on the left and the other on the right or vice versa. So for each distinct arrangement of the couples there are $2\times2\times2\times2=16$ possible seating arrangements.
Therefore there are $24\times16=384$ possible seating arrangements where the two members of each couple sit next to each other.

14 The number of ways to match the four balls is 1

The number of ways to draw four different balls from 10 balls is ${}^{10}C_4=\frac{10!}{6!4!}=210$

Hence the probability of winning a star prize is $\frac{1}{210}$

15 a There are $17\times16\times15=4080$ ways of selecting the three officers.

b i There are $10\times16\times15=2400$ of selecting the three officers such that the president is female.

ii There are $10\times9\times15+7\times6\times15=1980$ ways of selecting the officers such that both the president and the deputy are of the same gender.

iii There are 4080 ways of selecting three officers (answer to part **a**).
There are $10\times9\times8+7\times6\times5=930$ ways selecting all female or all male officers.
So, there are $4080-930=3150$ ways to select three officers that are not all of the same gender.

c It can be done in ${}^{17}C_3=680$ ways.

16 a The number of ways to select a 4-digit number with no repeated digits is $9\times8\times7\times6=3024$

b There number of ways to select a 3-digit number with no repeated digits is $9\times8\times7$
There are $9\times8\times7\times3$ ways to select 4-digit numbers, with one repeating number at the end.
The repeating digit can also be in two additional positions.
Hence there are $9\times8\times7\times3\times2=3024$ ways of making the 4-digit numbers with a repeated digit.

c Each 4-digit number has two distinct digits.
There are ${}^9C_2=36$ combinations of two distinct digits from nine numbers.
There are $\frac{4!}{2!2!}$ ways to arrange two pairs of repeated digits.
Hence there are $36\times\frac{4!}{2!2!}=216$ numbers with two repeated distinct digits.

17 a The total number of possible subsets of S is $2^5=32$

b The number of subsets that contain three elements is ${}^5C_3=10$

18 a $1000=111\times9+1$, hence there are 111 numbers that are <1000 and are divisible by 9.

b The number of subsets that contain three or fewer elements is:
${}^{111}C_3+{}^{111}C_2+{}^{111}C_1+{}^{111}C_0=228\,032$

19 a In total there are ${}^{21}C_4=5985$ possibilities.

b There are ${}^9C_2\times{}^{12}C_2=36\times66=2376$ possibilities.

c There number of possible teams that have three or four boys (i.e. more boys than girls) are:
${}^9C_3\times{}^{12}C_1+{}^9C_4\times{}^{12}C_0=84\times12+126\times1=1134$

20 a The total number of possible samples is ${}^{100}C_6=1\,192\,052\,400$

b The number of samples that contain no defective disks is ${}^{96}C_6=927\,048\,304$
So the probability that the sample contains at least one defective disk is $1-\frac{{}^6C_{96}}{{}^6C_{100}}=0.222$ (3 s.f.)

21 a If the positions are not assigned, the number of possible teams is ${}^{23}C_{11}=\frac{23!}{12!11!}=1\,352\,078$

b If any player can play in any position and each player is assigned to unique position, the number of different possible teams is ${}^{23}P_{11}=\frac{23!}{12!}=53\,970\,627\,110\,400$.

c If the specialisms are assigned, the number is ${}^3C_1\times{}^{10}C_5\times{}^5C_3\times{}^5C_2=3\times252\times10\times10=75\,600$

Challenge

a There are four steps up and four steps to the right. The choice at each node is whether to go up or go to the right. The number of these choices is equal to the number of routes and is given by ${}^{8}C_{4} = 70$

b For an $n \times n$ grid, the same method applies.

The number of routes is given by ${}^{2n}C_{n} = \frac{(2n)!}{n!n!}$