Number theory 1D

1 **a** $2+5+0+2=9$ and $9\,|\,9$, so $9\,|\,2502$

b $5+9+3+1=18$ and $3\,|\,18$, so $3\,|\,5931$

c $1-0+1-7+9-5=-1$ and 11 does not divide –1, hence 11 does not divide 101 795

d $60=4\times 15$, so $4\,|\,2\,000\,560$

e $51\,792$ is even and hence divisible by 2
$5+1+7+9+2=24$ and $3\,|\,24$
So $51\,792$ is divisible by 2 and 3, hence $6\,|\,51\,792$

f $1-3+2-6+0-9+4=-11$ and $11\,|\,-11$, so $11\,|\,1\,326\,094$

2 Let x be a positive integer with decimal digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$.
Then $x=10^n a_n+10^{n-1}a_{n-1}\dots+10^2 a_2+10a_1+a_0$
As $10\equiv 0 \pmod 2 \Rightarrow 10^k\equiv 0^k\equiv 0 \pmod 2$ for all $k\in\mathbb{N}$
Hence $x\equiv 10^n a_n+10^{n-1}a_{n-1}\dots+10^2 a_2+10a_1+a_0\equiv a_0 \pmod 2$
So $2\,|\,a_0 \Rightarrow x\equiv a_0\equiv 0 \pmod 2 \Rightarrow 2\,|\,x$
Conversely, $2\,|\,x \Rightarrow x\equiv 0 \pmod 2 \Rightarrow a_0\equiv 0 \pmod 2 \Rightarrow 2\,|\,a_0$
So x is divisible by 2 if and only if its last digit is divisible by 2.

3 Let x be a positive integer with decimal digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$.
Then $x=10^n a_n+10^{n-1}a_{n-1}\dots+10^2 a_2+10a_1+a_0$
As $10\equiv 0 \pmod 5 \Rightarrow 10^k\equiv 0^k\equiv 0 \pmod 5$ for all $k\in\mathbb{N}$ for all $k\in\mathbb{N}$
Hence $x\equiv 10^n a_n+10^{n-1}a_{n-1}\dots+10^2 a_2+10a_1+a_0\equiv a_0 \pmod 5$
So $5\,|\,a_0 \Rightarrow x\equiv a_0\equiv 0 \pmod 5 \Rightarrow 5\,|\,x$
Conversely, $5\,|\,x \Rightarrow x\equiv 0 \pmod 5 \Rightarrow a_0\equiv 0 \pmod 5 \Rightarrow 5\,|\,a_0$
So x is divisible by 5 if and only if its last digit is divisible by 5.

4 $10\equiv 1 \pmod 9$ and $100\equiv 1 \pmod 9$
Hence $N\equiv 100a+10b+c\equiv a+b+c \pmod 9$
So $N\equiv 0 \pmod 9$ if and only if $a+b+c\equiv 0 \pmod 9$
Therefore $9\,|\,N$ if and only if $9\,|\,a+b+c$

5 $10\equiv -1 \pmod{11}$, $100\equiv 1 \pmod{11}$, $1000\equiv -1 \pmod{11}$ and $10\,000\equiv 1 \pmod{11}$
Hence $N\equiv 10\,000a+1000b+100c+10d+e\equiv a-b+c-d+e \pmod{11}$
So $N\equiv 0 \pmod{11}$ if and only if $a-b+c-d+e\equiv 0 \pmod{11}$
Therefore $11\,|\,N$ if and only if $11\,|\,a-b+c-d+e$

6 Let N be a positive integer with decimal digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$.
Then $N = 10^n a_n + 10^{n-1} a_{n-1} \dots + 10^2 a_2 + 10a_1 + a_0$
$10 \equiv 1 \pmod 3 \Rightarrow 10^k \equiv 1 \pmod 3$ for all $k \in \mathbb{N}$
So $N \equiv 10^n a_n + 10^{n-1} a_{n-1} \dots + 10^2 a_2 + 10a_1 + a_0 \equiv a_n + a_{n-1} + \dots + a_2 + a_1 + a_0 \pmod 3$
If $3 \mid a_n + a_{n-1} \dots a_2 + a_1 + a_0 \Rightarrow a_n + a_{n-1} \dots a_2 + a_1 + a_0 \equiv 0 \pmod 3 \Rightarrow N \equiv 0 \pmod 3 \Rightarrow 3 \mid N$
So if the sum of the digits of N is divisible by 3, N is divisible by 3.

7 Let x be a positive integer with decimal digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$.
Then $x = 10^n a_n + 10^{n-1} a_{n-1} \dots + 10^2 a_2 + 10a_1 + a_0$
As $100 \equiv 0 \pmod 4$, $10^k \equiv 0^k \equiv 0 \pmod 4$ for all integers $k > 1$
So $x \equiv 10^n a_n + 10^{n-1} a_{n-1} \dots + 10^2 a_2 + 10a_1 + a_0 \equiv 10a_1 + a_0 \pmod 4$
Therefore $4 \mid x$ if $4 \mid a_0 + 10a_1$, that is x is only divisible by 4 if its last two digits are divisible by 4.

8 $6+1+5+9+2+8+5 = 36$ and $9 \mid 36$, so $9 \mid 6\,159\,285$
$6-1+5-9+2-8+5 = 0$ and $11 \mid 0$, therefore $11 \mid 6\,159\,285$

9 $1-0+2-x+5-7+6-1 = 6-x$
If the number if divisible by 11, then $11 \mid 6-x$, therefore as $0 \leqslant x \leqslant 9 \Rightarrow x = 6$.

10 Divisibility by 11 implies $2-a+8-4+5-5+b-8 = -2-a+b$ is divisible by 11
So $-2-a+b = 11p$ for some $p \in \mathbb{Z}$
As $a, b \geqslant 0$ and $a, b \leqslant 9, -9 \leqslant b-a \leqslant 9$
Only $b-a = -9$ and $b-a = 2$ can satisfy $-2-a+b = 11p$

Divisibility by 9 implies $2+a+8+4+5+5+b+8 = 32+a+b$ is divisible by 9
So $32+a+b = 9q$ for some $q \in \frown$
As $a, b \geqslant 0$ and $a, b \leqslant 9, 0 \leqslant a+b \leqslant 18$
Only $a+b = 4$ and $a+b = 13$ can satisfy $32+a+b = 9q$

Consider $b-a = -9 \Rightarrow b = 0, a = 9$. This result does not satisfy either condition $a+b = 4$ or $a+b = 13$ so it does not yield a solution.

Consider $b-a = 2 \Rightarrow b = 2+a$
If $a+b = 4 \Rightarrow a+(2+a) = 4 \Rightarrow a = 1, b = 3$
If $a+b = 13 \Rightarrow a+(2+a) = 13 \Rightarrow 2a = 11$, which does not yield a solution as a is an integer

So the only valid solution is $a = 1, b = 3$

11 Because 9 and 11 are coprime, any number divisible by both of the numbers must be divisible by 99.
The only 3-digit numbers that are multiples of 99 are: 198, 297, 396, 495, 594, 693, 792, 891 and 990

12 a If $m = 10a + b$ is divisible by 9, then $9 \mid a+b$
Since $0 < a \leqslant 9$ and $0 \leqslant b \leqslant 9$, $a+b$ is a multiple of 9 between 1 and 18
So $a+b = 9$ or 18.

b $10a + b \equiv -a + b \pmod{11} \Rightarrow b - a \equiv 5 \pmod{11}$
Since $0 < a \leqslant 9$ and $0 \leqslant b \leqslant 9$, $-9 \leqslant b-a \leqslant 8$ and hence the possible values of $b-a$ are 5 and –6.

12 c The solution must satisfy $a+b=9$ or 18 and $b-a=5$ or -6.
Because the sum and the difference of two integers will have the same parity, the only possible solutions come from $a+b=9$ and $b-a=5$, or $a+b=18$ and $b-a=-6$.
The first system of equations gives $a=2$, $b=7$ which is a valid solution.
The second system gives $a=12$ and $b=6$, but is not a valid solution as $0<a\leqslant 9$.
So the solution is $a=2, b=7 \Rightarrow m=27$.

13 As $a+b\equiv 0 \pmod 4$ and since $0<a\leqslant 9$ and $0\leqslant b\leqslant 9$, $a+b$ can be 4, 8, 12 or 16.
As $N\equiv 7 \pmod 8$, $N=7+8k$ for some $k\in\mathbb{N}$. numbers are
The only 2-digit numbers satisfying this condition are: 15, 23, 31, 39, 47, 55, 63, 71, 79, 87 and 95
Of these, the only ones satisfying $a+b$ equals 4, 8, 12 or 16 are:
31 ($a=3$, $b=1$, $a+b=4$)
39 ($a=3$, $b=9$, $a+b=12$)
71 ($a=7$, $b=1$, $a+b=8$)
79 ($a=7$, $b=9$, $a+b=16$)

14 Let $x=100a+10b+c$ where a, b and c are integers between 0 and 9, with $a\neq 0$.
Fact 1: Divisibility by 11 implies $11\,|\,a-b+c$
The restrictions on since a, b and c mean that $-8\leqslant a-b+c\leqslant 20$ so solutions are $a-b+c=0$ or $a-b+c=11$
Fact 2: As $a+b+c$ is odd, $a-b+c=(a+b+c)-2b$ must be odd, so $a-b+c\neq 0$, which leaves the possibility $a-b+c=11$
Fact 3: $x\equiv 8 \pmod 9$. As $10\equiv 1 \pmod 9 \Rightarrow 10^k\equiv 1 \pmod 9$ for all $k\in\mathbb{N}$, this gives:
$x\equiv 8 \pmod 9 \Rightarrow a+b+c\equiv 8 \pmod 9$
As $a-b+c=11\Rightarrow a-b+c\equiv 2 \pmod 9$, subtracting these two identities gives:
$2b\equiv 6 \pmod 9$
Given that $0<b\leqslant 9$, the only possible solution is
Adding $a+b+c\equiv 8 \pmod 9$ and $a-b+c\equiv 2 \pmod 9$ gives:
$2(a+c)\equiv 10 \pmod 9 \Rightarrow a+c\equiv 5 \pmod 9$
Additionally, as $b=3$, $a-b+c=11\Rightarrow a+c=14$
The only pairs of integers less than 10 that satisfy $a+c\equiv 5 \pmod 9$ and $a+c=14$ are:
5 and 9, 6 and 8, 7 and 7
Hence, all possible values of x are 539, 935, 638, 836, 737.

15 Let $Q = 1000a + 100b + 10c + d$ where a, b, c and d are integers between 0 and 9, with $a > 0$ and $a < b < c < d$.
Because the digits are strictly increasing the largest such number can be 6789. Furthermore, we have $10 \leqslant a + b + c + d \leqslant 30$.
If $Q \equiv 6 \pmod 9$, $a + b + c + d \equiv 6 \pmod 9$ and the possible values for $a + b + c + d$ are $6, 15, 24$.
We also have $a + b + c + d$ is even and $10 \leqslant a + b + c + d$, which leaves us only with $a + b + c + d = 24$.
If $Q \equiv 4 \pmod{11}$, $-a + b - c + d \equiv 4 \pmod{11}$ and we can use the fact that $a + b + c + d \equiv 24 \pmod{11} \equiv 2 \pmod{11}$. Adding these two identities yields $2(b + d) \equiv 6 \pmod{11} \Rightarrow b + d \equiv 3 \pmod{11}$, and $a < b < c < d \leqslant 9$ gives $b \geqslant 2$.
Therefore $b + d = 14$ and $b = 5, d = 9$ or $b = 6, d = 8$.
If $a = 1$, the largest number such that $a + b + c + d = 24$ gives $c = 9$ which is a contradiction because $c \leqslant 8$. If $a = 2$ we find a first solution 2589. If $a = 3$, we get 3579, and $a = 4$ gives 4569.
Because $a < b$, these are all possible solutions with the pair $b = 5, d = 9$.
For $b = 6, d = 8$ the only possibility is $c = 7$, which gives $a = 3$, and so 3678 is a solution.
Hence, all possible values of Q are 2589, 3579, 3678, 4569.

Challenge

a In base 8 any number with digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$ can be written as

$$x = 8^n a_n + 8^{n-1} a_{n-1} + \dots + 8a_1 + a_0.$$

As $8 \equiv 1 \pmod 7 \Rightarrow 8^k \equiv 1^k \equiv 1 \pmod 7$ for all $k \in \mathbb{N}$ then:

$$x \equiv a_n + a_{n-1} + a_{n-2} + \dots a_1 + a_0 \pmod 7$$

Therefore if $7 \mid x$ then $7 \mid a_n + a_{n-1} + a_{n-2} + \dots a_1 + a_0$; that is, if a number written in base 8 is divisible by 7 then the sum of its digits are divisible by 7.

b $x \equiv 8^n a_n + 8^{n-1} a_{n-1} + \dots + 8a_1 + a_0 \equiv a_0 \pmod 2$
Hence x is divisible by 2 if and only if the last digit is divisible by 2.
Similarly, $x \equiv 8^n a_n + 8^{n-1} a_{n-1} + \dots + 8a_1 + a_0 \equiv a_0 \pmod 4$
Hence x is divisible by 4 if and only if the last digit is divisible by 4, i.e. the last digit is 0 or 4
(It cannot be 8, as x is in base 8.)
Finally $x \equiv 8^n a_n + 8^{n-1} a_{n-1} + \dots + 8a_1 + a_0 \equiv a_0 \pmod 8$
Hence x is divisible by 8 if and only if the last digit is divisible by 8, i.e. the last digit is 0

c In base 7 any number with digits $a_n a_{n-1} a_{n-2} \dots a_1 a_0$

$$x = 7^n a_n + 7^{n-1} a_{n-1} + \dots + 7a_1 + a_0$$

As $7 \equiv 1 \pmod 3 \Rightarrow 7^k \equiv 1^k \equiv 1 \pmod 3$ for all $k \in \mathbb{N}$ then:

$$x \equiv a_n + a_{n-1} + a_{n-2} + \dots a_1 + a_0 \pmod 3$$

Therefore $3 \mid x$ if $3 \mid a_0 + a_1 + a_2 + \dots + a_n$, that is x is divisible by 3 if and only if the sum of its digits is divisible by 3.
Similarly, as $7 \equiv 1 \pmod 6 \Rightarrow 7^k \equiv 1^k \equiv 1 \pmod 6$ for all $k \in \mathbb{N}$ then:

$$x \equiv a_n + a_{n-1} + a_{n-2} + \dots a_1 + a_0 \pmod 6$$

Therefore $6 \mid x$ if $6 \mid a_0 + a_1 + a_2 + \dots + a_n$, that is x is divisible by 6 if and only if the sum of its digits is divisible by 6.