Number theory 1A

1 **a** $21 = 3\times 7 \Rightarrow 3\,|\,21$ so yes

b $8 > 2 \Rightarrow 8$ does not divide 2 so no

c $25 = (-1)\times 25 \Rightarrow -25\,|\,25$ so yes

d $11\times 12 = 132$ and $12\times 12 = 144$, so 12 does not divide 140 so no

2 Given $n \in \mathbb{Z}$ and $n\,|\,15$, because the prime factors of 15 are 3 and 5 ($15 = 3\times 5$),
$n = \pm 1, \pm 3, \pm 5, \pm 15$

3 **a** $12 = 1\times 2\times 2\times 3$
The divisors are all the possible combinations of products of the prime factors 1, 2, 2 and 3, and can also be negative, hence the solution is: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

b $20 = 1\times 2\times 2\times 5$,
The divisors are all the possible combinations of products of the prime factors 1, 2, 2 and 5, and can also be negative, hence the solution is: $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

c $6 = 1\times 2\times 3$, so all the divisors are: $\pm 1, \pm 2, \pm 3, \pm 6$

d All the divisors of 1 are ± 1.

4 Given $a, b, n \in \mathbb{Z}$, $a, b > 0$ and $n \neq 0$, if $a\,|\,b$, then $b = ka$ for some integer k.
So for any non-zero integer n, $bn = kan = k(an)$, therefore $an\,|\,bn$.

5 If $a\,|\,b$, then $b = ka$ for some integer k. If $b\,|\,c$, then $c = lb$ for some integer l.
So $c = lb = l(ka) = (lk)a \Rightarrow a\,|\,c$.

6 **a** $\frac{121}{9} = 13.44\ldots$, so $q = 13$
$r = a - bq = 121 - 9\times 13 = 121 - 117 = 4$
So the solution is $q = 13$, $r = 4$

b $-\frac{148}{12} = -12.33\ldots$, so $q = -13$
$r = a - bq = -148 - 12\times(-13) = 156 - 148 = 8$

c $\frac{51}{9} = 5.66\ldots$, so $q = 5$
$r = a - bq = 51 - 9\times 4 = 51 - 45 = 6$

d $-\frac{51}{9} = -5.66\ldots$, so $q = -6$
$r = a - bq = -51 - 9\times(-6) = 54 - 51 = 3$

6 e $\frac{544}{84} = 6.48\ldots$, so $q = 6$

$r = a - bq = 544 - 84 \times 6 = 544 - 504 = 40$

f $-\frac{544}{84} = -6.48\ldots$, so $q = -7$

$r = a - bq = -544 - 84 \times (-7) = 588 - 544 = 44$

g $44 < 84$, so $q = 0$ and $r = 44$, i.e. $44 = 0 \times 84 + 44$

h $\frac{5723}{100} = 57.23$, so $q = 57$

$r = a - bq = 5723 - 100 \times 57 = 5723 - 5700 = 23$

7 a $\frac{200}{7} = 28.57\ldots$, so $q = 28$

$r = a - bq = 200 - 7 \times 28 = 200 - 196 = 4$

So the quotient is 28 and remainder is 4.

b $-\frac{52}{3} = -17.33\ldots$, so $q = -18$

$r = a - bq = -52 - 3 \times (-18) = 54 - 52 = 2$

c $\frac{22\,000}{13} = 1692.30\ldots$, so $q = 1692$

$r = a - bq = 22\,000 - 13 \times 1692 = 22\,000 - 21996 = 4$

d $\frac{752}{57} = 13.19\ldots$, so $q = 13$

$r = a - bq = 752 - 57 \times 13 = 752 - 741 = 11$

8 By the division algorithm, any integer n can be written in one of the following forms: $3q, 3q+1$ or $3q+2$, where q is some integer. Cubing these expressions gives respectively:

$n^3 = (3q)^3 = 9(3q^3)$

$n^3 = (3q+1)^3 = 27q^3 + 27q^2 + 9q + 1 = 9(3q^3 + 3q^2 + q) + 1$

$n^3 = (3q+2)^3 = 27q^3 + 54q^2 + 36q + 8 = 9(3q^3 + 6q^2 + 4q) + 8$

Hence n^3 can be written in one of the forms: $9k, 9k+1, 9k+8,$ where k is some integer.

9 Any odd integer can be written as $n = 2m + 1$ for some integer m.

Squaring this expression gives $n^2 = 4(m^2 + m) + 1$

If m is odd, then m^2 is also odd, and $m^2 + m$ will be even and hence divisible by 2.

So $m^2 + m = 2k$ for some integer k, hence $n^2 = 8k + 1$ for some integer, k.

If m is even, $m^2 + m$ will be even and hence divisible by 2.

So $m^2 + m = 2k$ for some integer k, hence $n^2 = 8k + 1$ for some integer, k.

Therefore the square of any odd integer can be written $n^2 = 8k + 1$ for some integer, k.

10 By the division algorithm, any integer can be written as $n = 5q + r,\ r \in \{0,1,2,3,4\}$.

So:

$$\begin{aligned} n^4 &= 625q^4 + 500q^3r + 150q^2r^2 + 20qr^3 + r^4 \\ &= 5(125q^4 + 100q^3r + 30q^2r^2 + 4qr^3) + r^4 \\ &= 5m + r^4 \qquad \text{for some integer, } m \end{aligned}$$

Then for:

$r = 0$: $n^4 = 5m + 0^4 = 5m$

$r = 1$: $n^4 = 5m + 1^4 = 5m + 1$

$r = 2$: $n^4 = 5m + 2^4 = 5m + 16 = 5m + 15 + 1 = 5(m+3) + 1$

$r = 3$: $n^4 = 5m + 3^4 = 5m + 81 = 5m + 80 + 1 = 5(m+16) + 1$

$r = 4$: $n^4 = 5m + 4^4 = 5m + 256 = 5m + 255 + 1 = 5(m+51) + 1$

So all cases, n^4 is in either in the forms $5k$ or $5k+1$ for some $k \in \mathbb{Z}$.

11 By the division algorithm, any integer, a, can be written as $a = 3q + r$, where $r \in \{0,1,2\}$.

So:

$$\begin{aligned} a(a^2+2) &= (3q+r)(9q^2 + 6qr + r^2 + 2) \\ &= 27q^3 + 18q^2r + 3qr^2 + 6q + 9q^2r + 6qr^2 + r^3 + 2r \\ &= 27q^3 + 27q^2r + 9qr^2 + 6q + r^3 + 2r \\ &= 3(9q^3 + 9q^2r + 3qr^2 + 2q) + r^3 + 2r \\ &= 3k + r^3 + 2r \qquad \text{for some integer, } k \end{aligned}$$

Then for:

$r = 0$: $a(a^2+2) = 3k + 0^3 + 2 \times 0 = 3k$

$r = 1$: $a(a^2+2) = 3k + 1^3 + 2 \times 1 = 3k + 3 = 3(k+1)$

$r = 2$: $a(a^2+2) = 3k + 2^3 + 2 \times 2 = 3k + 12 = 3(k+4)$

So all cases, $a(a^2+2)$ is divisible by 3, hence $3 \mid a(a^2+2)$

Therefore $\dfrac{a(a^2+2)}{3}$ is an integer.

Challenge

a Given integers a, b, by the division algorithm, there are unique integers q and r such that $a = bq + r$ and $0 \leqslant r < |b|$.

If $0 \leqslant r \leqslant \frac{|b|}{2}$, then a can be written as $a = bp + s$ with $p = q$ and $s = r$.

If $\frac{|b|}{2} < r < |b|$ and $b > 0$, write $a = b(q+1) + (r-b)$. Then $-\frac{|b|}{2} < r - b < 0$, and so a can be written as $a = bp + s$ with $p = q + 1$, $s = r - b$.

If $b < 0$ write $a = b(q-1) + (r+b)$ and hence $\frac{|b|}{2} + b < r + b < |b| + b \Rightarrow -\frac{|b|}{2} < r + b < 0$, and so a can be written as $a = bp + s$ with $p = q - 1$, $s = r + b$.

b If $a = 49$ and $b = 26$, by the division algorithm, $49 = 1 \times 26 + 23$

In this case, $\frac{|b|}{2} < r < |b|$, i.e. $\frac{|26|}{2} < 23 < |26|$ and $b = 26$, so $b > 0$

So $p = q + 1 = 2$, and $s = r - b = 23 - 26 = -3$