Exam-style practice – A Level

1 a The auxiliary equation is

$m^2 - 6m + 9 = 0$

Hence $(m - 3)^2 = 0$

Hence $m = 3$ (repeated root)

Using the standard form for the general solution of recurrence relations with repeated roots:

$u_n = A(3)^n + Bn(3)^n$

b Since $g(x)$ is a constant and the roots of the auxiliary equation $\neq 1$, let λ be a particular solution to the recurrence relation.

Then

$\lambda - 6\lambda + 9\lambda = 15 \Rightarrow \lambda = \frac{15}{4}$

Hence the general solution is

$u_n = A(3)^n + Bn(3)^n + \frac{15}{4}$

$u_1 = \frac{21}{4} = 3A + 3B + \frac{15}{4} \Rightarrow A + B = \frac{1}{2}$

$u_2 = -\frac{93}{4} = 9A + 18B + \frac{15}{4} \Rightarrow A + 2B = -3$

Subtracting these equations gives

$B = -\frac{7}{2}$ hence $A = 4$

The particular solution of the recurrence relation is therefore:

$u_n = 4(3)^n - \frac{7}{2}n(3)^n + \frac{15}{4}$

2 a Every number in column 3 is smaller than the corresponding number in column 2 so Y would never play 2 and the column can be removed from the pay-off matrix:

	Y plays 1	**Y plays 3**
X plays 1	3	1
X plays 2	1	2
X plays 3	–2	1
X plays 4	1	4

b

	Y plays 1	**Y plays 3**	
X plays 1	3	1	1
X plays 2	1	2	1
X plays 3	–2	1	–2
X plays 4	1	4	1
	3	4	

Row maximin = 1

Column minimax = 3

Since these are not equal, there is no stable solution and mixed strategies should be used.

2 c Assume that player Y plays 1 with probability q and 3 with probability $(1 - q)$:

If X plays 1, the value of the game to Y is

$-3q - (1 - q) = -1 - 2q$

If X plays 2, the value of the game to Y is

$-q - 2(1 - q) = -2 + q$

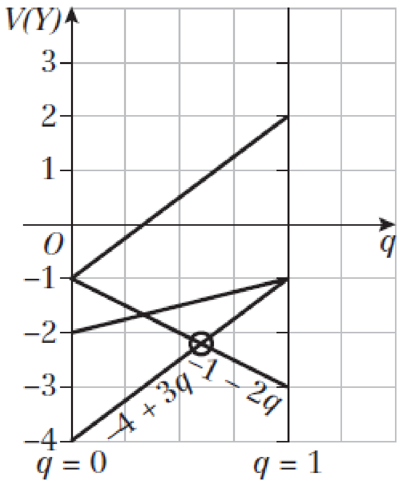
If X plays 3, the value of the game to Y is

$2q - (1 - q) = -1 + 3q$

If X plays 4, the value of the game to Y is

$-q - 4(1 - q) = -4 + 3q$

Graphing these gives:



Which leads to the optimum strategy for Y when $-4 + 3q = -1 - 2q \Rightarrow q = \frac{3}{5}$

The value of the game to Y is

$$-4 + 3\left(\frac{3}{5}\right) = -\frac{11}{5}$$

3 a Reducing rows gives:

	W	*X*	*Y*	*Z*
P	0	6	17	12
Q	0	2	5	3
R	0	11	8	13
S	0	8	2	6

Reducing columns gives:

	W	*X*	*Y*	*Z*
P	0	4	15	9
Q	0	0	3	0
R	0	9	6	10
S	0	6	0	3

Minimum number of lines required to cover all of the zeroes is 3:

	W	*X*	*Y*	*Z*
P	0	4	15	9
Q	0	0	3	0
R	0	9	6	10
S	0	6	0	3

The smallest uncovered element is 4 so subtract 4 from each uncovered element and add 4 to each element covered twice:

	W	*X*	*Y*	*Z*
P	0	0	11	5
Q	4	0	3	0
R	0	5	2	6
S	4	6	0	3

Four lines are now required to cover the zeroes so the solution is optimal:

P does *X*
Q does *Z*
R does *W*
S does *Y*

Total cost = 41 + 46 + 37 + 44 = £168

3 b $x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ does task } j \\ 0 & \text{otherwise} \end{cases}$

where:

$i \in \{P, Q, R, S\}$ and $j \in \{W, X, Y, Z\}$

The problem is to minimise the following:

$$C = 35x_{PW} + 41x_{PX} + 52x_{PY} + 47x_{PZ}$$
$$+43x_{QW} + 45x_{QX} + 48x_{QY} + 46x_{QZ}$$
$$+37x_{RW} + 48x_{RX} + 45x_{RY} + 50x_{RZ}$$
$$+42x_{SW} + 50x_{SX} + 44x_{SY} + 48x_{SZ}$$

Subject to:

$$\sum x_{iW} = 1, \sum x_{iX} = 1, \sum x_{iY} = 1, \sum x_{iZ} = 1$$
$$\sum x_{Pj} = 1, \sum x_{Qj} = 1, \sum x_{Rj} = 1, \sum x_{Sj} = 1$$

4 a

	P	***Q***	***R***	**Supply**
A	10			10
B	2	14	0	16
C			14	14
Demand	12	14	14	40

Note that the zero could be placed in cell *CQ*

Total cost = £460

b

Shadow costs		11	7	11	
		P	***Q***	***R***	**Supply**
0	***A***	11	12	15	10
3	***B***	14	10	14	16
2	***C***	12	16	13	14
	Demand	12	14	14	40

Improvement indices:

AQ = 12 – 7 = 5
AR = 15 – 11 = 4
CP = 12 – 11 – 2 = –1
CQ = 16 – 7 – 2 = 7

4 a (continued)

Entering cell: *CP*
$\theta = 2$
Exiting cell: *BP*

	P	*Q*	*R*	Supply
A	10			10
B	$2 - \theta$	14	θ	16
C	θ		$14 - \theta$	14
Demand	12	14	14	40

Improved solution:

	P	*Q*	*R*	Supply
A	10			10
B		14	2	16
C	2		12	14
Demand	12	14	14	40

Shadow costs		11	8	12	
		P	*Q*	*R*	Supply
0	*A*	**(11)**	12	15	10
2	*B*	14	**(10)**	**(14)**	16
1	*C*	**(12)**	16	**(13)**	14
	Demand	12	14	14	40

Improvement indices:

$AQ = 12 - 8 = 4$
$AR = 15 - 12 = 3$
$BP = 14 - 11 - 2 = 1$
$CQ = 16 - 8 - 1 = 7$

There are no negative improvement indices so the solution is optimal.
Total cost = £458

5 **a** $x = 31 + 7 - 14 - 14 = 10$
$w = y + 7$
Since max w is 24 and min y is 17, increasing y from this min will lead to w exceeding its maximum therefore
$w = 24$
$y = 17$
$z = w + 18 + 31 - 40 = 33$

b **i** $25 + 10 + 14 + 35 = 84$

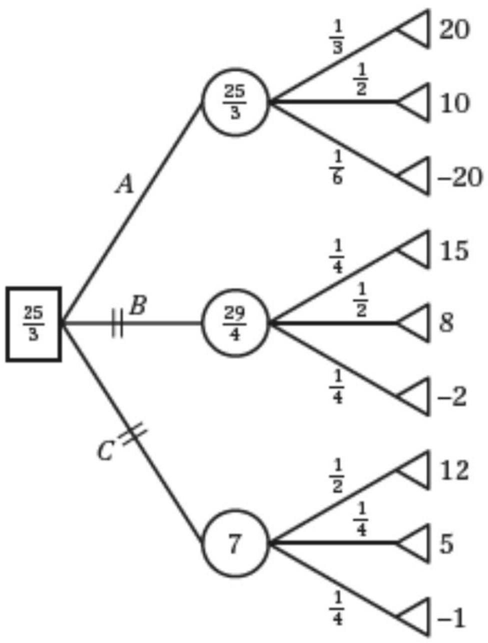
ii $35 - 4 + 14 + 35 = 80$

c The maximum flow is less than or equal to 80

d The flow through *SB* can be increased by 2 units leading to a flow-augmenting path *SBADT*.
The flow through *SC* can be increased by 2 units leading to a flow-augmenting path *SCFT*.

e The initial flow has value 73 and the augmented flow has value 77. Since the cut through *SA*, *SB* and *SC* has maximum value 24 + 33 + 20 = 77, by the maximum flow-minimum cut theorem, the flow is now maximal.

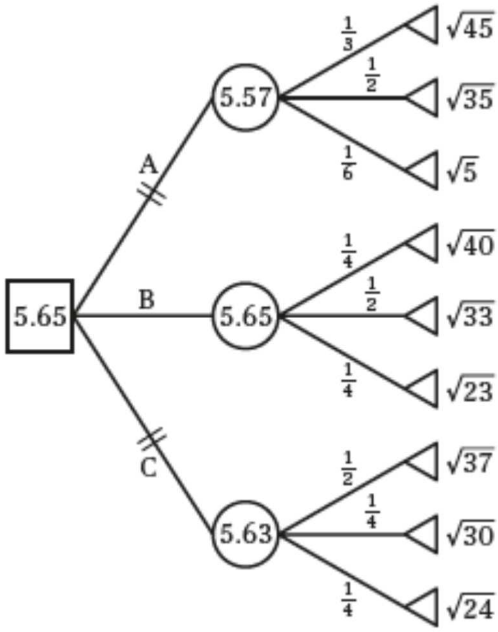
6 **a** Amounts shown are in £1000s



The highest EMV is $\frac{25}{3}$ so Project *A* is the best option.

b A small start-up company may not have the resources to withstand the possible loss associated with project *A*. A utility function could be chosen to reflect a degree of risk aversion, making it more appropriate than the EMVs alone.

6 c



d Project B is the favoured option using the utility function.

7

Stage (demand)	State	Action	Destination	Value (£s)
Oct (15)	2	13	0	300 + 10000 = 10300*
	1	14	0	150 + 10000 = 10150*
	0	15	0	800 + 10000 = 10800*
Sept (17)	2	15	0	300 + 800 + 10000 + 10800 = 21900
		16	1	300 + 800 + 10000 + 10150 = 21250*
		17	2	300 + 800 + 10000 + 10300 = 21400
	1	16	0	150 + 800 + 10000 + 10800 = 21750
		17	1	150 + 800 + 10000 + 10150 = 21100*
	0	17	0	800 + 10000 + 10800 = 21600*
Aug (13)	2	11	0	300 + 10000 + 21600 = 31900
		12	1	300 + 10000 + 21100 = 31400*
		13	2	300 + 10000 + 21250 = 31550
	1	12	0	150 + 10000 + 21600 = 31750
		13	1	150 + 10000 + 21100 = 31250*

7 (continued)

		14	2	150 + 10000 + 21250 = 31400
	0	13	0	10000 + 21600 = 31600
		14	1	10000 + 21100 = 31100*
		15	2	800 + 10000 + 21250 = 32050
July (15)	0	15	0	800 + 10000 + 31100 = 41900*
		16	1	800 + 10000 + 31250 = 42050
		17	2	800 + 10000 + 31400 = 42200

The minimum production cost is £41900

The production schedule should be to make 15 in July, 14 in August, 17 in September and 14 in October.