Exam-style Practice – AS Level

1 Maximum profit is required so subtract each element from 62, the largest value in the original matrix, to give:

	1	**2**	**3**	**4**	**5**
A	11	15	0	12	7
B	8	11	2	9	11
C	13	10	4	7	9
D	10	6	1	4	5
E	6	14	3	7	6

Reducing rows gives:

	1	**2**	**3**	**4**	**5**
A	11	15	0	12	7
B	6	9	0	7	9
C	9	6	0	3	5
D	9	5	0	3	4
E	3	11	0	4	3

Reducing columns gives the reduced cost matrix:

	1	**2**	**3**	**4**	**5**
A	8	10	0	9	4
B	3	4	0	4	6
C	6	1	0	0	2
D	6	0	0	0	1
E	0	6	0	1	0

The zeroes in the reduced cost matrix can be covered by 4 lines (as shown above). The smallest uncovered element is 3. So adding 3 to the element covered by two lines and subtracting 3 from the uncovered elements gives:

	1	**2**	**3**	**4**	**5**
A	5	7	0	6	1
B	0	1	0	1	3
C	6	1	3	0	2
D	6	0	3	0	1
E	0	6	3	1	0

Five lines are now required to cover the zeroes so the solution is optimal.
The solution is: A does task 3, B – 1, C – 4, D – 2, E – 5
Maximum profit = 62 + 54 + 55 + 56 + 56 = £283

2 a A zero-sum game is a game in which each player's gain or loss is balanced by the losses or gains of the other player (or players).

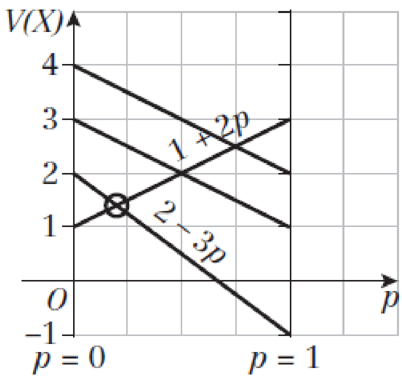
b

	***Y* plays 1**	***Y* plays 2**	***Y* plays 3**	***Y* plays 4**	**Row minimum**
***X* plays 1**	3	–1	2	1	–1
***X* plays 2**	1	2	4	3	1
Column maximum	3	2	4	3	

Column minimax = 2
Row maximin = 1
Since these are not equal, there is no stable solution.

c Assume that player X plays 1 with probability p and 2 with probability $(1 - p)$:
If Y plays 1, the value of the game to X is $3p + 1 - p = 2p + 1$
If Y plays 2, the value of the game to X is $-p + 2(1 - p) = 2 - 3p$
If Y plays 3, the value of the game to X is $2p + 4(1 - p) = 4 - 2p$
If Y plays 4, the value of the game to X is $p + 3(1 - p) = 3 - 2p$

Showing these options on a graph gives:



The optimum strategy for X is when $1 + 2p = 2 - 3p \Rightarrow p = \frac{1}{5}$

The value of the game to $X = 1 + 2\left(\frac{1}{5}\right) = \frac{7}{5}$

3 a K is not a cut since it does not separate the source S from the sink T. Water can still flow along $SBEGT$ for example.

b i Capacity of cut $C = 60 + 14 + 16 = 90$

ii Using conservation of flow at B: $12 + x + 12 = 38 \Rightarrow x = 14$

iii Flow through the network = flow leaving the source = 42 + 38 = 80
Flow through the network = flow entering the sink = 60 + 20 = 80

c The flow-augmenting path $SACDGT$ increases the flow by 3 as arcs SA and DG have maximum spare capacities of 3. The flow from D to C is reduced from 4 to 1.

3 d The initial flow was 80 (part **b iii**) and has been increased by 3 (part **c**) so it is now 83.
The cut through *CF*, *DF*, *DG* and *EG* must be a minimum cut because it only passes through saturated arcs. (The increase flow of 3 down *SACDGT* means arc DG is saturated.) The value of the cut is 58 + 2 + 14 + 9 = 83. As the minimum cut value equals the maximum flow value, by the maximum flow–minimum cut theorem the flow of 83 is maximal.

4 a $L_n = \left(1+\frac{r}{100}\right)L_{n-1} - p$

b The recurrence relation is of the form $u_n = au_{n-1} + \mathrm{g}(n)$ with $\mathrm{g}(n)$ constant so let λ be a particular solution to the recurrence relation.

Then $\lambda = \left(1+\frac{r}{100}\right)\lambda - p = \lambda + \frac{r\lambda}{100} - p$

$\Rightarrow \frac{r\lambda}{100} = p \Rightarrow \lambda = \frac{100p}{r}$

c General solution = complementary function + particular solution, hence

$L_n = k\left(1+\frac{r}{100}\right)^n + \frac{100p}{r}$

$L_0 = X \Rightarrow X = k + \frac{100p}{r} \Rightarrow k = X - \frac{100p}{r}$

Hence the general solution is $L_n = \left(X - \frac{100p}{r}\right)\left(1+\frac{r}{100}\right)^n + \frac{100p}{r}$

d After *n* payments, $L_n = 0$, so using part **c**:

$\left(X - \frac{100p}{r}\right)\left(1+\frac{r}{100}\right)^n + \frac{100p}{r} = 0$

Writing each bracket as a single fraction gives:

$\left(\frac{Xr - 100p}{r}\right)\left(\frac{r+100}{100}\right)^n + \frac{100p}{r} = 0$

Multiplying through by r × 100*n*:

$(Xr - 100p)(r+100)^n + 100^{n+1}p = 0$

Expanding brackets and collecting terms in *p*:

$Xr(r+100)^n = p(100(r+100)^n - 100^{n+1})$

Finding an expression for p:

$p = \frac{Xr(r+100)^n}{100(r+100)^n - 100^{n+1}}$

Dividing top and bottom by $(r+100)^n$ gives:

$p = \frac{Xr}{100 - 100^{n+1}(r+100)^{-n}}$ as required.