Review exercise 2

1

Stage	State	Action	Value
1	G	GT	33*
	H	HT	25*
	I	IT	28*
2	D	DG	32+33=65
		DH	28+25=53*
	E	EG	34+33=67
		EH	30+25=55
		EI	25+28=53*
	F	FG	31+25=56
		FH	27+28=55*
3	A	AD	33+53=86*
		AE	35+53=88
	B	BD	28+53=81
		BE	23+53=76*
		BF	26+55=81
	C	CE	28+53=81*
		CF	31+55=86
4	S	SA	22+86=108
		SB	31+76=107*
		SC	27+81=108

Shortest route from S to T is SBEIT, length 107

2 a Minimax

2 b

Stage	State	Action	Value
1	D	DT	8*
	E	ET	10*
	F	FT	6*
2	A	AD	max(7,8)=8*
	A	AE	max(8,10)=10
	B	BE	max(9,10)=10
		BF	max(3,6)=6*
	C	CE	max(6,10)=10
		CF	max(9,6)=9*
3	S	SA	max(9,8)=9
		SB	max(7,6)=7*
		SC	max(6,9)=9

Optimum route is SBFT

c Minimum required fuel capacity is 7 tonnes

d The aircraft may be required to divert onto a different route for some reason (for example, weather).

3 a The route from start to finish in which the arc of minimum length is as large as possible.

Example must be practical, involve choice of route, have arc ‘costs’.

3 b

Stage	State	Action	Value
1	H	HK	18*
	I	IK	19*
	J	JK	21*
2	F	FH	min(16,18)=16
		FI	min(23,19)=19*
		FJ	min(17,21)=17
	G	GH	min(20,18)=18
		GI	min(15,19)=15
		GJ	min(28,21)=21*
3	B	BG	min(18,21)=18*
	C	CF	min(25,19)=19*
		CG	min(16,21)=16
	D	DF	min(22,19)=19*
		DG	min(19,21)=19*
	E	EF	min(14,19)=14
4	A	AB	min(24,18)=18
		AC	min(25,19)=19*
		AD	min(27,19)=19*
		AE	min(23,14)=14

c Routes: ACFIK, ADFIK, ADGJK

4 a

Stage	State	Action	Value
1	I	IT	4.6*
	J	JT	5.2*
2	G	GI	min(5.3,4.6)=4.6*
		GJ	min(4.4,5.2)=5.2
	H	HI	min(5.1,4.6)=4.6
		HJ	min(4.7,5.2)=4.7*
3	D	DG	min(4.5,4.6)=4.5
		DH	min(4.6,4.7)=4.6*
	E	EG	min(5.3,4.6)=4.6
		EH	min(5.8,4.7)=4.7*
	F	FG	min(5.1,4.6)=4.6*
		FH	min(4.5,4.7)=4.5
4	A	AD	min(4.2,4.6)=4.2
		AE	min(6.2,4.7)=4.7
	B	BD	min(4.9,4.6)=4.6*
		BE	min(4.5,4.7)=4.5
		BF	min(4.7,4.6)=4.6*
	C	CE	min(4.8,4.7)=4.7*
		CF	min(4.2,4.6)=4.2
5	S	SA	min(6.5,4.7)=4.7*
		SB	min(6.7,4.6)=4.6
		SC	min(5.8,4.7)=4.7*

Optimal routes are SAEHJT and SCEHJT,

b Minimum lane width is 4.7 m

c HJ becomes 6.7 m. SAEHJT now has min width 5.2 m (from JT) and SCEHJT has min width 4.8 m (from CE). So the optimal solution is SAEHJT with min width 5.2 m

5 e.g.

Stage	State	Action	Dest	Value
1 (Sept)	2	2	0	200 + 200 = 400*
	1	3	0	200 + 100 = 300*
	0	4	0	200 = 200*
2 (Aug)	2	5	2	200 + 200 + 500 + 400 = 1300
		4	1	200 + 200 + 300 = 700
		3	0	200 + 200 + 200 = 600*
	1	5	1	200 + 100 + 500 + 300 = 1100
		4	0	200 + 100 + 200 = 500*
	0	5	0	200 + 500 + 200 = 900*
3 (July)	2	5	0	200 + 200 + 500 + 900 = 1800*
4 (June)	2	3	2	200 + 200 + 1800 = 2200*
	1	4	2	200 + 100 + 1800 = 2100*
	0	5	2	200 + 500 + 1800 = 2500*
5(May)	0	5	2	200 + 500 + 2200 = 2900
		4	1	200 + 2100 = 2300*
		3	0	200 + 2500 = 2700

Month	May	June	July	August	September
Production schedule	4	4	5	5	4

Cost: £2300

6 a Total cost $= 2 \times 40 + 350 + 200 = £630$

b

Stage	Demand	State	Action	Destination	Value
(2) Oct	(5)	(1)	(4)	(0)	(590 + 200 = 790)*
		(2)	(3)	(0)	280 + 200 = 480*
			(4)	(1)	630 + 240 = 870
		(3)	(2)	0	320 + 200 = 520*
			3	1	320 + 240 = 560
			4	2	670 + 80 = 750
3 Sept	3	0	4	1	550 + 790 = 1340*
		1	3	1	240 + 790 = 1030*
			4	2	590 + 480 = 1070
4 Aug	3	0	3	0	200 + 1340 = 1540*
			4	1	550 + 1030 = 1580

Month	August	September	October	November
Make	3	4	4	2

Cost: £1540

6 c Profit per cycle $= 13 \times 1400$ cost of Kris' time = £2000

$= 18\,200$ cost of production = £1540

$\therefore$ total profit $= 18\,200 - 3540$

$= £14\,660$

7 a Stage – Number of weeks to finish

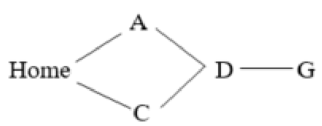
State – Show being attended

Action – Next journey to undertake

b

Stage	State	Action	Value
1	F	F – Home	500 – 80 = 420*
	G	G – Home	700 – 90 = 610*
	H	H – Home	600 – 70 = 530*
2	D	DF	1500 – 200 + 420 = 1720
		DG	1500 – 160 + 610 = 1950*
		DH	1500 – 120 + 530 = 1910
	E	EF	1300 – 170 + 420 = 1550
		EG	1300 – 100 + 610 = 1810*
		EH	1300 – 110 + 530 = 1720
3	A	AD	900 – 180 + 1950 = 2670*
		AE	900 – 150 + 1810 = 2560
	B	BD	800 – 140 + 1950 = 2610*
		BE	800 – 120 + 1810 = 2490
	C	CD	1000 – 200 + 1950 = 2750*
		CE	1000 – 210 + 1810 = 2600
4	Home	Home – A	–70 + 2670 = 2600*
		Home – B	–80 + 2610 = 2530
		Home – C	–150 + 2750 = 2600*

c



Total profit: £2600

8 **a** In a play-safe strategy, each player looks for the worst that could happen for each choice they can make and chooses the option with the least worst option.

b

	B plays 1	B plays 2	B plays 3	B plays 4	row min
A plays 1	−4	−5	−2	4	−5
A plays 2	−1	1	−1	2	−1
A plays 3	0	5	−2	-4	−4
A plays 4	−1	3	−1	1	−1
col max	0	5	−1	4	

The play safe strategies are: A plays 2 or 4 and B plays 3. The payoff for A is –1.

c The row maximin is equal to the col minimax so there is a stable point and therefore a stable solution.

d Value of game to B is –(–1) = 1

9 **a** In a zero-sum game, the winnings of one player are equal to the loss of the other.

b Col max $\begin{pmatrix} -4 & -1 & 3 \\ 2 & 1 & -2 \end{pmatrix}$ row min: $-4 \leftarrow \text{max}$, -2

$$\begin{matrix} 2 & 1 & 3 \\ & \uparrow & \\ & \text{min} & \end{matrix}$$

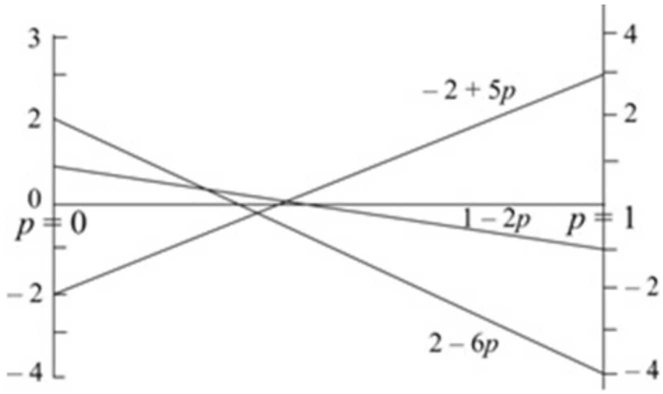
$-2 \neq 1 \quad \therefore$ not stable

9 c Let Emma play R_1 with probability p

If Freddie plays C_1 Emma's winnings are $-4p+2(1-p)=2-6p$

If Freddie plays C_2 Emma's winnings are $-p+1(1-p)=1-2p$

If Freddie plays C_3 Emma's winnings are $3p-2(1-p)=-2+5p$



need intersection of $2-6p$ and $-2+5p$

$$2-6p=-2+5p$$
$$4=11p$$
$$p=\frac{4}{11}$$

So Emma should play R_1 with probability $\frac{4}{11}$

R_2 with probability $\frac{7}{11}$

The value of the game is $\frac{-2}{11}$ to Emma

d Value to Freddie $\frac{2}{11}$, matrix $\begin{pmatrix} 4 & -2 \\ 1 & -1 \\ -3 & 2 \end{pmatrix}$

10 a A saddle is the smallest entry in the row and the largest entry in the column. 5 is not the smallest in the row.

b The col minimax is 3 and the row maximin is 0 so there is no saddle and no stable solution.

10 c i Let B play 1 with probability p and 2 with probability $1-p$

For A, 3 dominates 4, so remove row 4

If A plays 1, B's expected winnings are

$-5p+2(1-p)=2-7p$

If A plays 2, B's expected winnings are

$2p-3(1-p)=-3+5p$

If A plays 3, B's expected winnings are

$-4p$

The optimal solution is the intersection of $-3+5p$ and $-4p$

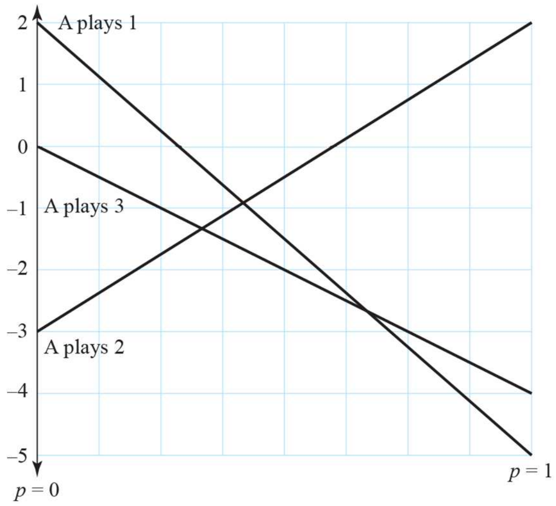
$-3+5p=-4p,\ 9p=3$

$p=\frac{1}{3}$

B should play 1 with probability $\frac{1}{3}$

B should play 2 with probability $\frac{2}{3}$

ii the value to A is $4p=\frac{4}{3}$



11 a

	B plays 4	B plays 5
A plays 4	−16	20
A plays 5	20	−25

11 b The col minimax is 20 and the row maximin is –16 so there is no saddle and therefore no stable solution.

Let A play 4 with probability p and 5 with probability $1 - p$
If B plays 4, A's expected winnings are
$-16p + 20(1 - p) = 20 - 36p$
If B plays 4, A's expected winnings are
$20p - 25(1 - p) = -25 + 45p$
The optimal solution is the intersection of $20 - 36p$ and $-25 + 45p$
$-20 + 36p = -25 + 45p$, $45 = 81p$
$p = \frac{5}{9}$
Amir should play 4 with probability $\frac{5}{9}$

and 5 with probability $\frac{4}{9}$

c The value of the game to Amir is $20 - 36p = 0$.

12 a Row 1 dominates row 2 so A will never choose R2

Column 1 dominates column 3 so B will never choose C3

Thus Row 2 and column 3 may be deleted.

b Let A play row 1 with probability p and hence row 2 with probability $(1 - p)$
If B plays 1 A's expected gain is $3p+6(1-p)=6-3p$
If B plays 2 A's expected gain is $5p+3(1-p)=2p+3$

Optimal when $6-3p=2p+3$
$5p=3$
$p=\frac{3}{5}$

Hence A should play row 1 with probability $\frac{3}{5}$ and row 3 with probability $\frac{2}{5}$ and row 2 never
Similarly, let B play column 1 with probability q
$3q+5(1-q)=6q+3(1-q) \Rightarrow 5-2q=3q+3$
$5q=2$
$q=\frac{2}{5}$

So B should play column 1 with probability $\frac{2}{5}$ and column 2 with probability $\frac{3}{5}$ and column 3 never
Value of game is $4\frac{1}{5}$ to A.

13 a Player A: Row minima are –1, 0, –3 so maximin choice is play 2

Player B: column maxima are 2, 3, 3 so minimax choice is play 1

b Since A's maximin (0) ≠ B's minimax (2) no stable solution

13 c For player A row 2 dominates row 3, (so A will never play 3), since
$1 > 0 \quad 3 > 1 \quad 0 > -3$

d Let A play 1 with probability p and 2 with probability $1 - p$
If B plays 1, A's expected winnings are
$2p + (1 - p) = 1 + p$
If B plays 2, A's expected winnings are
$-p + 3(1 - p) = 3 - 4p$
If B plays 3, A's expected winnings are
$3p$
The optimal solution is the intersection of $3p$ and $3 - 4p$
$3p = 3 - 4p$, $7p = 3$
$p = \frac{3}{7}$
A should play 1 with probability $\frac{3}{7}$
and should play 2 with probability $\frac{4}{7}$

the value of A is $3p = \frac{9}{7}$

14 a In a pure strategy, a player always makes the same choice. In a mixed strategy, each action is played with a determined probability (but at least two options have non-zero probability!)

b For player B, 3 dominates 1. For player A, no row dominates another.

c Col minimax is 1, row maximin is −1 so there is no saddle and no stable solution

14 d Let B play 2 with probability p and 3 with probability $1-p$

If A plays 1, B's expected winnings are

$-p+(1-p)=1-2p$

If A plays 2, B's expected winnings are

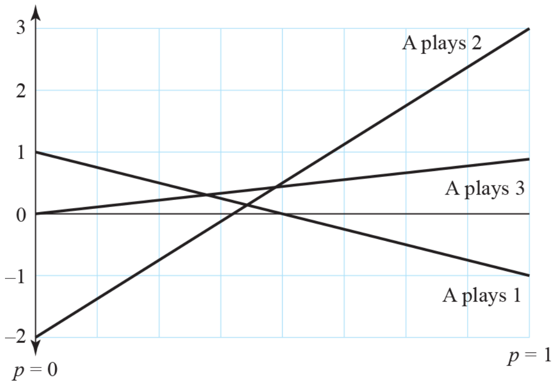
$3p-2(1-p)=-2+5p$

If A plays 3, B's expected winnings are p

The optimal solution is the intersection of $1-2p$ and $-2+5p$

$1-2p=-2+5p,\ 3=7p$

$p=\frac{3}{7}$



B should play 2 with probability $\frac{3}{7}$

and should play 3 with probability $\frac{4}{7}$

the value to B is $1-2p=\frac{1}{7}$

15 a

	A(I)	A(II)
B(I)	3	–4
B(II)	–2	1
B(III)	–5	4

15 b Add 6 to each element to make all terms positive

	A(I)	A(II)
B(I)	9	2
B(II)	4	7
B(III)	1	10

Let q_1 be the probability that B plays row 1

Let q_2 be the probability that B plays row 2

Let q_3 be the probability that B plays row 3

Let value of the game be v and let $V = v + 6$

where $q_1, q_2, q_3 \geqslant 0$

e.g. maximise $P = V$

Subject to $V - 9q_1 - 4q_2 - q_3 + r = 0$

$V - 2q_1 - 7q_2 - 10q_3 + s = 0$

$q_1 + q_2 + q_3 + t = 1$

16 a Row minimums (–2, –1, –4, –2) row maximin = –1
Column maximums (1, 3, 3, 3) column minimax = 1
Since $1 \neq -1$ not stable

b Row 2 dominates Row 3
column 1 dominates column 4

c Let A play row R, with probability P_1, R_2 with probability P_2 and "R_3" with probability P_3

$$\begin{pmatrix} -2 & 1 & 3 \\ -1 & 3 & 2 \\ 1 & -2 & -1 \end{pmatrix} \underset{+3}{\overset{\text{e.g.}}{\rightarrow}} \begin{pmatrix} 1 & 4 & 6 \\ 2 & 6 & 5 \\ 4 & 1 & 2 \end{pmatrix}$$

e.g. maximise $P = V$

subject to $V - P_1 - 2P_2 - 4P_3 \leqslant 0$

$V - 4P_1 - 6P_2 - P_3 \leqslant 0$

$V - 6P_1 - 5P_2 - 2P_3 \leqslant 0$

$P_1 + P_2 + P_3 \leqslant 1$

$V_1 \; P_1 \; P_2 \; P_3 \geqslant 0$

17 a A zero-sum game is one in which the sum of the gains for all players is zero.

b

	I	II	III	
I	5	2	3	min 2
II	3	5	4	min 3 ← max
	max 5	5	4	
			↑	
			min	

Since $3 \neq 4$ not stable

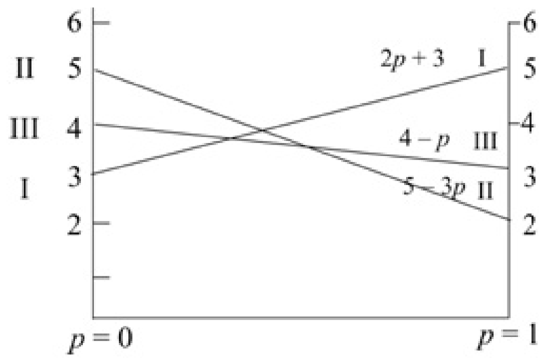
c Let A play I with probability p

Let A play II with probability $(1 - p)$

If B play I A's gains are $5p+3(1-p)=2p+3$

If B plays II A's gains are $2p+5(1-p)=5-3p$

If B plays III A's gains are $3p+4(1-p)=4-p$



Intersection of $2p+3$ and $4-p \Rightarrow p=\frac{1}{3}$

$\therefore$ A should play I $\frac{1}{3}$ of time and II $\frac{2}{3}$ of time; value (to A) $=3\frac{2}{3}$

d Let B play I with probability q_1, II with probability q_2 and III with probability q_3

e.g. $\begin{bmatrix} -5 & -3 \\ -2 & -5 \\ -3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 3 & 2 \end{bmatrix}$

maximise $P = V$

$V - q_1 - 4q_2 - 3q_3 \leqslant 0$

Subject to $V - 3q_1 - q_2 - 2q_3 \leqslant 0 \quad q_1 + q_2 + q_3 \leqslant 1$

$V, q_1, q_2, q_3 \geqslant 0 \quad \text{or} = 1$

18 a $u_n = 4u_{n-1} + 1,\ n \geqslant 1$

associated homogeneous recurrence $u_n = 4u_{n-1}$

has complementary function $u_n = c \times 4^n$

particular solution has the form $u_n = \lambda$

$\lambda = 4\lambda + 1,\ \lambda = -\frac{1}{3}$

$u_n = c \times 4^n - \frac{1}{3}$

b $u_0 = 7 = c - \frac{1}{3}$ so $u_n = \frac{22}{3} \times 4^n - \frac{1}{3}$

19 a For p_n, we need to add n to p_{n-1} for the top left side, then $n-1$ to complete the top right side and finally $n-1$ to finish the pentagon

so $p_5 = 22 + 5 + 4 + 4 = 35,\ p_6 = 51$

b $p_n = p_{n-1} + n + (n-1) + (n-1)$

$= p_{n-1} + 3n - 2$

c $p_n = p_{n-1} + 3n - 2$

associated homogeneous recurrence $u_n = u_{n-1}$

has complementary function $u_n = c \times 1^n = c$

particular solution has the form $u_n = \lambda n^2 + \mu n$

$\lambda n^2 + \mu n = \lambda(n-1)^2 + \mu(n-1) + 3n - 2$

$0 = -2\lambda n + \lambda + \mu + 3n - 2$

coefficient of n: $0 = -2\lambda + 3,\ \lambda = \frac{3}{2}$

constant term: $0 = \lambda + \mu - 2,\ \mu = -\frac{1}{2}$

$p_n = c + \frac{3}{2}n^2 - \frac{1}{2}n$

$p_1 = c + \frac{3}{2} - \frac{1}{2} = 1,\ c = 0$

$p_n = \frac{3}{2}n^2 - \frac{1}{2}n$

d $p_{100} = \frac{1}{2}(30000 - 100) = 14950$

20 $u_n = 2u_{n-1} + 3n + 1$

associated homogeneous recurrence $u_n = 2u_{n-1}$

has complementary function $u_n = c \times 2^n$

particular solution has the form $u_n = \lambda n + \mu$

$\lambda n + \mu = 2\lambda(n-1) + 2\mu + 3n + 1$

coefficient of n: $\lambda = 2\lambda + 3,\ \lambda = -3$

constant term: $\mu = -2\lambda + 2\mu + 1,\ \mu = -7$

$u_n = c \times 2^n - 3n - 7$

$u_0 = 11 = c - 7$ so $u_n = 18 \times 2^n - 3n - 7 = 9 \times 2^{n+1} - 3n - 7$

21 a $u_{n+1} - 3u_n = 10,\ u_1 = 7$

$u_{n+1} = 3u_n + 10$

$u_2 = 21 + 10 = 31$

$u_3 = 93 + 10 = 103$

b i $u_{n+1} = 3u_n + 10$

associated homogeneous recurrence $u_n = 3u_{n-1}$

has complementary function $u_n = c \times 3^n$

particular solution has the form $u_n = \lambda$

$\lambda = 3\lambda + 10,\ \lambda = -5$

$u_n = c \times 3^n - 5$

$u_1 = 3c - 5 = 7,\ c = 4$

$u_n = 4 \times 3^n - 5$

ii $u_n = 4 \times 3^n - 5 \leqslant 1\,000\,000$

$n \log 3 \leqslant \log \dfrac{1\,000\,005}{4}$

$n \leqslant \log \dfrac{1\,000\,005}{4} / \log 3 = 11.31$ (2 d.p.)

$n = 12$ is the smallest integer

22 a $u_n = (0.8)^4 u_{n-1} + 100$ $\quad\quad u_1 = 100$ (or $u_0 = 0$)

b $u_n = 0.4096u_{n-1} + 100$

complementary function $u_n = c \times 0.4096^n$

particular solution has the form $u_n = \lambda$

$\lambda = 0.4096\lambda + 100,\ \lambda = \dfrac{100}{0.5904} = 169.38$ (2 d.p.)

$u_n = c \times 0.4096^n + 169.38 = 169.38(d \times 0.4096^n + 1)$

$u_1 = 100 = 169.38(1 + 0.4096d)$

$0.5094 = 1 + 0.4096d,\ d = -1$

$u_n = 169.38(1 - 0.4096^n) = 169.38(1 - 0.8^{4n})$

22 c $u_n = 169.38(1 - 0.4096^n) = 169.38(1 - 0.8^{4n})$

$u_n \leqslant 160$

$9.38 \geqslant 169.38 \times 0.8^{4n}$

$0.8^{4n} \leqslant 0.05536$

$4n \log 0.8 \leqslant \log 0.05536$

$n \leqslant \dfrac{\log 0.05536}{4 \log 0.8} = 3.24$

the drug exceeds 160 mg on the 4th dose so 3 doses are the maximum

23 a $u_{n+2} = 4u_{n+1} + 5u_n$

let $u_n = Ar^n$

then $r^2 = 4r + 5$

$r^2 - 4r - 5 = (r - 5)(r + 1) = 0$

general solution is $u_n = A \times 5^n + B \times (-1)^n$

b $u_0 = 8, u_1 = -20$

$u_0 = 8 = A + B$

$u_1 = -20 = 5A - B$

$A = -2,\ \mathrm{B} = 10$

$u_n = -2 \times 5^n + 10(-1)^n$

24 a $3u_{n+2} + 10u_{n+1} - 8u_n = 20$

$u_n = k$ is a particular solution

$3k + 10k - 8k = 20,\ k = 4$

b Associated homogenous recurrence relation

$3u_{n+2} + 10u_{n+1} - 8u_n = 0$

let $u_n = Ar^n$

then $3r^2 + 10r - 8 = (3r - 2)(r + 4) = 0$

general solution is $u_n = A\left(\dfrac{2}{3}\right)^n + B(-4)^n + 4$

$u_0 = A + B + 4 = 4A + 4B + 16 = 0$

$u_1 = \dfrac{2}{3}A - 4B + 4 = 1$

$\dfrac{14}{3}A + 20 = 1,\ A = -\dfrac{57}{14},\ B = \dfrac{1}{14}$

$u_n = -\dfrac{57}{14}\left(\dfrac{2}{3}\right)^n + \dfrac{1}{14}(-4)^n + 4$

25 a There are two ways to make a row of length $n + 2$ – add a horizontal domino (length 2) to a row of length n or add a vertical domino (length 1) to a row of length $n + 1$.
There is 1 way to get a row of length 1 (1 vertical domino) and 2 ways to get a row of length 2 (1 horizontal or 2 vertical).

25 b $x_{n+2} = x_{n+1} + x_n,\ x_1 = 1,\ x_2 = 2$

$x_3 = x_2 + x_1 = 3,\ x_4 = 5$

$x_5 = 8,\ x_6 = 13,\ x_7 = 21,\ x_8 = 34$

c i $x_{n+2} = x_{n+1} + x_n$

let $x_n = Ar^n$

then $r^2 = r + 1$

$r^2 - r - 1 = 0,\ \left(r - \frac{1}{2}\right)^2 = \frac{5}{4}$

$r = \frac{1 \pm \sqrt{5}}{2}$

general solution is $x_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$x_1 = 1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$x_2 = 2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2$

$= A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right)$

$A = \frac{5+\sqrt{5}}{10},\ B = \frac{5-\sqrt{5}}{10}$

$x_n = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$

ii 2 feet is 24 inches

$x_{24} = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{24} + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^{24}$

$= 75025$

26 a The probability that all 3 dice are the same is

$\left(1\times\frac{1}{6}\times\frac{1}{6}\right)=\frac{1}{36}$ (the first dice can be anything, the second and third dice have to be equal to it)

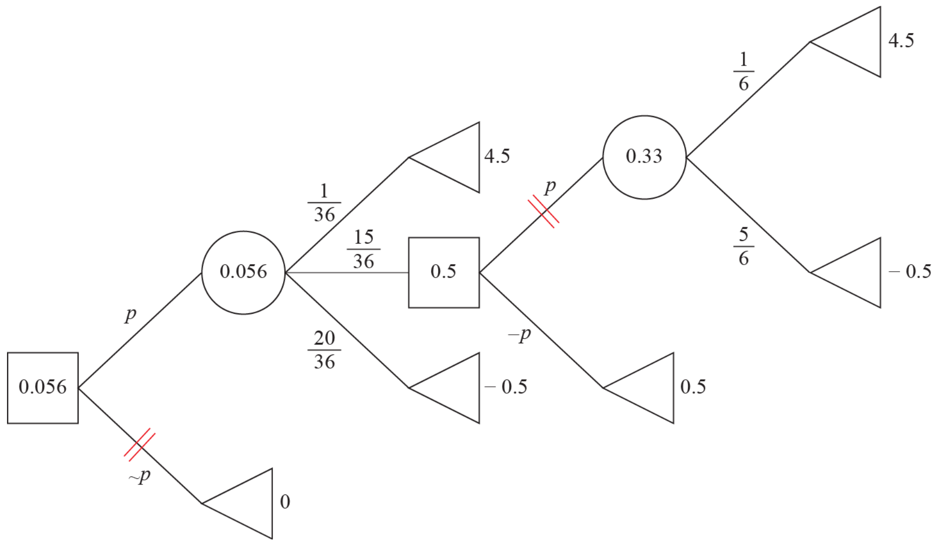
The probability that 2 of 3 dice are the same is

$3\left(1\times\frac{1}{6}\times\frac{5}{6}\right)=\frac{15}{36}$ (the first dice can be anything, the second has to be equal and third dice has to be different, and there are 3 combinations of dice order with 2 equal – AAB, ABA and BAA)

EMV calculations:

second choice EMV $=\frac{1}{6}\times 4.5-\frac{5}{6}\times 0.5=\frac{1}{3}$

first choice EMV $=\frac{1}{36}\times 4.5+\frac{15}{36}\times 0.5-\frac{20}{36}\times 0.5=\frac{1}{18}$



b EMV = 5.6p − play the game but don't continue if only two dice are equal

27 a EMV calculations:

top EMV $=0.2 \times 75 + 0.5 \times 45 - 0.3 \times 30 = 28.5$

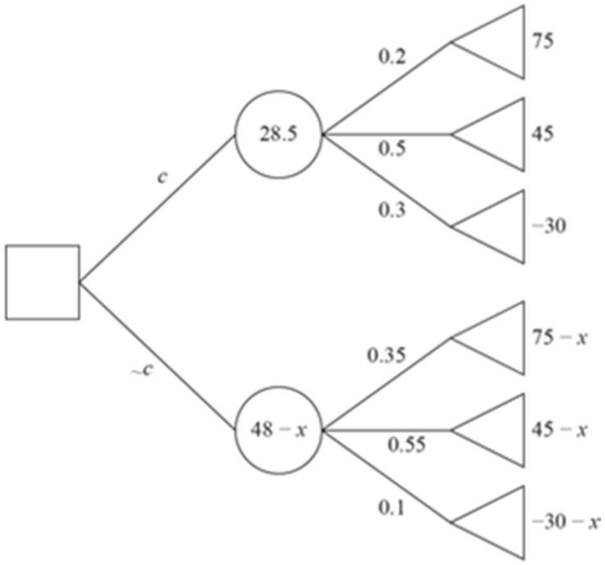
bottom EMV $=0.35(75 - x) + 0.55(45 - x)$

$+0.1(-30 - x) = 48 - x$

EMV is $\max(48 - x, 28.5)$

hire if $x \leqslant 48 - 28.5 = 19.5$

(in £1000s)



b, c

EMV calculations:

top EMV $=0.2 \times 75 + 0.5 \times 45 - 0.3 \times 30 = 28.5$

bottom EMV $=0.35(75 - x) + 0.55(45 - x)$

$+0.1(-30 - x) = 48 - x$

EMV is $\max(48 - x, 28.5)$

hire if $x \leqslant 48 - 28.5 = 19.5$

(in £1000s)

28 a Probability of red 5,6,7 is $\frac{6}{52}$,

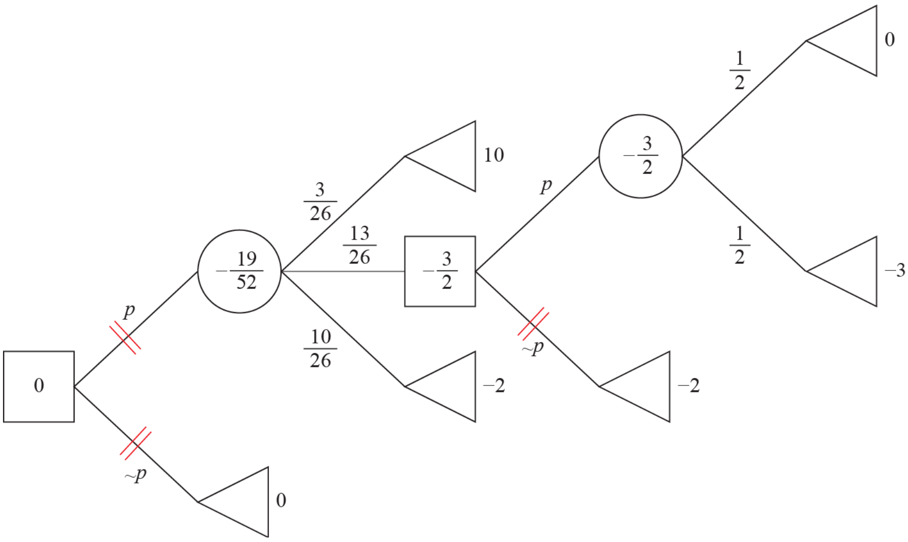
probability of black is $\frac{26}{52}$

EMV calculations:

second choice EMV $= -\frac{1}{2} \times 3 = -\frac{3}{2}$

first choice EMV $= \frac{3}{26} \times 10 + \frac{13}{26} \times -\frac{3}{2} + \frac{10}{26} \times -2$

$= -\frac{19}{52}$



The best strategy is to not play but if the player does play and selects a black card, then they should continue

b The best strategy is to not play but if the player does play and selects a black card, then they should continue. The EMV of playing the game is $-\frac{19}{52}$.

29 For simplicity, the tree has been collapsed slightly – each branch should have a further branch (only one has been shown)

EMV calculations:

furthest left EMV $=0.7\times5+0.2\times4+0.1\times3=4.6$

first date EMV $=0.5\times4.6+0.3\times4.14$

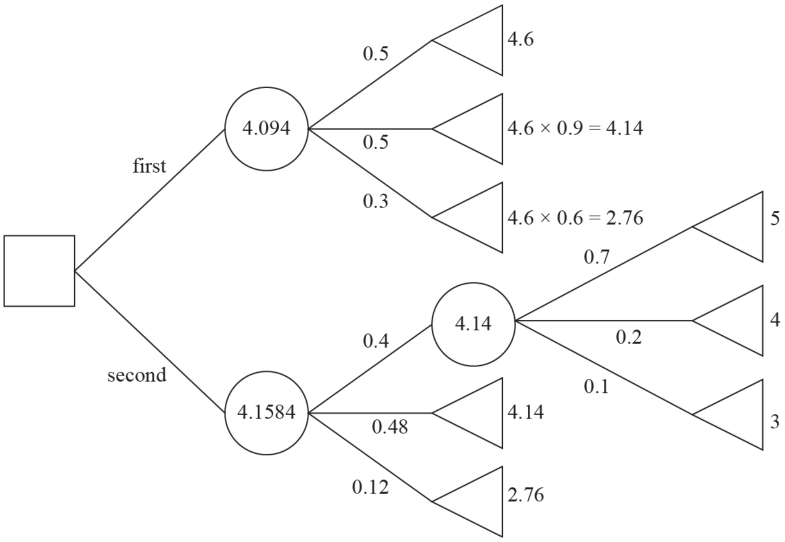
$+0.2\times2.76=4.094$

second date EMV $=0.4\times4.6+0.48\times4.14$

$+0.12\times2.76=4.1584$

choose the second saturday

(in £1000s)

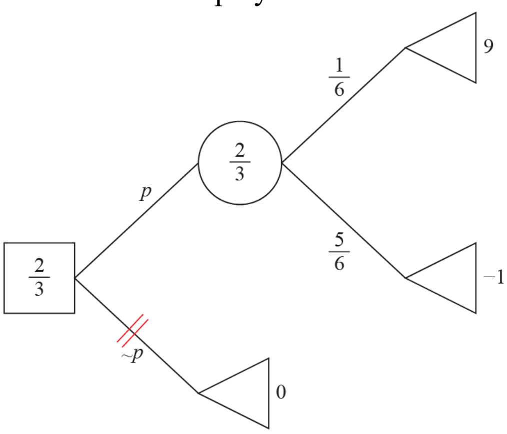


30 a There are 6 possibilities for scoring 10 or more: (4,6),(5,5),(5,6),(6,4),(6,5),(6,6)

so the probability of winning is $\frac{1}{6}$

$\text{EMV}=\frac{1}{6}\times9-\frac{5}{6}\times1=\frac{2}{3}$

so he should play



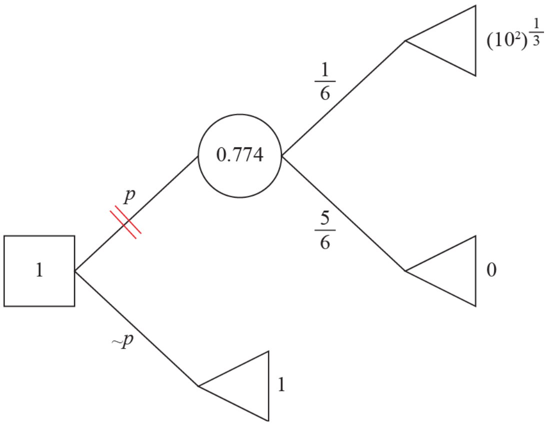
30 b play and win: $E(U) = \sqrt[3]{10^2} = 4.641$

play and lose: $E(U) = \sqrt[3]{0^2} = 0$

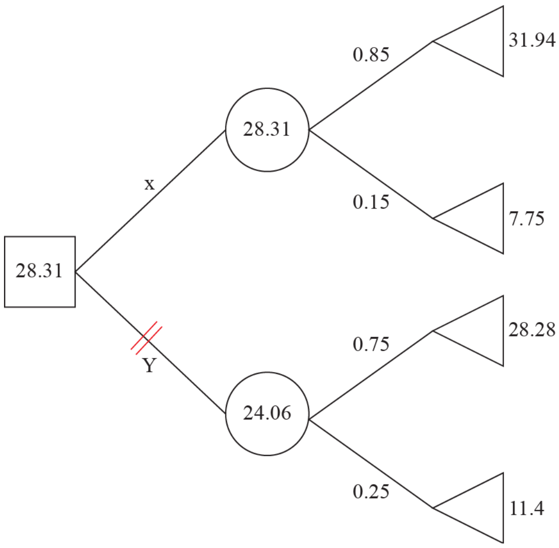
play: $E(U) = \frac{1}{6}\sqrt[3]{10^2} = 0.774$

don't play: $E(U) = \sqrt[3]{1^2} = 1$

so he shouldn't play



31 a



b X: $E(U) = 0.85\sqrt{1020} + 0.15\sqrt{60} = 28.31$

Y: $E(U) = 0.75\sqrt{800} + 0.25\sqrt{130} = 24.06$

so X is the better option, the expected profit is $(0.85 \times 720 - 0.15 \times 240) \times £1000 = £576\,000$

Challenge

1 a There are 4 options at each vertex so 4^n options for a walk of length n

b A close walk of length n from A is any walk of length $n-1$ that doesn't end at A, plus a final walk from the penultimate vertex to A.
So,
$u_n = 4^{n-1} - u_{n-1}$

c $u_n + u_{n-1} = 4^{n-1}$
complementary function $u_n = c(-1)^n$
particular solution has the form $u_n = \lambda 4^n$
$\lambda 4^n + \lambda 4^{n-1} = 4^{n-1},\ 5\lambda = 1$
$u_n = c(-1)^n + \frac{1}{5}4^n$
$u_2 = 4$ (ABA, ACA, ADA and AEA)
$u_2 = 4 = c + \frac{16}{5},\ c = \frac{4}{5}$
$u_n = \frac{4(-1)^n + 4^n}{5}$

d Similarly, the number of walks of length $n-1$ on K_p is $(p-1)^{n-1}$ and a closed walk of length n from the first vertex A is $u_n = (p-1)^{n-1} - u_{n-1}$
complementary function $u_n = c(-1)^n$
particular solution has the form $u_n = \lambda(p-1)^n$
$\lambda(p-1)^n + \lambda(p-1)^{n-1} = (p-1)^{n-1},\ p\lambda = 1$
$u_n = c(-1)^n + \frac{1}{p}(p-1)^n$
$u_2 = p - 1 = c + \frac{1}{p}(p-1)^2,\ c = \frac{1}{p}(p-1)$
$u_n = \frac{(p-1)(-1)^n + (p-1)^n}{p}$
There are p starting vertices so the total number of closed walks of length n is
$pu_n = (p-1)(-1)^n + (p-1)^n$

Challenge

2 a Let A play 1 with probability p and 2 with probability $\frac{1}{3}-p$ and 3 with probability $\frac{2}{3}$

If B plays 1, A's expected winnings are

$2p+3\left(\frac{1}{3}-p\right)+\frac{2}{3}=\frac{5}{3}-p$

If B plays 2, A's expected winnings are

$p-2\left(\frac{1}{3}-p\right)+\frac{4}{3}=\frac{2}{3}+3p$

If B plays 3, A's expected winnings are

$-p+5\left(\frac{1}{3}-p\right)+\frac{4}{3}=3-6p$

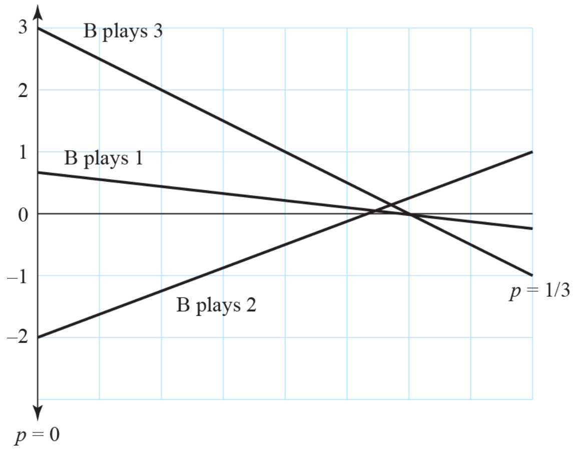
The optimal solution (see the zoomed in diagram) is the intersection

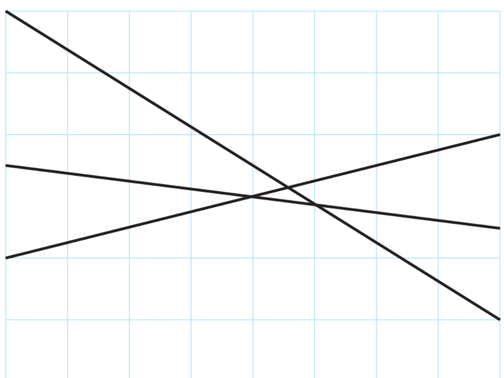
of $\frac{5}{3}-p$ and $\frac{2}{3}+3p$

$\frac{5}{3}-p=\frac{2}{3}+3p,\ 1=4p$

$p=\frac{1}{4}$

A plays 1 with probability $\frac{1}{4}$ and 2 with probability $\frac{1}{12}$ and 3 with probability $\frac{2}{3}$





Challenge

2 b We need to turn this problem into a linear programming problem that we can solve using the simplex algorithm.

Let B play 1 with probability p_1 and 2 with probability p_2 and 3 with probability p_3

Convert the table to B's point of view by transposing the matrix and changing signs and add 6 to each entry so all are strictly positive.

	B plays 1	B plays 2	B plays 3
A plays 1	2	1	−1
A plays 2	3	−2	5
A plays 3	1	2	2

	A plays 1	A plays 2	A plays 3
B plays 1	−2	−3	−1
B plays 2	−1	2	−2
B plays 3	1	−5	−2

	A plays	A plays 2	A plays 3
B plays 1	4	3	5
B plays 2	5	8	4
B plays 3	7	1	4

In this transformed game, if A plays 1, B's expected winnings are

$4p_1 + 5p_2 + 7p_3$

If A plays 2, B's expected winnings are

$3p_1 + 8p_2 + p_3$

If A plays 3, B's expected winnings are

$5p_1 + 4p_2 + 4p_3$

Challenge
2 b (continued)

The additional constraint that B's expected losses are minimised when A plays 1 gives

$4p_1 + 5p_2 + 7p_3 \geqslant 3p_1 + 8p_2 + p_3$

$4p_1 + 5p_2 + 7p_3 \geqslant 5p_1 + 4p_2 + 4p_3$

The linear programming problem is max $P = -V$ subject to

$V - 4p_1 - 5p_2 - 7p_3 + r = 0$

$V - 3p_1 - 8p_2 - p_3 + s = 0$

$V - 5p_1 - 4p_2 - 4p_3 + t = 0$

$-p_1 + 3p_2 - 6p_3 + u = 0$

$p_1 - p_2 - 3p_3 + v = 0$

$p_1 + p_2 + p_3 + w = 1$

$V, p_1, p_2, p_3, r, s, t, u, v, w \geqslant 0$

The simplex tableaus for this are given below

bv	V	p1	p2	p3	r	s	t	u	v	w	value	row op
r	1	−4	−5	−7	1	0	0	0	0	0	0	
s	1	−3	−8	−1	0	1	0	0	0	0	0	
t	1	−5	−4	−4	0	0	1	0	0	0	0	
u	0	−1	3	−6	0	0	0	1	0	0	0	
v	0	1	−1	−3	0	0	0	0	1	0	0	
w	0	1	1	1	0	0	0	0	0	1	1	
P	−1	0	0	0	0	0	0	0	0	0	0	

bv	V	p1	p2	p3	r	s	t	u	v	w	value	row op
V	1	−4	−5	−7	1	0	0	0	0	0	0	R1
s	0	1	−3	6	1	1	0	0	0	0	0	R2-R1
t	0	−1	1	3	1	0	1	0	0	0	0	R3-R1
u	0	−1	3	−6	0	0	0	1	0	0	0	R4
v	0	1	−1	−3	0	0	0	0	1	0	0	R5
w	0	1	1	1	0	0	0	0	0	1	1	R5
P	0	−4	−5	−7	1	0	0	0	0	0	0	R6+R1

Challenge
2 b (continued)

bv	V	p1	p2	p3	r	s	t	u	v	w	value	row op
V	1	$-\frac{19}{3}$	$-\frac{8}{3}$	0	1	0	0	0	$-\frac{7}{3}$	0	0	R1-7/3R5
s	0	3	−5	0	1	1	0	0	2	0	0	R2+2R5
t	0	0	0	0	1	0	1	0	1	0	0	R3+R5
u	0	−3	5	0	0	0	0	1	−2	0	0	R4-2R5
p3	0	$-\frac{1}{3}$	$\frac{1}{3}$	1	0	0	0	0	$-\frac{1}{3}$	0	0	-1/3R5
w	0	$\frac{4}{3}$	$\frac{2}{3}$	0	0	0	0	0	$\frac{1}{3}$	1	1	R6+1/3R5
P	0	$-\frac{19}{3}$	$-\frac{8}{3}$	0	1	0	0	0	$-\frac{7}{3}$	0	0	R1-7/3R5

bv	V	p1	p2	p3	r	s	t	u	v	w	value	row op
V	1	0	$-\frac{119}{9}$	0	$\frac{28}{9}$	$\frac{19}{9}$	0	0	$\frac{17}{9}$	0	0	R1+19/9R2
p1	0	1	$-\frac{5}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	0	1/3R2
t	0	0	0	0	1	0	1	0	1	0	0	R3
u	0	0	0	0	1	1	0	1	0	0	0	R4+R2
p3	0	0	$-\frac{2}{9}$	1	$\frac{1}{9}$	$\frac{1}{9}$	0	0	$-\frac{1}{9}$	0	0	R5+1/9R2
w	0	0	$\frac{26}{9}$	0	$-\frac{4}{9}$	$-\frac{4}{9}$	0	0	$-\frac{2}{9}$	1	1	R6-4/9R2
P	0	0	$-\frac{119}{9}$	0	$\frac{28}{9}$	$\frac{19}{9}$	0	0	$\frac{17}{9}$	0	0	R7+19/9R2

Challenge
2 b (continued)

bv	V	p1	p2	p3	r	s	t	u	v	w	value	row op
V	1	0	0	0	$\frac{14}{13}$	$\frac{1}{13}$	0	0	$\frac{34}{13}$	0	$\frac{119}{26}$	R1+119/26R6
p1	0	1	0	0	$\frac{1}{13}$	$\frac{1}{13}$	0	0	$\frac{7}{13}$	0	$\frac{15}{26}$	R2+15/26R6
t	0	0	0	0	1	0	1	0	1	0	0	R3
u	0	0	0	0	1	1	0	1	0	0	0	R4
p3	0	0	0	1	$\frac{1}{13}$	$\frac{1}{13}$	0	0	$-\frac{5}{39}$	0	$\frac{1}{13}$	R5+2/16R6
p2	0	0	1	0	$-\frac{2}{13}$	$-\frac{2}{13}$	0	0	$-\frac{1}{13}$	$\frac{9}{26}$	$\frac{9}{26}$	9/26R6
P	0	0	0	0	$\frac{14}{13}$	$\frac{1}{13}$	0	0	$\frac{34}{13}$	0	$\frac{119}{26}$	R1+51/8R3

Thus, we see that B should play 1 with probability $\frac{15}{26}$, 2 with probability $\frac{9}{26}$ and 3 with probability $\frac{1}{13}$.

Challenge

3 (i): $\text{E}(U) = 0.6\log 11200 + 0.4\log 10500$

(ii): $\text{E}(U) = 0.7\log 12000 + 0.3\log(10000(1-p))$

the second option is more favourable if

$0.7\log 12000 + 0.3\log(10000(1-p)) \geqslant$

$0.6\log 11200 + 0.4\log 10500$

$\log(10000(1-p)) \geqslant$

$\frac{10}{3}\log\left(11200^{0.6} \times 10500^{0.4} \times 12000^{-0.7}\right)$

$1 - p \geqslant \frac{1}{10000}\left(11200^{0.6} \times 10500^{0.4} \times 12000^{-0.7}\right)^{\frac{10}{3}}$

$p \leqslant 0.1251$ (4 d.p.)

12.5%