Recurrence relations Mixed exercise

1 $u_n = 2u_{n-1} - 1$

CF is $A(2^n)$

PS is λ

$\lambda - 2\lambda + 1 = 0 \Rightarrow \lambda = 1$

GS is $A(2^n) + 1$

$A + 1 = 4 \Rightarrow A = 3$

$u_n = 3(2^n) + 1$

2 a $u_n = 2000 - \sum_{r=1}^{n} r$

$= 2000 - \frac{1}{2}n(n+1)$

b $n(n+1) > 4000$

$\Rightarrow n > 62$ since $62 \times 63 = 3906$

and $63 \times 64 = 4032$

$u_{63} = -16$

3 a CF is $A(3^n)$

PS is λ

$\lambda = 3\lambda + 5 \Rightarrow \lambda = -\frac{5}{2}$

GS is $A(3^n) - \frac{5}{2}$

$A - \frac{5}{2} = 0 \Rightarrow A = \frac{5}{2}$

$u_n = \frac{5}{2}\left((3^n) - 1\right)$

b 147620

c $\frac{5}{2}(3^n - 1) > 10\,000\,000$

$\Rightarrow 3^n > 4\,000\,001$

$n > \frac{\ln 4\,000\,001}{\ln 3} = 13.83...$

So $u_{14} = 11\,957\,420$

4 a T_0 = number of trees planted in first year = 12 000
Removing 20% of the trees compared to year $n - 1$ leaves 80% of this number of trees, i.e. $0.8T_{n-1}$, then to represent the 1000 trees planted, add 1000 to this to get $T_n = 0.8T_{n-1} + 1000$

4 b CF is $A(0.8)^n$

PS is λ

$\lambda = 0.8\lambda + 1000 \Rightarrow \lambda = 5000$

$A + 5000 = 12\,000 \Rightarrow A = 7000$

$T_n = 7000(0.8)^n + 5000$

c 5000

5 a $b_n = 1.0025b_{n-1} - 1200$, $b_0 = 175\,000$

b CF is $A(1.0025^n)$

PS is λ

$\lambda = 1.0025\lambda - 1200 \Rightarrow \lambda = 480\,000$

GS is $A(1.0025^n) + 480\,000$

$A + 480\,000 = 175\,000 \Rightarrow A = -305\,000$

$b_n = -305\,000(1.0025^n) + 480\,000$

Balance = 0 when

$$n = \frac{\ln \frac{480}{305}}{\ln 3} = 181.616$$

Just over 15 years so 2033

6 a 6

b $P_n = P_1 + \sum_{r=2}^{n}(r-1) = 0 + \sum_{r=2}^{n} r - (n-1)$

$= \frac{1}{2}n(n+1) - 1 - n + 1 = \frac{1}{2}n(n-1)$

c 4950

7 a $t_5 = 25$, $t_6 = 36$, $t_7 = 49$

b $t_n = t_{n-1} + 2n - 1$

c $t_n = \sum_{r=1}^{n}(2r-1) = n(n+1) - n = n^2$

$t_{100} = 10\,000$

8 a $\begin{pmatrix} 1 & 4+3p \\ 0 & 3q \end{pmatrix}$

b $a_n = 3a_{n-1} + 4$, $b_n = 3b_{n-1}$

8 c $a_n = 3a_{n-1} + 4$

CF is $A(3^n)$

PS is λ

$\lambda = 3\lambda + 4 \Rightarrow \lambda = -2$

GS is $A(3^n) - 2$

$3A - 2 = 4 \Rightarrow A = 2$

$a_n = 2(3^n - 1)$

$b_n = 3^n$

So $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & 2(3^n - 1) \\ 0 & 3^n \end{pmatrix}$

9 a $S_5 = 55$, $S_6 = 91$, $S_7 = 140$

b $S_n = S_{n-1} + n^2$

c $S_n = \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

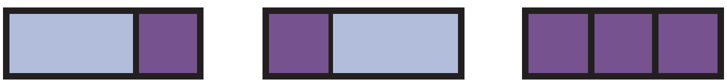
10 a $u_n = 1.2u_{n-1} - k(2^n)$, $u_0 = 100$

b C.F. is $A(1.2^n)$ and P.S. is $-\frac{5k}{2}(2^n)$,

so $u_n = A(1.2n) - \frac{5k}{2}(2^n)$

Using $u_0 = 100$, $A = 100 + \frac{5k}{2}$, and hence

$u_n = \left(100 + \frac{5k}{2}\right)(1.2^n) - \frac{5k}{2}(2^n)$

11 a 

b There are f_{n-1} paths of length n ending in a small flagstone and f_{n-2} paths of length n ending in a long flagstone. This gives a total of $f_n = f_{n-1} + f_{n-2}$ paths of length n. There is one path of length 1 m and there are two paths of length 2 m, so $f_1 = 1$ and $f_2 = 2$.

c Solving the recurrence relation gives

$u_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$

So for $n = 200$, $u_{200} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{201} - \left(\frac{1-\sqrt{5}}{2}\right)^{201}\right)$

12 a $t_2 = 8$, $t_3 = 22$

b If the final digit of the string is not 0, then there are t_{n-1} possibilities for the rest of the string for each of final digits 1 and 2. If the final digit is zero, then the penultimate digit must not be zero, i.e. can be either 1 or 2, and then there are t_{n-2} possibilities for the rest of the string for each of these two cases.
Thus, $t_n = 2t_{n-1} + 2t_{n-2}$

c $t_6 = 448$

d i $r^2 - 2r - 2 = 0$

$r = 1 \pm \sqrt{3}$

$t_n = A\left(1+\sqrt{3}\right)^n + B\left(1-\sqrt{3}\right)^n$

$A\left(1+\sqrt{3}\right) + B\left(1-\sqrt{3}\right) = 3$

$A\left(4+2\sqrt{3}\right) + B\left(4-2\sqrt{3}\right) = 8 \Rightarrow A\left(2+\sqrt{3}\right) + B\left(2-\sqrt{3}\right) = 4$

Subtracting

$A + B = 1$

$A\left(1+\sqrt{3}\right) + (1-A)\left(1-\sqrt{3}\right) = 3$

$\Rightarrow A = \dfrac{2+\sqrt{3}}{2\sqrt{3}},\ B = \dfrac{\sqrt{3}-2}{2\sqrt{3}}$

$t_n = \dfrac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)\left(1+\sqrt{3}\right)^n + \left(\sqrt{3}-2\right)\left(1-\sqrt{3}\right)^n\right)$

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13 a $r^2 - r - 2 = 0$

$r = -1, 2$

$u_n = A(2)^n + B(-1)^n$

b $2A - B = 1$

$4A + B = 2$

$A = \dfrac{1}{2}, B = 0$

PS is $u_n = 2^{n-1}$

14 a $r^2 - 7r + 10 = 0$

CF is $A(5^n) + B(2^n)$

PS is λ

$\lambda = 7\lambda - 10\lambda + 3$

$\lambda = \dfrac{3}{4}$

GS is $A(5^n) + B(2^n) + \dfrac{3}{4}$

14 b $5A + 2B + \frac{3}{4} = 1$

$25A + 4B + \frac{3}{4} = 2$

$A = \frac{1}{20}, B = 0$

$x_n = \frac{1}{20}(5^n) + \frac{3}{4} = \frac{1}{4}\left(5^{n-1} + 3\right)$

15 $r^2 - 2r - 15 = 0, r = 5, -3$

CF is $A(5^n) + B(-3)^n$

PS is $C(2^n)$

$C(2^n) = C(2^n) + 15C(2^{n-2}) + (2^n)$

$15C + 4 = 0 \Rightarrow C = -\frac{4}{15}$

GS is $A(5^n) + B(-3)^n - \frac{4}{15}(2^n)$

$5A - 3B - \frac{8}{15} = 2$

$25A - 9B - \frac{16}{15} = 4$

$A = \frac{19}{60}, B = -\frac{19}{60}$

$u_n = \frac{1}{60}\left(19(5^n) - 19(-3)^n + 2^{n+4}\right)$

16 a $r^2 - \sqrt{2}r + 1 = 0$

$r = \frac{\sqrt{2} \pm \sqrt{2}i}{2} = e^{\pm\frac{\pi}{4}}$

So GS is $A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}$

$u_0 = 1 \Rightarrow A = 1$

$u_1 = 1 \Rightarrow \frac{1}{\sqrt{2}}A + \frac{1}{\sqrt{2}}B = 1$

$\Rightarrow B = \sqrt{2} - 1$

$u_n = \cos\frac{n\pi}{4} + \left(\sqrt{2} - 1\right)\sin\frac{n\pi}{4}$

b cos and sin are periodic of period 2π, so period for u_n is $\frac{2\pi}{\frac{\pi}{4}} = 8$

17 a $S_{n+2} = S_{n+1} + S_n$, $S_1 = 1$, $S_2 = 2$

b $r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{5}}{2}$

$t_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$\frac{1+\sqrt{5}}{2}A + \frac{1-\sqrt{5}}{2}B = 2 \Rightarrow \left(1+\sqrt{5}\right)A + \left(1-\sqrt{5}\right)B = 2$

$\frac{3+\sqrt{5}}{2}A + \frac{3-\sqrt{5}}{2}B = 2 \Rightarrow \left(3+\sqrt{5}\right)A + \left(3-\sqrt{5}\right)B = 4$

Subtracting

$A + B = 1$

$A\left(1+\sqrt{5}\right) + (1-A)\left(1-\sqrt{5}\right) = 2$

$\Rightarrow A = \frac{1+\sqrt{5}}{2\sqrt{5}}, B = -\frac{1-\sqrt{5}}{2\sqrt{5}}$

$s_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$

Challenge

1 a 31

b Consider one of the points already on the circumference of the circle (A), and one of the existing points of intersection of two diagonals (B).
The line through these two points will meet the circle at two points, A and C. Since there are finitely many pairs $\{A, B\}$, there will be finitely many points C on the circumference of the circle such that the line AC goes through an existing intersection point.
However, since there are infinitely many points on the circumference of the circle, it is possible to choose one, D, which doesn't coincide with any of the points C, and thus the chords AD do not go through any of the existing intersection points.

c Looking at the sequence $C_n - C_{n-1}$ gives

1,2,4,8,15

Since 3rd differences are equal this is a cubic

sequence of the form $an^3 + bn^2 + cn + d$

$a + b + c + d = 1$

$8a + 4b + 2c + d = 2$

$27a + 9b + 3c + d = 4$

$64a + 16b + 4c + d = 8$

$a = \frac{1}{6}, b = -1, c = \frac{17}{6}, d = -2$

$\Rightarrow C_n = C_{n-1} + \frac{1}{6}n^3 - n^2 + \frac{17}{6}n - 2$

Challenge

1 d $C_n = 1 + \sum_{r=1}^{n}\left(\frac{1}{6}r^3 - r^2 + \frac{17}{6}r - 2\right)$

$= \frac{1}{24}n^2(n+1)^2 - \frac{1}{6}n(n+1)(2n+1) + \frac{17}{12}n(n+1) - 2n + 1$

$= \frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$

3926176 ways for 100 points

2 a Walks of length 1 start at A and end at any of the other points; the spider cannot return to A.

b 6 (ABCA, ABDA, ACBA, ACDA, ADBA, ADCA)

c If we have a walk of length $n-2$ ending in A we can add BA, CA or DA to give a walk of length n
If we have a walk of length $n-1$ ending in A we can find 2 walks for each by replacing e.g. BA with BCA or BDA etc.

So $C_n = 2C_{n-1} + 3C_{n-2}$

$r^2 - 2r = 3 = 0,\ r=3,-1$

$C_n = A(3^n) + B(-1)^n$

$3A - B = 0$

$9A + B = 3$

$A = \frac{1}{4}, B = \frac{3}{4}$

$C_n = \frac{1}{4}\left((3^n) + 3(-1)^n\right)$