Recurrence relations 7C

1 a n even: $5(-1)^{n-1} + 6(-1)^{n-2} = -5 + 6 = 1 = (-1)^n$
n odd: $5(-1)^{n-1} + 6(-1)^{n-2} = 5 - 6 = -1 = (-1)^n$

b $5 \times 6^{n-1} + 6 \times 6^{n-2} = 6^{n-1}(5 + 1) = 6^n$

c $5(A(-1)^{n-1} + B(6^{n-1})) + 6(A(-1)^{n-2} + B(6^{n-2}))$
$= -5A(-1)^{n-2} + 6A(-1)^{n-2} + 5B(6^{n-1}) + 6B(6^{n-2})$
$= A(-1)^{n-2} + 6B(6^{n-1}) = A(-1)^{n-2} + B(6^n)$

2 a $5(3^n) - 6 \times 5(3^{n-1}) + 9 \times 5(3^{n-2}) = (45 - 90 + 45)\,3^{n-2} = 0$

b $-n3^n - 6(-(n-1)3^{n-1}) + 9(-(n-2)3^{n-2})$
$= -n3^n + (6n - 6)3^{n-1} + (18 - 9n)3^{n-2}$
$= (-9n + 18n - 18 + 18 - 9n)3^{n-2} = 0$

c Follows from parts **a** and **b**.

3 a $\cos\left((n+2)\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) = \cos\left(\pi + n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right)$

$$= -\cos\left(n\frac{\pi}{2}\right) + \cos\left(n\frac{\pi}{2}\right) = 0$$

b $\sin\left((n+2)\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right) = \sin\left(\pi + n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right)$

$$= -\sin\left(n\frac{\pi}{2}\right) + \sin\left(n\frac{\pi}{2}\right) = 0$$

c Follows from parts **a** and **b**.

4 $au_{n-1} + bu_{n-2}$
$= a(cF(n-1) + dG(n-1)) + b(cF(n-2) + dG(n-2))$
$= c(aF(n-1) + bF(n-2)) + d(aG(n-1) + bG(n-2))$
$= cF(n) + dG(n) = u_n$

5 a $r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$
GS is $a_n = A + Bn$

b $r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0$
GS is $u_n = A + B(2^n)$

c $r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0$
GS is $a_n = (A + Bn)3^n$

d $r^2 - 4r + 5 = 0 \Rightarrow r = 2 \pm i$
GS is $a_n = A(2+i)^n + B(2-i)^n$

6 $u_0 = D + E, u_1 = D + 7E, u_2 = D + 49E$

$\Rightarrow D + 49E + a(D + 7E) + b(D + E) = 0$

$a + b + 1 = 0$

$7a + b + 49 = 0$

$a = -8, b = 7$

7 **a** $r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$

GS is $a_n = A(2^n) + B(3^n)$

$A + B = 2$

$3A + 2B = 5$

$A = 1, B = 1$

$a_n = 2^n + 3^n$

b $r^2 - 6r + 9 = 0 \Rightarrow r = 3$

GS is $a_n = (An + B)(3^n)$

$3(A + B) = 2$

$9(2A + B) = 5$

$A = -\frac{1}{9}, B = \frac{7}{9}$

$u_n = \left(\frac{7}{9} - \frac{1}{9}n\right)3^n = (7 - n)3^{n-2}$

c $r^2 - 7r + 10 = 0 \Rightarrow r = 2, 5$

GS is $s_n = A(2^n) + B(5^n)$

$A + B = 4$

$2A + 5B = 17$

$A = 1, B = 3$

$s_n = 2^n + 3(5^n)$

d $r^2 - 2r + 5 = 0 \Rightarrow r = 1 \pm 2i$

GS is $s_n = A(1 + 2i)^n + B(1 - 2i)^n$

$A + B = 1$

$A(1 + 2i) + B(1 - 2i) = 5$

$A = \frac{1 - 2i}{2} B = \frac{1 + 2i}{2}$

$u_n = \frac{1 - 2i}{2}(1 + 2i)^n + \frac{1 + 2i}{2}(1 - 2i)^n$

$= \frac{5}{2}\left((1 + 2i)^{n-1} + (1 - 2i)^{n-1}\right)$

8 a $r^2 - 5r + 4 = 0 \Rightarrow r = 1, 4$

GS is $s_n = A(4^n) + B$

$A + B = 20$

$4A + B = 19$

$A = -\frac{1}{3}, B = \frac{61}{3}$

$u_n = -\frac{1}{3}(4^n) + \frac{61}{3} = \frac{61}{3} - \frac{1}{3}(4^n)$

b $u_{n+1} - u_n = -4^n < 0 \Rightarrow u_n$ is decreasing

$u_n < 0 \Rightarrow 4n > 61 \Rightarrow n \geqslant 3$

9 a $r^2 - \sqrt{2}r + 1 = 0 \Rightarrow r = \frac{1}{\sqrt{2}}(1 \pm i) = 2e^{\pm\frac{\pi}{4}}$

GS is $u_n = 2^n\left(A\cos\frac{n\pi}{4} + B\sin\frac{n\pi}{4}\right)$

$u_0 = 1 \Rightarrow A = 1$

$u_1 = 1 \Rightarrow 2\left(\frac{1}{\sqrt{2}} + \frac{B}{\sqrt{2}}\right) = 1$

$\Rightarrow B = \frac{\sqrt{2}}{2} - 1$

$u_n = 2^n\left(\cos\frac{n\pi}{4} + \left(\frac{\sqrt{2}}{2} - 1\right)\sin\frac{n\pi}{4}\right)$

b cos and sin are periodic of period 2π, so period for u_n is $\frac{2\pi}{\frac{\pi}{4}} = 8$

10 a $L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, L_5 = 11, L_6 = 18, L_7 = 29$

b Auxiliary equation is $r^2 - r - 1 = 0$, so $r = \frac{1 \pm \sqrt{5}}{2}$ and

$L_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

$L_1 = \frac{1}{2}A + \frac{\sqrt{5}}{2}A + \frac{1}{2}B - \frac{\sqrt{5}}{2}B = 1$

$L_2 = \frac{3}{2}A + \frac{\sqrt{5}}{2}A + \frac{3}{2}B - \frac{\sqrt{5}}{2}B = 3$

Solving these equations gives $A = B = 1$, so the closed form is $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

11 a $r^2 - 5r + 6 = 0 \Rightarrow r = 2, 3$

CF is $A(2^n) + B(3^n)$

PS is λ

$\lambda = 5\lambda - 6\lambda + 1 \Rightarrow \lambda = \frac{1}{2}$

$x_n = A(2^n) + B(3^n) + \frac{1}{2}$

b $r^2 - r - 2 = 0 \Rightarrow r = 2, -1$

CF is $A(2^n) + B(-1)^n$

PS is $\lambda n + \mu$

$\lambda n + \mu - \left(\lambda(n-1) + \mu\right) - 2(\lambda(n-2) + \mu) = 2n$

$\lambda = -1, \mu = -\frac{5}{2}$

$u_n = A(2^n) + B(-1)^n - n - \frac{5}{2}$

c $r^2 + 4r + 3 = 0 \Rightarrow r = -1, -3$

CF is $A(-1)^n + B(-3)^n$

PS is $C(-2)^n$

$C(-2)^{n+2} + 4C(-2)^{n+1} + 3C(-2)^n = 5(-2)^n$

$4C - 8C + 3C = 5 \Rightarrow C = -5$

$a_n = A(-3)^n + B(-1)^n - 5(-2)^n$

d $r^2 + 4r + 3 = 0 \Rightarrow r = -1, -3$

CF is $A(-3)^n + B(-1)^n$

PS is $Cn(-3)^n$

$C\left(n+2\right)(-3)^{n+2} + 4C\left(n+1\right)(-3)^{n+1} + 9Cn(-3)^n = 12(-3)^n$

$9C(n+2) - 12C(n+1) + 3Cn = 12$

$\Rightarrow 6C = 12 \Rightarrow C = 2$

$a_n = A(-3)^n + B(-1)^n + 2n\left(-3\right)^n$

11 e $r^2 - 6r + 9 = 0 \Rightarrow r = 3$

CF is $A(3^n) + Bn(3^n)$

PS is $Cn^2(3)^n$

$C(n+2)^2(3^{n+2}) - 6C(n+1)^2(3^{n+1}) + 9Cn^2(3^n) = 3^n$

$9C(n+2)^2 - 18C(n+1)^2 + 9Cn^2 = 1$

$\Rightarrow 36C - 18C = 1 \Rightarrow C = \dfrac{1}{18}$

$a_n = A(3^n) + Bn(3^n) + \dfrac{1}{18}n^2(3^n)$

$= \left(A + Bn + \dfrac{n^2}{18}\right)3^n$

f $r^2 - 7r + 10 = 0 \Rightarrow r = 2, 5$

CF is $A(2^n) + B(5^n)$

PS is $C + Dn$

$C + Dn = 7(C + D(n-1) - 10(C + D(n-2)) + 6 + 8n$

$C = 8, D = 2$

$u_n = A(2^n) + B(5^n) + 8 + 2n$

12 a $r^2 - 2r - 3 = 0 \Rightarrow r = -1, 2$

CF is $A(-1)^n + B(3^n)$

PS is λ

$\lambda = 2\lambda + 3\lambda + 1 \Rightarrow \lambda = -\dfrac{1}{4}$

GS is $A(-1)^n + B(3^n) - \dfrac{1}{4}$

$-A + 3B - \dfrac{1}{4} = 3$

$A + 9B - \dfrac{1}{4} = 7$

$A = -\dfrac{5}{8}, B = \dfrac{7}{8}$

$u_n = \dfrac{1}{8}\left(7(3^n) - 5(-1)^n - 2\right)$

12 b $r^2 - 3r + 2 = 0 \Rightarrow r = 1, 2$

CF is $A + B(2^n)$

PS is $C(-1)^n$

$-C - 3C - 2C = 6 \Rightarrow C = -1$

GS is $A + B(2^n) + (-1)^{n+1}$

$A + B - 1 = 12$

$A + 2B + 1 = 12$

$A = 15, B = -2$

$a_n = 15 - 2^{n+1} + (-1)^{n+1}$

c $r^2 - 3r - 10 = 0 \Rightarrow r = 5, -2$

CF is $A(5^n) + B(-2)^n$

PS is $Cn(5^n)$

$Cn(5^n) = 3C(n-1)(5^{n-1}) + 10C(n-2)(5^{n-2}) + 7 \times 5^n$

$\Rightarrow 25Cn = 15C(n-1) + 10C(n-2) + 175$

$C = 5$

$A + B = 4$

$5A - 2B + 25 = 3$

$A = -2, B = 6$

$u_n = n5^{n+1} - 2 \times 5^n + 6(-2)^n$

d $r^2 - 10r + 25 = 0 \Rightarrow r = 5$

CF is $A(5^n) + Bn(5^n)$

PS is $Cn^2(5^n)$

$Cn^2(5^n) = 10C(n-1)^2(5^{n-1}) - 25C(n-2)^2(5^{n-2}) + 8 \times 5^n$

$\Rightarrow 25Cn^2 = 50C(n-1)^2 - 25C(n-2)^2 + 200$

$C = 4$

GS is $A(5^n) + Bn(5^n) + 4n^2(5^n)$

$A = 6$

$5A + 5B + 20 = 10 \Rightarrow B = -8$

$x_n = (4n^2 - 8n + 6)5^n$

13 a $k+4k+4k=7 \Rightarrow k=\frac{7}{9}$

b $r^2+4r+4=0 \Rightarrow r=-2$

CF is $A(-2)^n+Bn(-2)^n$

GS is $A(-2)^n+Bn(-2)^n+\frac{7}{9}$

$A+\frac{7}{9}=1 \Rightarrow A=\frac{2}{9}$

$-2A-2B+\frac{7}{9}=2 \Rightarrow B=-\frac{5}{6}$

$b_n=\frac{2}{9}(-2)^n-\frac{5}{6}n(-2)^n+\frac{7}{9}$

14 a $r^2-7r+6=0 \Rightarrow r=6,1$

CF is $A(6^n)+B$

PS is λn

$\lambda n=7(\lambda(n-1))-6(\lambda(n-2))+75$

$\lambda=-15$

$u_n=A(6^n)+\mathrm{B}-15n$

b $A+6B=17$

$A+B=2$

$A=-1, B=3$

$u_n=3(6^n)-15n-1$

15 a $k(n+2)^2 3^{n+2}-6k(n+1)^2 3^{n+1}+9kn^2 3^n=7\times 3^n$

$\Rightarrow 9k(n+2)^2-18k(n+1)^2+9kn^2=7$

$\Rightarrow 18k=7$

$\Rightarrow k=\frac{7}{18}$

b $r^2-6r+9=0 \Rightarrow r=3$

$u_n=(A+Bn)3^n$

c GS is $(A+Bn)3^n+\frac{7}{18}n^2 3^n$

$A=1$

$(A+B)3+\frac{21}{18}=4 \Rightarrow B=-\frac{1}{18}$

$u_n=\left(1-\frac{1}{18}n+\frac{7}{18}n^2\right)3^n$

16 a Auxiliary equation is $r^2 - r + 1 = 0$, so $r = e^{\pm\frac{\pi}{3}}$, and the general solution has the form

$$u_n = A\cos\left(\frac{\pi}{3}n\right) + B\sin\left(\frac{\pi}{3}n\right)$$

$$u_0 = 0 \Rightarrow A = 0, \text{ and } u_1 = 3 \Rightarrow \frac{\sqrt{3}}{2}B = 3 \Rightarrow B = 2\sqrt{3}$$

b We have that $\sin\left(\frac{\pi}{3}n\right) = \sin\left(\frac{\pi}{3}n + 2\pi\right) = \sin\left(\frac{\pi(n+6)}{3}\right)$ so the sequence is periodic with period 6.

17 a 22 (ABA, ABB, ABC, ACA, ACB, ACC, BAB, BAC, BBA, BBB, BBC, BCA, BCB, BCC, CAB, CAC, CBA, CBB, CBC, CCA, CCB, CCC.)

b If we have any string of length $n-1$ we can add a B or C to make a string of length n.
This gives $2s_{n-1}$. If we have a string of length n-1which doesn't end in A, we can add an A to make a string of length n.
This gives $2s_{n-2}$ because we have a string of length n-2 followed by B or C. So
$s_n = 2s_{n-1} + 2s_{n-2}$, $s_0 = 1, s_1 = 3$

c i $r^2 - 2r - 2 = 0, r = 1 \pm \sqrt{3}$

GS is $s_n = A(1+\sqrt{3})^n + B\left(1-\sqrt{3}\right)^n$

$A + B = 1$

$A\left(1+\sqrt{3}\right) + B\left(1-\sqrt{3}\right) = 3$

$A = \frac{3+2\sqrt{3}}{6}$, $B = \frac{3-2\sqrt{3}}{6}$

$$s_n = \frac{1}{6}\left(\left(3+2\sqrt{3}\right)\left(1+\sqrt{3}\right)^n + \left(3-2\sqrt{3}\right)\left(1-\sqrt{3}\right)^n\right)$$

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Challenge

1 Take $\log_2$ of both sides

$2\text{v}_n + \text{v}_{n-1} - \text{v}_{n-2} = 0$

$2r^2 + r - 1 = 0, r = \frac{1}{2}, -1$

$\text{v}_n = A(-1)^n + B\left(\frac{1}{2}\right)^n$

$v_0 = 3, v_1 = -\frac{3}{2}$

$A + B = 3$

$-A + \frac{1}{2}B = -\frac{3}{2}$

$A = 2, B = 1$

$\text{v}_n = 2(-1)^n + \left(\frac{1}{2}\right)^n$

$\Rightarrow u_n = 2^{\left(\frac{1}{2}\right)^n + 2(-1)^n}$

2 To have period 12 it must be based on cos and sin of $\frac{n\pi}{6}$

$r^2 - ar - b = 0 \Rightarrow r = \frac{a \pm \sqrt{a^2 + 4b}}{2}$

So to give the desired roots of $\frac{\sqrt{3} \pm i}{2}$

$a = \sqrt{3}, b = -1$

$u_n = 2k \sin \frac{n\pi}{6}$

max = 2k, min -2k