Recurrence relations 7B

1 a $u_n = 5(2^n)$

b $b_n = 4\left(\frac{5}{2}\right)^{n-1}$

c $d_n = 10\left(-\frac{11}{10}\right)^{n-1}$

d $x_n = 2(-3)^n$

2 a $u_n = 5 + 3n$

b $x_n = 2 + \sum_{i=1}^{n} i = 2 + \frac{1}{2}n(n+1) = 2 + \frac{1}{2}n + \frac{1}{2}n^2$

c $y_n = 3 + \sum_{i=1}^{n}\left(i^2 - 2\right) = 3 + \frac{1}{6}n(n+1)(2n+1) - 2n$

d $s_n = 1 + 2\sum_{i=1}^{n}(i-1) - n = 1 + n(n-1) - n = 1 - 2n + n^2$

3 a CF is $\mathrm{a}_n = c(2^n)$

PS is $\mathrm{a}_n = \lambda$ so $\lambda = 2\lambda + 1 \Rightarrow \lambda = -1$

GS is $\mathrm{a}_n = c(2^n) - 1$

Since $\mathrm{a}_1 = 1,\ 2c - 1 = 1 \Rightarrow c = 1$

So $\mathrm{a}_n = 2^n - 1$

b CF is $u_n = c(-1)^n$

PS is $u_n = \lambda$ so $\lambda = -\lambda + 2 \Rightarrow \lambda = 1$

GS is $u_n = c(-1)^n + 1$

Since $u_1 = 3,\ -c + 1 = 3 \Rightarrow c = -2$

So $u_n = 2(-1)^{n-1} + 1$

c CF is $\mathrm{h}_n = c(3^n)$

PS is $\mathrm{h}_n = \lambda$ so $\lambda = 3\lambda + 5 \Rightarrow \lambda = -\frac{5}{2}$

GS is $\mathrm{h}_n = c(3^n) - \frac{5}{2}$

Since $\mathrm{h}_0 = 1,\ \mathrm{c} - \frac{5}{2} = 1 \Rightarrow c = \frac{7}{2}$

So $\mathrm{h}_n = \frac{7}{2}(3^n) - \frac{5}{2} = \frac{1}{2}(7 \times 3^n - 5)$

3 d CF is $b_n = c(-2)^n$

PS is $b_n = \lambda$ so $\lambda = -2\lambda + 6 \Rightarrow \lambda = 2$

GS is $b_n = c(-2)^n + 2$

Since $b_1 = 3,\ -2c+2=3 \Rightarrow c = -\frac{1}{2}$

So is $b_n = -\frac{1}{2}(-2)^n + 2 = 2 + (-2)^{n-1}$

4 a $n - 1$ teams play each other g_{n-1} times. When an nth team is added, this team has to play each of the other $n - 1$ teams once, so there are $g_{n-1} + n - 1$ games in total. i.e. $g_n = g_{n-1} + n - 1$. $g_1 = 0$.

b $g_n = g_1 + \sum_{r=2}^{n} r - \sum_{r=2}^{n} 1$

$= 0 + \frac{n(n+1)}{2} - 1 - (n-1) = \frac{n(n+1)}{2}$

5 a $u_n = c(4^n) + \lambda, \lambda = 4\lambda - 1 \Rightarrow \lambda = \frac{1}{3}$

So GS is $u_n = c(4^n) + \frac{1}{3}$

b i $4c + \frac{1}{3} = 3 \Rightarrow c = \frac{2}{3}$

PS is $u_n = \frac{2}{3}(4^n) + \frac{1}{3} = \frac{1}{3}(2 \times 4^n + 1)$

ii $4c + \frac{1}{3} = 0 \Rightarrow c = -\frac{1}{12}$

PS is $u_n = -\frac{1}{12}(4^n) + \frac{1}{3} = \frac{1}{3}(1 - 4^{n-1})$

iii $4c + \frac{1}{3} = 200 \Rightarrow c = \frac{599}{12}$

PS is $u_n = \frac{599}{12}(4^n) + \frac{1}{3} = \frac{1}{3}(599 \times 4^{n-1} + 1)$

6 a CF is $u_n = c(3^n)$

PS is $u_n = \lambda n + \mu$

$u_n = 3u_{n-1} + n \Rightarrow \lambda n + \mu = 3(\lambda(n-1) + \mu) + n$

$\Rightarrow 0 = (2\lambda + 1)n + (2\mu - 3\lambda)$

$\Rightarrow \lambda = -\frac{1}{2}, \mu = -\frac{3}{4}$

So GS is $u_n = c(3^n) - \frac{1}{2}n - \frac{3}{4}$

6 b $3c-\frac{1}{2}-\frac{3}{4}=5 \Rightarrow c=\frac{25}{12}$

PS is $u_n=\frac{25}{12}(3^n)-\frac{1}{2}n-\frac{3}{4}=\frac{1}{4}(25\times 3^{n-1}-2n-3)$

7 a Repeating 3 times gives 7, 8.2, 8.92, 9.352 so $u_3=9.352$

b CF is $u_n=c(0.6)^n$

PS is λ

$\lambda=0.6\lambda+4 \Rightarrow \lambda=10$

$c+10=7 \Rightarrow c=-3$

$u_n=10-3(0.6)^n$

c $10-3\times(0.6)^n>9.9$

$\Rightarrow 3\times(0.6)^n<0.1$

Repeating gives n=7

8 a $D_n = 0.95D_{n-1} + 20$, $D_0 = 200$

b $D_n=c(0.95)^n+\lambda$

$\lambda=0.95\lambda+20 \Rightarrow \lambda=400$

$D_0=200 \Rightarrow c+400=200 \Rightarrow c=-200$

So $D_n=-200(0.95)^n+400=200(2-0.95^n)$

c As $n \to \infty$, $0.95n \to 0$, so the deer population approaches 400 in the long term.

9 $u_n=4u_{n-1}-3$

GS is $u_n=c(4^n)+\lambda$

$\lambda=4\lambda-3 \Rightarrow \lambda=1$

$c+1=7 \Rightarrow c=6$

So $u_n=6(4^n)+1$

10 CF is $u_n=c$

PS is $u_n=\lambda(2^n)$

$2\lambda=\lambda+2 \Rightarrow \lambda=2$

$u_n=c+2\times 2^n=2^{n+1}+c$

$5=c+4 \Rightarrow c=1$

So $u_n=2^{n+1}+1$

11 CF is $c \times 4^n$

PS is $\lambda n + \mu$

$\lambda n + \mu = 4(\lambda(n-1) + \mu) + 2n$

$\Rightarrow n(-3\lambda - 2) + (4\lambda - 3\mu) = 0$

$\Rightarrow \lambda = -\frac{2}{3}, \mu = -\frac{8}{9}$

$7 = c - \frac{8}{9} \Rightarrow c = \frac{71}{9}$

$u_n = \frac{71}{9}(4^n) - \frac{2}{3}n - \frac{8}{9} = \frac{1}{9}(71 \times 4^n - 6n - 8)$

12 a CF is $c(2^n)$

PS is λ

$\lambda = 2\lambda - 1005 \Rightarrow \lambda = 1005$

$1000 = c + 1005 \Rightarrow c = -5$

$u_n = -5 \times 2^n + 1005 = 5(201 - 2^n)$

b $201 - 2^n < 0 \Rightarrow n = 8$

$u_8 = 5(201 - 256) = -275$

13 a CF is $c(2^n)$

PS is $\lambda n(2^n)$

$u_2 = 2u_1 - 4 \Rightarrow 2\lambda \times 4 = 2 \times 2\lambda - 4 \Rightarrow \lambda = -1$

GS is $2^n(c - n)$

b $2(c-1) = 3 \Rightarrow c = \frac{5}{2}$

$u_n = 2^n(\frac{5}{2} - n)$

14 a $u_1 = k \times 0 + 1 = 1$

$u_2 = k \times 1 + 1 = k + 1$

$u_3 = k(k+1) + 1 = k^2 + k + 1$

b CF is $c(k^n)$

PS is λ

$\lambda = k\lambda - 1 \Rightarrow \lambda = -\frac{1}{k-1}$

$0 = c - \frac{1}{k-1} \Rightarrow c = \frac{1}{k-1}$

$u_n = \frac{k^n - 1}{k - 1}$

14 c i k^n gets very large so u_n tends to ∞

ii k^n tends to 0 so u_n tends to $\frac{1}{1-k}$

iii $k-1=-2$

k^n alternates between ± 1

so $k^n - 1$ alternates between 0 and -2

so u_n alternates between 0 and 1

iv k^n diverges to $\pm\infty$ alternating in sign

so u_n also diverges to $\pm\infty$ alternating in sign

15 a $\sum_{r=1}^{n}(6r+1) = 6\sum_{r=1}^{n} r + n = 3n(n+1)+n$

$= 3n^2 + 4n$

b $u_n = 2 + \sum_{r=1}^{n}(6r+1) = 3n^2 + 4n + 2$

c $3n^2 + 4n + 2 = 561$

$\Rightarrow 3n^2 + 4n = 559 = 0$

$\Rightarrow n = 13, -14.33..$

$n = 13$

16 a $u_n = 89 - 6 \times \sum_{r=1}^{n} r^2 = 89 - n(n+1)(2n+1)$

b $n(n+1)(2n+1) > 89 \Rightarrow n = 4$

$u_4 = 59 = 4 \times 5 \times 9 = -91$

c Adding an odd number, 89, to an even number $n\ (n+1)\ (2n+1)$ gives an odd number.

17 a $u_n = 3 - n(n+1)$

b $3 - n(n+1) = -103$

$\Rightarrow n(n+1) = 106$

But no 2 consecutive integers multiply to give 106 so this is not possible

c $3 - k^2 - k = -459$

$\Rightarrow k^2 + k - 462 = 0$

$\Rightarrow k = -22, 21$

$k = 21$

18 a $u_n = 1.015u_{n-1} - P$, $u_0 = 2000$

b CF is $c(1.025^n)$
PS is λ

$$1.015\lambda - P = \lambda \Rightarrow \lambda = \frac{P}{0.015} = \frac{200P}{3}$$

GS is $c(1.015)^n + \frac{200P}{3}$

$$c + \frac{200P}{3} = 2000 \Rightarrow c = 2000 - \frac{200P}{3}$$

$$= \frac{200}{3}(30 - P)$$

$$u_n = \frac{200}{3}(1.015^n(30-P) + P)$$

c $u_{18} = 0 \Rightarrow 1.015^{18}(30 - P) + P = 0$

$$\Rightarrow P = \frac{30 \times 1.015^{18}}{1.015^{18} - 1} = 127.612...$$

$P = £127.61$

Challenge

a Disk cannot be moved from *A* to *C* in one jump, so must move from *A* to *B*, then *B* to *C*.

b A→B, B→C, A→B, C→B, B→A, B→C, A→B, B→C

c Transfer $n - 1$ disks from *A* to *C* (H_{n-1} moves), then move nth disk from *A* to *B* (1 move), then transfer $n - 1$ disks from *C* to *A* (H_{n-1} moves), then move nth disk from *B* to *C* (1 move), then transfer $n - 1$ disks from *A* to *C* (H_{n-1} moves). In total, $H_n = 3H_{n-1} + 2$.

d i CF is $c(3^n)$
PS is λ

$\lambda = 3\lambda + 2 \Rightarrow \lambda = -1$

$3c - 1 = 2 \Rightarrow c = 1$

$H_n = 3^n - 1$

ii $H_{10} = 3^{10} - 1 = 59048$ moves