Game theory 6D

1 a First, check if we can reduce the pay-off matrix. The row maximin = –2.
The column minimax = 2, so the game has no stable solution. We cannot use row or column dominance to reduce the matrix. To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 6 to every value in the table:

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	1	10	7
***A* plays 2**	9	3	8
***A* plays 3**	7	4	5

Now, assume *A* plays 1 with probability p_1, 2 with probability p_2 and 3 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:

If *B* plays 1 = $p_1 + 9p_2 + 7p_3$
If *B* plays 2 = $10p_1 + 3p_2 + 4p_3$
If *B* plays 3 = $7p_1 + 8p_2 + 5p_3$

Now let *V* be the value of the transformed game. *A* will want probabilities such that:
$V \leqslant p_1 + 9p_2 + 7p_3$
$V \leqslant 10p_1 + 3p_2 + 4p_3$
$V \leqslant 7p_1 + 8p_2 + 5p_3$

Rearrange these inequalities and add slack variables *r*, *s*, *t* and *u*. Now, maximise
$P = V$, subject to:
$V - p_1 - 9p_2 - 7p_3 + r = 0$
$V - 10p_1 - 3p_2 - 4p_3 + s = 0$
$V - 7p_1 - 8p_2 - 5p_3 + t = 0$
$p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$

1 b First, check if we can reduce the pay-off matrix. The row maximin = –2. The column minimax = 1, so the game has no stable solution. We cannot use row or column dominance to reduce the matrix. To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 5 to every value in the table:

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	2	7	4
***A* plays 2**	4	3	6
***A* plays 3**	7	1	3

Now, assume *A* plays 1 with probability p_1, 2 with probability p_2 and 3 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:
If *B* plays 1 = $2p_1 + 4p_2 + 7p_3$
If *B* plays 2 = $7p_1 + 3p_2 + p_3$
If *B* plays 3 = $4p_1 + 6p_2 + 3p_3$

Now let V be the value of the transformed game. *A* will want probabilities such that:
$V \leqslant 2p_1 + 4p_2 + 7p_3$
$V \leqslant 7p_1 + 3p_2 + p_3$
$V \leqslant 4p_1 + 6p_2 + 3p_3$

Rearrange these inequalities and add slack variables *r*, *s*, *t* and *u*. Now, maximise $P = V$, subject to:
$V - 2p_1 - 4p_2 - 7p_3 + r = 0$
$V - 7p_1 - 3p_2 - p_3 + s = 0$
$V - 4p_1 - 6p_2 - 3p_3 + t = 0$
$p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$

2 First, check if we can reduce the pay-off matrix. The row maximin = –1. The column minimax = 1, so the game has no stable solution. We cannot use row or column dominance to reduce the matrix. To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 4 to every value in the table:

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	6	1	3
***A* plays 2**	2	8	5
***A* plays 3**	5	3	4

Now, assume *A* plays 1 with probability p_1, 2 with probability p_2 and 3 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:

If *B* plays 1 = $6p_1 + 2p_2 + 5p_3$

If *B* plays 2 = $p_1 + 8p_2 + 3p_3$

If *B* plays 3 = $3p_1 + 5p_2 + 4p_3$

Now let *V* be the value of the transformed game. *A* will want probabilities such that:

$V \leqslant 6p_1 + 2p_2 + 5p_3$

$V \leqslant p_1 + 8p_2 + 3p_3$

$V \leqslant 3p_1 + 5p_2 + 4p_3$

Rearrange these inequalities and add slack variables *r*, *s*, *t* and *u*. Now, maximise $P = V$, subject to:

$V - 6p_1 - 2p_2 - 5p_3 + r = 0$

$V - p_1 - 8p_2 - 3p_3 + s = 0$

$V - 3p_1 - 5p_2 - 4p_3 + t = 0$

$p_1 + p_2 + p_3 + u = 1$

$p_1, p_2, p_3, r, s, t, u \geqslant 0$

3 a First, check if we can reduce the pay-off matrix. The row maximin = –2.
The column minimax = 2, so the game has no stable solution. We cannot use row or column dominance to reduce the matrix. Next, we re-write the game from *B*'s perspective:

	***A* plays 1**	***A* plays 2**	***A* plays 3**
***B* plays 1**	5	–3	–1
***B* plays 2**	–4	3	2
***B* plays 3**	–1	–2	1

To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 5 to every value in the table:

	***A* plays 1**	***A* plays 2**	***A* plays 3**
***B* plays 1**	10	2	4
***B* plays 2**	1	8	7
***B* plays 3**	4	3	6

Now, assume *B* plays 1 with probability p_1, 2 with probability p_2 and 3 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:

If *A* plays 1 = $10p_1 + p_2 + 4p_3$

If *A* plays 2 = $2p_1 + 8p_2 + 3p_3$

If *A* plays 3 = $4p_1 + 7p_2 + 6p_3$

Now let V be the value of the transformed game. *B* will want probabilities such that:

$V \leqslant 10p_1 + p_2 + 4p_3$

$V \leqslant 2p_1 + 8p_2 + 3p_3$

$V \leqslant 4p_1 + 7p_2 + 6p_3$

Rearrange these inequalities and add slack variables r, s, t and u. Now, maximise $P = V$, subject to:

$V - 10p_1 - p_2 - 4p_3 + r = 0$

$V - 2p_1 - 8p_2 - 3p_3 + s = 0$

$V - 4p_1 - 7p_2 - 6p_3 + t = 0$

$p_1 + p_2 + p_3 + u = 1$

$p_1, p_2, p_3, r, s, t, u \geqslant 0$

3 b First, check if we can reduce the pay-off matrix. The row maximin = –2.
The column minimax = 1, so the game has no stable solution. We cannot use row or column dominance to reduce the matrix. Next, we re-write the game from *B*'s perspective:

	A plays 1	*A* plays 2	*A* plays 3
***B* plays 1**	3	1	–2
***B* plays 2**	–2	2	4
***B* plays 3**	1	–1	2

To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 3 to every value in the table:

	A plays 1	*A* plays 2	*A* plays 3
***B* plays 1**	6	4	1
***B* plays 2**	1	5	7
***B* plays 3**	4	2	5

Now, assume *B* plays 1 with probability p_1, 2 with probability p_2 and 3 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:
If *A* plays 1 = $6p_1 + p_2 + 4p_3$
If *A* plays 2 = $4p_1 + 5p_2 + 2p_3$
If *A* plays 3 = $p_1 + 7p_2 + 5p_3$

Now let *V* be the value of the transformed game. *B* will want probabilities such that:
$V \leqslant 6p_1 + p_2 + 4p_3$
$V \leqslant 4p_1 + 5p_2 + 2p_3$
$V \leqslant p_1 + 7p_2 + 5p_3$

Rearrange these inequalities and add slack variables *r*, *s*, *t* and *u*. Now, maximise $P = V$, subject to:
$V - 6p_1 - p_2 - 4p_3 + r = 0$
$V - 4p_1 - 5p_2 - 2p_3 + s = 0$
$V - p_1 - 7p_2 - 5p_3 + t = 0$
$p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$

3 c First, check if we can reduce the pay-off matrix. The row maximin = –1. The column minimax = 1, so the game has no stable solution. We cannot use row or column dominance to reduce the matrix. Next, we re-write the game from *B*'s perspective:

	***A* plays 1**	***A* plays 2**	***A* plays 3**
***B* plays 1**	–2	2	–1
***B* plays 2**	3	–4	1
***B* plays 3**	1	–1	0

To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 5 to every value in the table:

	***A* plays 1**	***A* plays 2**	***A* plays 3**
***B* plays 1**	3	7	4
***B* plays 2**	8	1	6
***B* plays 3**	6	4	5

Now, assume *B* plays 1 with probability p_1, 2 with probability p_2 and 3 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:

If *A* plays 1 = $3p_1 + 8p_2 + 6p_3$

If *A* plays 2 = $7p_1 + p_2 + 4p_3$

If *A* plays 3 = $4p_1 + 6p_2 + 5p_3$

Now let V be the value of the transformed game. *B* will want probabilities such that:

$V \leqslant 3p_1 + 8p_2 + 6p_3$

$V \leqslant 7p_1 + p_2 + 4p_3$

$V \leqslant 4p_1 + 6p_2 + 5p_3$

Rearrange these inequalities and add slack variables *r*, *s*, *t* and *u*. Now, maximise $P = V$, subject to:

$V - 3p_1 - 8p_2 - 6p_3 + r = 0$

$V - 7p_1 - p_2 - 4p_3 + s = 0$

$V - 4p_1 - 6p_2 - 5p_3 + t = 0$

$p_1 + p_2 + p_3 + u = 1$

$p_1, p_2, p_3, r, s, t, u \geqslant 0$

4 a Add 5 to all elements

	B plays 1	B plays 2
A plays 1	4	6
A plays 2	8	1
A plays 3	3	7

Let A play 1 with probability p_1

and A plays 2 with probability p_2

and A play 3 with probability p_3

Let the value of the game to A be v and $V = v + 5$

Maximise $P = V$

Subject to $4p_1 + 8p_2 + 3p_3 \geqslant V \Rightarrow V - 4p_1 - 8p_2 - 3p_3 + r = 0$

$6p_1 + p_2 + 7p_3 \geqslant V \Rightarrow V - 6p_1 - p_2 - 7p_3 + s = 0$

$p_1 + p_2 + p_3 \leqslant 1 \Rightarrow p_1 + p_2 + p_3 + t = 1$

$p_1,\ p_2,\ p_3,\ r,\ s,\ t \geqslant 0$

b Using part **a** we can write

b.v	V	p_1	p_2	p_3	r	s	t	Value
r	1	–4	–8	–3	1	0	0	0
s	1	–6	–1	–7	0	1	0	0
t	0	1	1	1	0	0	1	1
P	–1	0	0	0	0	0	0	0

c The value of the game for A when the strategy is to play 1 with probability $\frac{7}{9}$ and 2 with probability $\frac{2}{9}$ can be calculated as follows:

If B plays 1 = $4\cdot\frac{7}{9} + 8\cdot\frac{2}{9} = \frac{44}{9}$

If B plays 2 = $6\cdot\frac{7}{9} + 1\cdot\frac{2}{9} = \frac{44}{9}$

The value of the game for A is thus $\frac{44}{9} - 5 = -\frac{1}{9}$

5 a

	A plays 1	A plays 2			A plays 1	A plays 2
B plays 1	5	–1	Adding	B plays 1	9	3
B plays 2	–2	3	4 to all	B plays 2	2	7
B plays 3	–3	4	elements	B plays 3	1	8

Let B play 1 with probability q_1

Let B play 2 with probability q_2

Let B play 3 with probability q_3

Let the value of the game to B be v and $V = v + 4$

Maximise $P = V$

Subject to

$$9q_1 + 2q_2 + q_3 \geqslant V \quad V - 9q_1 - 2q_2 - q_3 + r = 0$$
$$3q_1 + 7q_2 + 8q_3 \geqslant V \quad V - 3q_1 - 7q_2 - 8q_3 + s = 0$$
$$q_1 + q_2 + q_3 \leqslant 1 \quad q_1 + q_2 + q_3 + t = 1$$
$$q_1, q_2, q_3, r, s, t \geqslant 0$$

b Using part **a** we can write

b.v	V	p_1	p_2	p_3	r	s	t	**Value**
r	1	–9	–2	–1	1	0	0	0
s	1	–3	–7	–8	0	1	0	0
t	0	1	1	1	0	0	1	1
P	–1	0	0	0	0	0	0	0

c Since A has only the two choices of playing 1 or 2, the value of the game for this strategy for B can be calculated as follows:

If A plays 1 $= 5\cdot\frac{7}{13} - 3.\frac{6}{13} = \frac{17}{13}$

If A plays 2 $= -\frac{7}{13} + 4\cdot\frac{6}{13} = \frac{17}{13}$

Hence the value of the game for B is $\frac{17}{13}$

6 a The row maximin = 2. The column minimax = 4. These are not the same, so by the stable solution theorem we know that this game has no stable solution.

b We notice that row 1 dominates row 3, so row 3 can be removed. Updated game:

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	5	3	–1
***A* plays 2**	4	5	2
***A* plays 4**	7	–2	4

c To use linear programming, we need to make sure all entries in the pay-off matrix are positive. Hence, we add 3 to every value in the table:

	***B* plays 1**	***B* plays 2**	***B* plays 3**
***A* plays 1**	8	6	2
***A* plays 2**	7	8	5
***A* plays 4**	10	1	7

Now, assume *A* plays 1 with probability p_1, 2 with probability p_2 and 4 with probability p_3.
We have $p_1 + p_2 + p_3 = 1$. The expected pay-offs are:
If *B* plays 1 = $8p_1 + 7p_2 + 10p_3$
If *B* plays 2 = $6p_1 + 8p_2 + p_3$
If *B* plays 3 = $2p_1 + 5p_2 + 7p_3$

Now let *V* be the value of the transformed game. *A* will want probabilities such that:
$V \leqslant 8p_1 + 7p_2 + 10p_3$
$V \leqslant 6p_1 + 8p_2 + p_3$
$V \leqslant 2p_1 + 5p_2 + 7p_3$

Rearrange these inequalities and add slack variables *r*, *s*, *t* and *u*. Now, maximise $P = V$, subject to:

$V - 8p_1 + 7p_2 + 10p_3 + r = 0$
$V - 6p_1 + 8p_2 + p_3 + s = 0$
$V - 2p_1 + 5p_2 + 7p_3 + t = 0$
$p_1 + p_2 + p_3 + u = 1$
$p_1, p_2, p_3, r, s, t, u \geqslant 0$