Game theory 6A

1 a In a zero-sum game the winnings of player *A* and player *B* add up to zero. This is not the case in this example; hence the game is not zero-sum.

b If player *A* plays 2 and player *B* plays 3, player *A* will win 1 and player *B* will win 2.

c To find the play-safe strategy, we need to find the worst outcomes for both *A* and *B:*

	***B* plays 1**	***B* plays 2**	***B* plays 3**	Worst outcome for *A*
***A* plays 1**	(2, 5)	(3, 1)	(2, 3)	2
***A* plays 2**	(4, 1)	(3, 5)	(1, 2)	1
***A* plays 3**	(3, 6)	(5, 4)	(7, 2)	3
***A* plays 4**	(1, 4)	(5, 2)	(3, 4)	1
Best outcome for *B*	6	5	4	

The best worst (maximin) outcome for *A* is to play 3. Similarly, the worst best (minimax) outcome for *B* is to play 3. Hence the play-safe strategy for both players is to play 3.

2 a The play-safe strategy for the row player is the choice which corresponds to the row maximin. The play-safe strategy for the column player is the choice which corresponds to the row minimax.

b If *A* plays 2 and *B* plays 3, player *A* will win 7 and player *B*, since this is a zero-sum game, will win –7 (or lose 7).

c The minima of rows 1 – 3 are: –3, 4, –5 respectively. Hence the row maximin is 4. The maxima of columns 1 – 3 are: 6, 5, 8 respectively. Hence the column minimax is 5.

d Based on part **c**, the play-safe strategy for player *A* is to play 2. The play-safe strategy for *B* is to play 2.

e By the stable solution theorem we conclude that this game has no stable solution. This is because the row maximin (which is 4) $\neq$ column minimax (which is 5).

3 a To find the play-safe strategy for each player, we need to find the row maximin and the column minimax. The minima of the rows are: –1, –4, –5 and –3. So the row maximin is –1 (row 1). The maxima of the columns are: 7, 6 and 8. So the column minimax is 6 (column 2). Thus the play-safe strategy for player *A* is to play 1, and for player *B* to play 2.

b If player *A* assumes that *B* plays safe, she can maximise her winnings if she plays 2.

4 a

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	3	2	3	2	←
A plays 2	–2	1	3	–2	
A plays 3	4	2	1	1	
Column max	4	2	3		
		↑			

A should play 1 (row maximin = 2)

B should play 2 (column minimax = 2)

b row maximin = 2 = column minimax

∴ game is stable

c The value of the game for player *A* is equal to the row maximin. Hence the value of the game for player *A* equals 2.

d To rewrite the matrix from *B*'s point of view, we need to multiply all the numbers by –1:

	***A* plays 1**	***A* plays 2**	***A* plays 3**
***B* plays 1**	–3	2	–4
***B* plays 2**	–2	–1	–2
***B* plays 3**	–3	–3	–1

Now that we switched perspectives, the value of the game for player *B* can be determined by finding the row maximin (as in part **c**). Thus the value of the game for player *B* is –2.

5 a

	S plays 1	S plays 2	S plays 3	S plays 4	Row min	
R plays 1	–2	–1	–3	1	–3	
R plays 2	2	3	1	–2	–2	
R plays 3	1	1	–1	3	–1	←
Column max	2	3	1	3		
			↑			

R should play 3 (row maximin = –1)

S should play 3 (column minimax = 1)

5 b row maximin ≠ column minimax

$-1 \neq 1$

so game is not stable

c To write out the matrix from Steve's perspective, we need to multiply all the numbers by –1:

	***R* plays 1**	***R* plays 2**	***R* plays 3**
***S* plays 1**	2	–2	–1
***S* plays 2**	1	–3	–1
***S* plays 3**	3	–1	1
***S* plays 4**	–1	2	–3

6 a A play-safe strategy maximises the worse possible outcome. This means that after considering the worse possible scenarios for each choice, the player chooses the best of them.

	B plays 1	B plays 2	B plays 3	Row min	
A plays 1	–3	–2	2	–3	
A plays 2	–1	–1	3	–1	←
A plays 3	4	–3	1	–3	
A plays 4	3	–1	–1	–1	
Column max	4	–1	3		←
		↑			

A should play 2 or 4 (row maximin –1)

B should play 2 (column minimax –1)

b Since row maximin = column minimax

$-1 = -1$

game is stable

Saddle points are (A2, B2) and (A4, B2).

c Value of the game is –1 to A (if A plays 2 or 4 and B plays 2 the value of the game is –1).

7 a In a zero-sum game, one player's winnings are the other player's losses. This means that if player *A* wins *x*, then player *B* has to lose *x*, or, in other words, win –*x*. So the winnings of players *A* and *B* add up to 0, which is why we call this a zero-sum game.

b

	D plays 1	D plays 2	D plays 3	D plays 4	Row min	
C plays 1	7	2	–3	5	–3	
C plays 2	4	–1	1	3	–1	←
C plays 3	–2	5	2	–1	–2	
C plays 4	3	–3	–4	2	–4	
Column max	7	5	2	5		
			↑			

C plays 2 (row maximin = –1)

D plays 3 (column minimax = 2)

c $-1 \neq 2$

row maximin ≠ column minimax

so no stable solution

d If both players play safe (see part **b**), the pay-off for Claire is 1.

e Again, using the answer to part **b**, we determine that if both players play safe, the pay-off for David is –1. We can also use the fact that this is a zero-sum game: in part **d** we showed that the pay-off for Claire is 1, so the pay-off for David must be –1.

f

	C plays 1	C plays 2	C plays 3	C plays 4
D plays 1	–7	–4	2	–3
D plays 2	–2	1	–5	3
D plays 3	3	–1	–2	4
D plays 4	–5	–3	1	–2

8 a A saddle point in a pay-off matrix is a value which is the smallest in its row and the largest in its column, and corresponds directly with stable solutions in two-person zero-sum games.

b A saddle point in a zero-sum pay-off matrix is the value which is both the smallest in its row and the largest in its column.

	D plays 1	D plays 2	D plays 3	D plays 4	D plays 5	Row min
H plays 1	2	1	0	0	2	0 ←
H plays 2	4	0	0	0	2	0 ←
H plays 3	1	4	–1	–1	3	–1
H plays 4	1	1	–1	–2	0	–2
H plays 5	0	–2	–3	–3	–1	–3
Column max	4	4	0	0	3	
			↑	↑		

H plays 1 or 2

D plays 3 or 4

c row maximin = column minimax

0 = 0

so game stable

saddle points (H1, D3) (H2, D3) (H1, D4) (H2, D4)

d The value of the game to Hilary = 0

e The value of the game to Denis = 0

f

	H plays 1	H plays 2	H plays 3	H plays 4	H plays 5
D plays 1	–2	–4	–1	–1	0
D plays 2	–1	0	–4	–1	2
D plays 3	0	0	1	1	3
D plays 4	0	0	1	2	3
D plays 5	–2	–2	–3	0	1

9 a To see that this is a zero-sum game, think about each players winnings. Whatever the outcome, one player will pay the other one x, which means that she loses x and the other person wins x. Hence this is a zero-sum game.

9 b

	B plays 1	*B* plays 2	*B* plays 3
***A* plays 1**	–1	1	2
***A* plays 2**	1	–2	1
***A* plays 3**	2	1	–3

c The row maximin is –1. The column minimax is 1. Since $-1 \neq 1$, this matrix has no saddle point and by the stable solution theorem we know that this means the game has no stable solution.

d To write out the pay-off matrix for Beth, we need to multiply all numbers by –1:

	A plays 1	*A* plays 2	*A* plays 3
***B* plays 1**	1	–1	–2
***B* plays 2**	–1	2	–1
***B* plays 3**	–2	–1	3

10 a The game has a stable solution, so the row maximin = column minimax. Currently the row minima are: 1, x or 5, 0. The column maxima are: 5, 3 or x. For the game to have a stable solution we need to consider what happens in the row and column containing x:
If the row minimum is 5, then $x \geqslant 5$. The row maximin is then 5. Since $x \geqslant 5$, the column minimax is also 5. So the game has a stable solution for all $x \geqslant 5$.
Let's now consider the situation when $x < 5$. For $x \geqslant 1$, the row maximin will be x. The column minimax is then x or 3, so for row maximin = column minimax we need $x \geqslant 3$.
Thus, for the game to have a stable solution we need $x \geqslant 3$.
(Note that if $x < 1$, the row maximin is 1 and this cannot be matched by the column minimax, which would be 3).

b We will follow the same two cases as in **a**. If $x \geqslant 5$, the value of the game is 5 (we determined that this was the column minimax and row maximin in part **a**). If, however, $5 > x \geqslant 3$, the value of the game will be x (again, refer back to part **a**).

Challenge

Assume a zero-sum game has multiple saddle points. Notation: let $x_{r,s}$ represent the entry in row r, column s. Now let's assume that two of the saddle points are $x_{a,b}$ and $x_{c,d}$. This means that $x_{a,b}$ is the smallest value in row a and the largest value in column b. Similarly, $x_{c,d}$ is the smallest value in row c and the largest value in column d. Now, consider the entry $x_{a,d}$. Since it is in row a, we must have $x_{a,d} \geqslant x_{a,b}$. Similarly, since it is in column d, we must have $x_{c,d} \geqslant x_{a,d}$. From this we deduce $x_{c,d} \geqslant x_{a,b}$. Next, consider $x_{c,b}$. By the same reasoning, we have $x_{c,b} \geqslant x_{c,d}$ and $x_{a,b} \geqslant x_{c,b}$. Thus we deduce $x_{a,b} \geqslant x_{c,d}$. So we have $x_{c,d} \geqslant x_{a,b}$ and $x_{a,b} \geqslant x_{c,d}$, thus it must be $x_{a,b} = x_{c,d}$. So both saddle points have the same value. This reasoning can be applied to any pair of saddle points. Thus we conclude that all saddle points must be equal.