Dynamic Programming Mixed exercise

1 a Minimax

Stage	State	Action	Destination	Value
1	F	FT	T	20*
	G	GT	T	19*
	H	HT	T	23*
2	D	DF	F	$\text{Max}(19, 20) = 20*$
		DG	G	$\text{Max}(20, 19) = 20*$
		DH	H	$\text{Max}(21, 23) = 23$
	E	EF	F	$\text{Max}(16, 20) = 20$
		EG	G	$\text{Max}(19, 19) = 19*$
		EH	H	$\text{Max}(18, 23) = 23$
	A	AD	D	$\text{Max}(23, 20) = 23*$
		AE	E	$\text{Max}(29, 19) = 29$
	B	BD	D	$\text{Max}(24, 20) = 24$
		BE	E	$\text{Max}(21, 19) = 21*$
	C	CD	D	$\text{Max}(24, 20) = 24$
		CE	E	$\text{Max}(22, 19) = 22*$
3	S	SA	A	$\text{Max}(20, 23) = 23$
		SB	B	$\text{Max}(25, 21) = 25$
		SC	C	$\text{Max}(18, 22) = 22*$

Minimax route is SCEGT value 22

b Minimal route is SCEGT
By Bellman's principle of optimality, any part of an optimal path is also optima, so CEGT is an optimal path from C to T

2 Maximin

Stage	State	Action	Destination	Value
1	F	FT	T	20*
	G	GT	T	19*
	H	HT	T	23*
2	D	DF	F	$\text{Min}(19,20)=19$
		DG	G	$\text{Min}(20,19)=19$
		DH	H	$\text{Min}(21,23)=21*$
	E	EF	F	$\text{Min}(16,20)=16$
		EG	G	$\text{Min}(19,19)=19*$
		EH	H	$\text{Min}(18,23)=18$
	A	AD	D	$\text{Min}(23,21)=21*$
		AE	E	$\text{Min}(29,19)=19$
	B	BD	D	$\text{Min}(24,21)=21*$
		BE	E	$\text{Min}(21,19)=19$
	C	CD	D	$\text{Min}(24,21)=21*$
		CE	E	$\text{Min}(22,19)=19$
3	S	SA	A	$\text{Min}(20,21)=20$
		SB	B	$\text{Min}(25,21)=21*$
		SC	C	$\text{Min}(18,21)=18$

The maximin route is SBDHT of value 21

3 a Maximum

Stage	State	Action	Destination	Value
Yoghurt	5	5	0	50*
	4	4	0	40*
	3	3	0	30*
	2	2	0	20*
	1	1	0	10*
	0	0	0	0*
Cheese	5	0	5	$0+50=50$
		1	4	$12+40=52$
		2	3	$30+30=60$*
		3	2	$40+20=60$*
		4	1	$45+10=55$
		5	0	$49+0=49$
	4	0	4	$0+40=40$
		1	3	$12+30=42$
		2	2	$30+20=50$*
		3	1	$40+10=50$*
		4	0	$45+0=45$
	3	0	3	$0+30=30$
		1	2	$12+20=32$
		2	1	$30+10=40$*
		3	0	$40+0=40$*
	2	0	2	$0+20=20$
		1	1	$12+10=22$
		2	0	$30+0=30$*
	1	0	1	$0+10=10$
		1	0	$12+0=12$*
	0	0	0	$0+0=0$*
Butter	5	0	5	$0+60=60$
		1	4	$14+50=64$
		2	3	$25+40=65$*
		3	2	$34+30=64$
		4	1	$41+12=53$
		5	0	$47+0=47$

3 b There are two possible courses of action each of value £65

Product	Butter	Cheese	Yoghurt
Units to be used	2	2	1
Product	Butter	Cheese	Yoghurt
Units to be used	2	3	0

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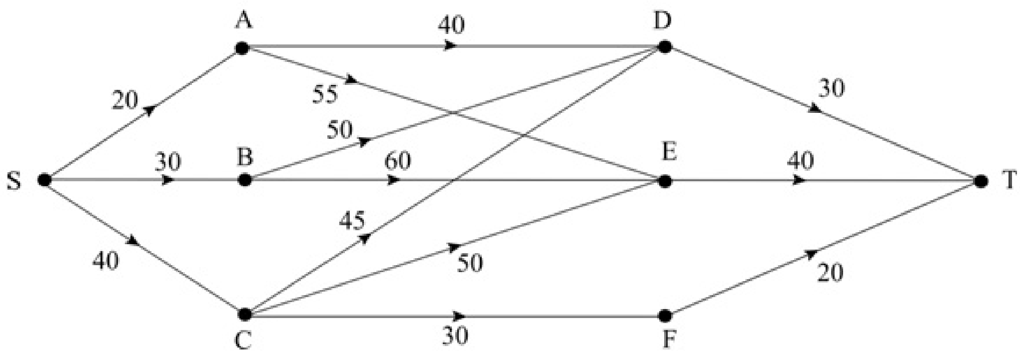

The states are the vertices.

Stage	Initial State	Action	Final state	Value
1	D	DT	T	30
	E	ET	T	40
	F	FT	T	20
2	A	AD	D	$\text{Max}(40,30)=40*$
		AE	E	$\text{Max}(55,40)=55$
	B	BD	D	$\text{Max}(50,30)=50*$
		BE	E	$\text{Max}(60,40)=60$
	C	CD	D	$\text{Max}(45,30)=45$
		CE	E	$\text{Max}(50,40)=50$
		CF	F	$\text{Max}(30,20)=30*$
3	S	SA	A	$\text{Max}(40,20)=40*$
		SB	B	$\text{Max}(50,30)=50$
		SC	C	$\text{Max}(40,30)=40*$

Tracing back there are two routes

SC, CF, FT, $\Rightarrow$ SCFT

SA, AD, DT $\Rightarrow$ SADT

Maximum altitude on these routes is $40\ (\times 100\,\text{ft}) = 4000\,\text{ft}$

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Stage	State	Action	Cost	Total cost
2	0	*A*	2	2
		C	3	3*
	1	*A*	2	2
		B	3	3
		C	6	6*
	2	*A*	1	1
		B	2	2*
1	0	*A*	2	2 + 3 = 5
		C	3	3 + 6 = 9*
	1	*A*	1	1 + 3 = 4
		B	3	3 + 6 = 9*
		C	6	6 + 2 = 8
	2	*A*	5	5 + 6 = 11*
		B	5	5 + 2 = 7
0	0	*A*	4	4 + 9 = 13
		B	3	3 + 9 = 12
		C	5	5 + 11 = 16*

Decisions CAC

b Hence maximum profit is £16 000

Tracing back through calculations the optimal strategy is CAC

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Stage	State	Action	Destination	Value
July	2	1	0	2 + 12.5 = 14.5*
	1	2	0	1 + 12.5 = 13.5*
	0	3	0	12.5*
June	2	5	2	2 + 12.5 + 3 + 14.5 = 32
		4	1	2 + 12.5 + 13.5 = 28
		3	0	2 + 12.5 + 12.5 = 27*
	1	6	2	1 + 12.5 + 3 + 14.5 = 31
		5	1	1 + 12.5 + 3 + 13.5 = 30
		4	0	1 + 12.5 + 12.5 = 26*
	0	6	1	12.5 + 3 + 13.5 = 29
		5	0	12.5 + 3 + 12.5 = 28*
May	2	6		2 + 12.5 + 3 + 26 = 43.5*
		5	1	2 + 12.5 + 3 + 28 = 45.5
	1	6	0	1 + 12.5 + 3 + 28 = 44.5*
April	0	5	2	12.5 + 3 + 43.5 = 59
		4	1	12.5 + 44.5 = 57*

Costs are in £100s
Minimum cost is £5700 with optimal schedule 4,6,5,3

7 a Minimax problem

7 b

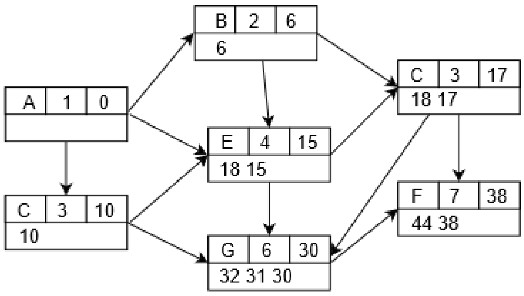
Stage	State	Action	Value
1	J	JT	5*
	K	KT	6*
	L	LT	2*
2	G	GJ	Max(6,5) = 6*
		GK	Max(8,6) = 8
	H	HJ	Max(6,5) = 6*
		HK	Max(9,6) = 9
	I	IK	Max(8,6) = 8
		IL	Max(5,2) = 5*
3	D	DG	Max(8,6) = 8*
		DH	Max(10,6) = 10
	E	DH	Max(7,6) = 7*
	F	FH	Max(9,6) = 9*
		FI	Max(10,5) = 10
4	A	AE	Max(5,7) = 7*
	B	BD	Max(7,8) = 8
		BE	Max(6,7) = 7*
	C	CE	Max(8,7) = 8*
		CF	Max(5,9) = 9
5	S	SA	Max(3,7) = 7*
		SB	Max(8,7) = 8*
		SC	Max(5,8) = 8

The optimal route is SAEHJT.

c FH is reassessed to be difficulty 6. The optimal path from F to T is still FHJT, with value 6. The optimal path from C to T (the only stage 4 state with a path through F) is now CFHJT with value 6 and the optimal path from S to T is SCFHJT with value 6.

Challenge

1 The shortest route is ABECGF. The complication in trying to use dynamic programming here is that the stages are not defined.



2 Let x_1, x_2, x_3, x_4 be the number of trailers built in April, May, June and July respectively

Construction costs are:

$$\sum_{i=1}^{4} 1250 \times 1_{\{x_i > 0\}} + 300 \times 1_{\{x_i > 5\}}$$

where $1_{\{x_i > a\}} = \begin{matrix} 0 \text{ if } x_i \leqslant a \\ 1 \text{ if } x_i > a \end{matrix}$

as $0 \leqslant x_i \leqslant 6$, $\left\lceil \frac{x_i}{6} \right\rceil = \begin{matrix} 0 \text{ if } x_i = 0 \\ 1 \text{ if } x_i > 0 \end{matrix}$

and $\left\lfloor \frac{x_i}{5} \right\rfloor = \begin{matrix} 0 \text{ if } x_i < 5 \\ 1 \text{ if } x_i \geqslant 5 \end{matrix}$

Number of trailers in storage in

May: $x_1 - 3$

June: $x_1 + x_2 - 3 - 7$

July: $x_1 + x_2 + x_3 - 3 - 7 - 5$

So storage costs are $100(3x_1 + 2x_2 + x_3 - 28)$

Subject to constraints:

$0 \leqslant x_i \leqslant 6$ (number of trailers that can be built)

$3 \leqslant x_1 \leqslant 3 + 2 = 5$

$10{=}3{+}7 \leqslant x_1 + x_2 \leqslant \min(10 + 2, 6 + 6) = 12$

$15{=}3{+}7{+}5 \leqslant x_1 + x_2 + x_3 \leqslant \min(15 + 2, 12 + 6) = 17$

$x_1 + x_2 + x_3 + x_4 = 18$