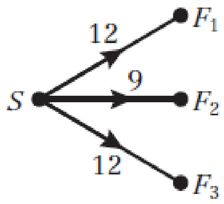
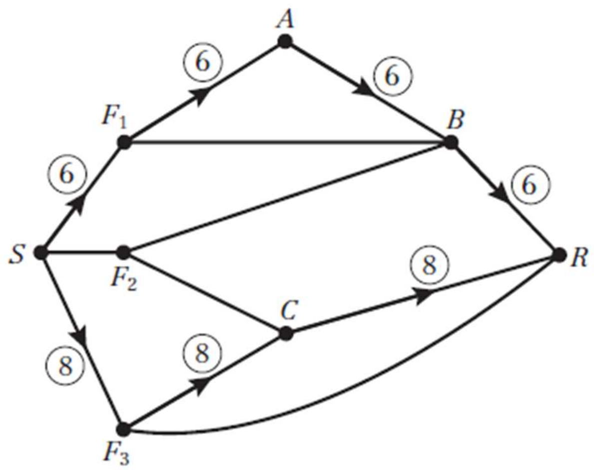
Flows in networks Mixed exercise

1 a

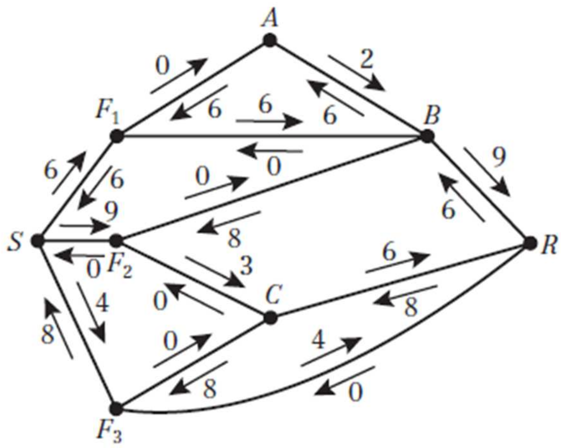


b i SF_1 A B R – 6

ii SF_3 C R – 8



c i



SF_1 B R – 6	SF_3 R – 4	
SF_2 B R – 3	SF_2 B R – 6	
e.g. SF_2C R – 3	or e.g. SF_2 C R – 3	Total flow: 30
SF_3 R – 4	SF_1 B R – 3	

ii max flow – min cut theorem

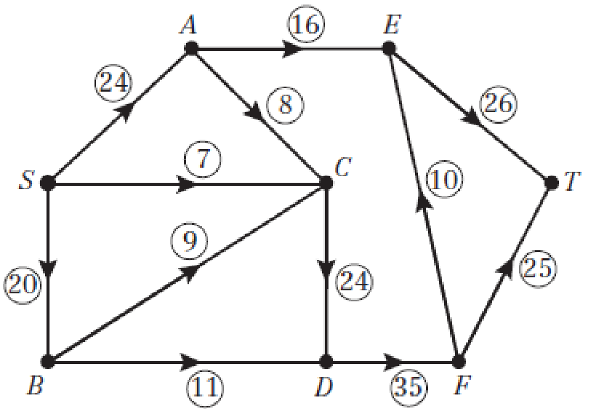
e.g. cut BR, F_2C, F_3C, F_3R

(accept BR, F_2C, S F_3)

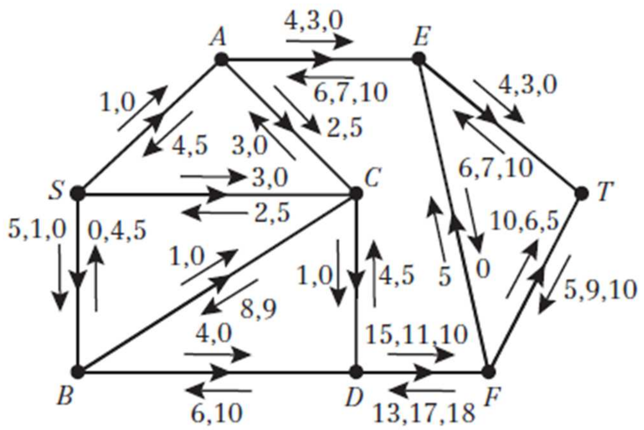
2 a The value of a cut is equal to the sum of all upper capacities of arcs flowing into the cut, minus the sum of all lower capacities of arcs flowing out of the cut. Thus:
$C_1 = AE + DF = 20 + 50 = 70$
$C_2 = ET + FT = 30 + 35 = 65$

b By the maximum flow – minimum cut theorem, the maximal flow of this network is at most 65.

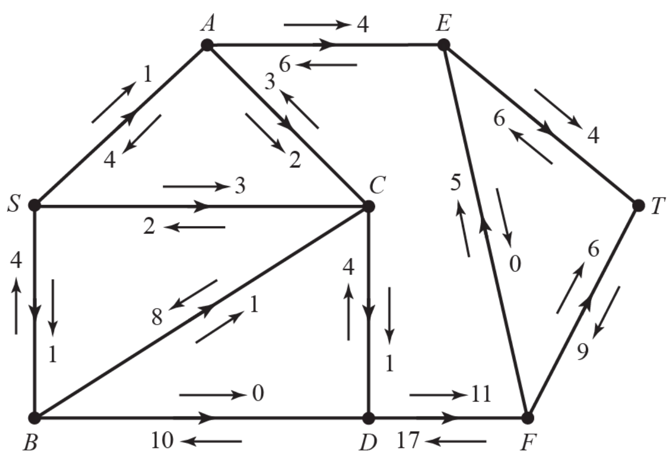
c First, consider vertex *E*. The flow out of *E* is 26, so, by flow conservation, flow into *E* must also be equal to 26. Hence *FE* = 10. Next, consider vertex *F*. Flow into *F* is 36, so flow out of *F* must also be 36. Hence *FT* = 25. Now consider *D*. Flow out of *D* is 35, so flow into *D* must also be 35. Thus *CD* = 24. Next, consider *B*. Flow into *B* is 20, thus flow out of *B* must also be 20. Hence *BC* = 9. Finally, consider *A*. Flow out of *A* is 24, so *SA* = 24 too. The flow value =51. Feasible flow.



d Initial capacities:

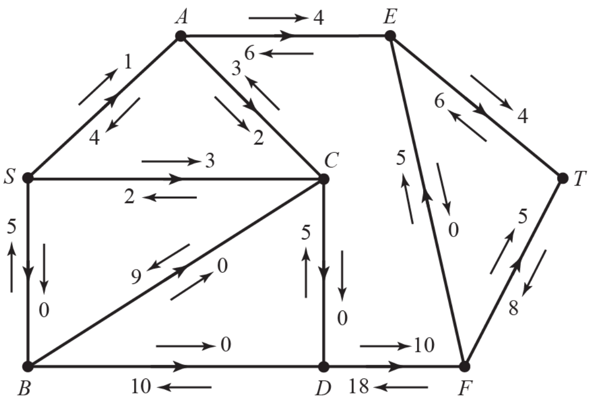


First, we notice a spare capacity of 4 along *SBDFT*. Updated network:

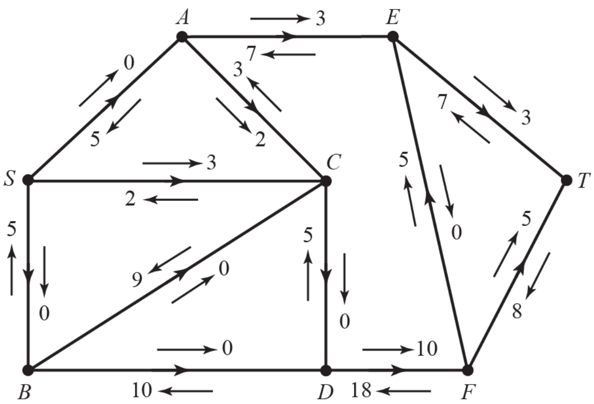


2 d (continued)

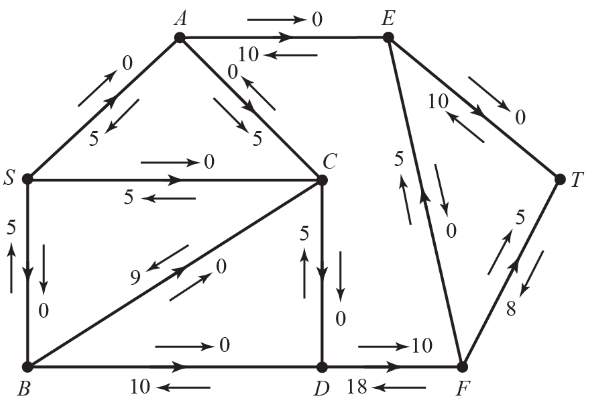
Next, we see a route with capacity 1 along *SBCDFT*. Updated network:



Now we notice that we can increase the flow by 1 along *SAET*. Updated network:

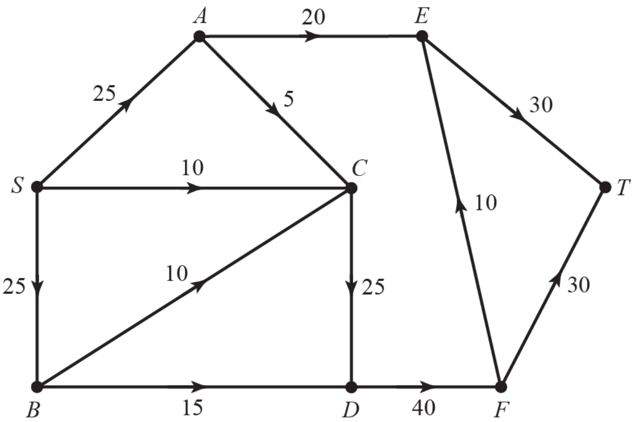


Finally, we can increase the flow along *SCAET* by 3. Updated network:



2 d (continued)

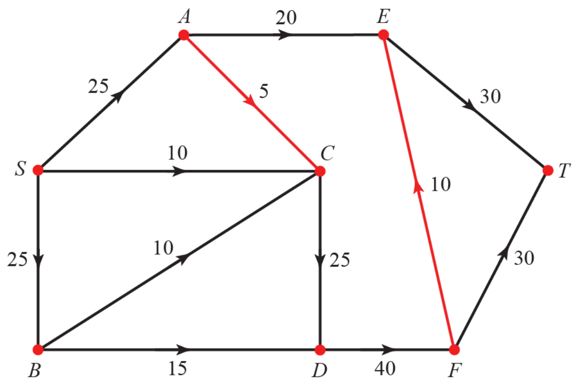
By inspecting arcs coming out of S we see that we cannot increase the flow any more. Hence, this is the final network:

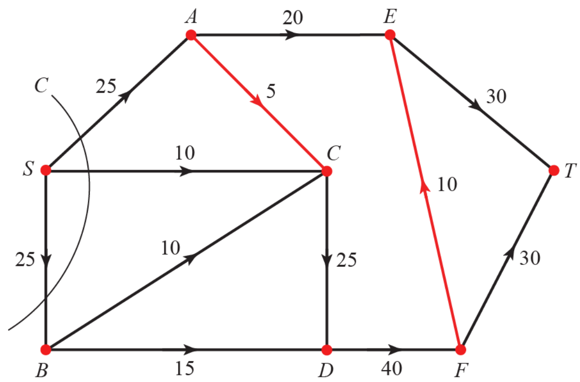


The flow of this network is equal to
$Flow = SA + SB + SC = 60$.

2 e First, identify all saturated arcs (purple) and all arcs at their lowest capacity (red)
Now, there are several cuts which could be drawn here, but the most obvious is probably SB, SC and SA: The value of this cut is 60 (remember to refer back to the lower and upper capacities of the arcs flowing into and out of the cut!) which is the same as the flow of this network. Hence, by the maximum flow – minimum cut theorem, this flow is maximal.

3



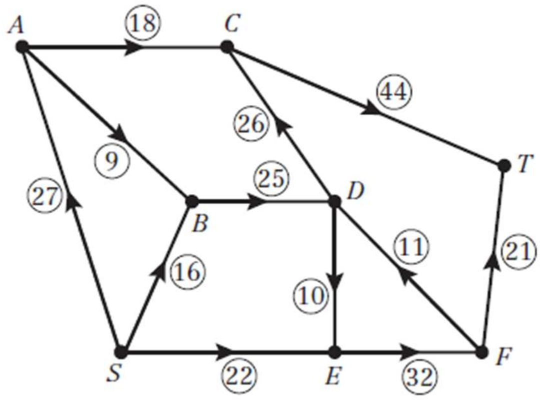


a By flow conservation, we know that the flow into a vertex must be equal to the flow out of a vertex. The upper capacity of CT is 44. The sum of the lower capacities of CD and AC is also 44. This means that we cannot increase the flow along these arcs, as CT would not be able to handle it. Hence, both CD and AC must be at their lowest values.

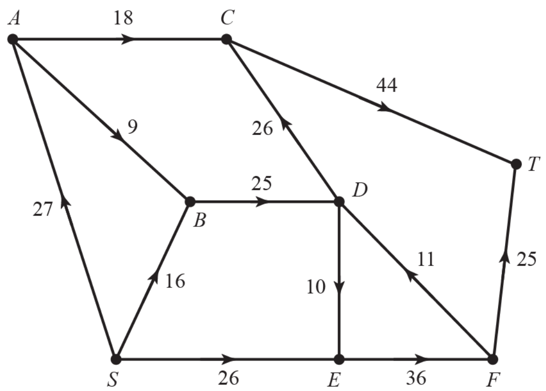
b We know that the flow through AC is equal to 18 (see part **a**). The source S can supply a flow of at most 27. This means that the arc AB must flow at its lowest capacity, or else the flow conservation will not be satisfied. Hence $AB = 9$.

c The value of a cut is equal to the sum of all upper capacities of arcs flowing into the cut, minus the sum of all lower capacities of arcs flowing out of the cut.
Thus:
$X = CT_{max} + EF_{max} - FD_{min} = 73$
By the maximum flow – minimum cut theorem, the flow through this network cannot exceed 73.

3 d Using the answers to parts **a** and **b**, we can immediately fill in the values at *AC, CD, AB* and *SA*. Next, by considering vertex *B* and applying flow conservation, we see that *SB* = 16. Similarly, by using flow conservation at vertex *E* we obtain *DE* = 10. Finally, by considering the vertex *F* and applying flow conservation we find *FD* = 11. Flow is 65.



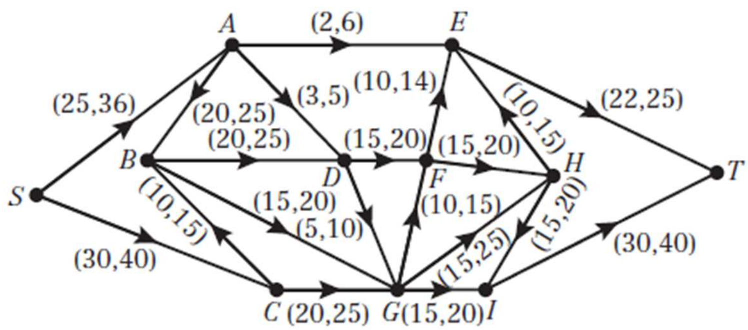
e Since *CT* is already saturated, we can only try and increase the total flow by increasing the flow through *FT*. Given that its upper capacity is 25, we can add at most 4. By examining the diagram, we see that it is possible to add 4 along *SEFT*. Updated network:



The flow through this network is now equal to 69 (just consider the flow into *T*). By part **c** we know that this is the maximal flow of this network.

4 a Sources are these vertices which have no arcs flowing into them. Sinks are those vertices which have no arcs flowing out of them. In our example we have
Sources: *A* and *C*, sinks: *E* and *I*.

b The super source *S* connects to both sources. Its lower capacity is equal to the sum of lower capacities of the sources and its upper capacity is equal to the sum of the upper capacities of the sources. Both sinks connect to the super sink *T*. Its lower capacity is equal to the sum of the lower capacities of these sinks, and its upper capacity is equal to the sum of their upper capacities.

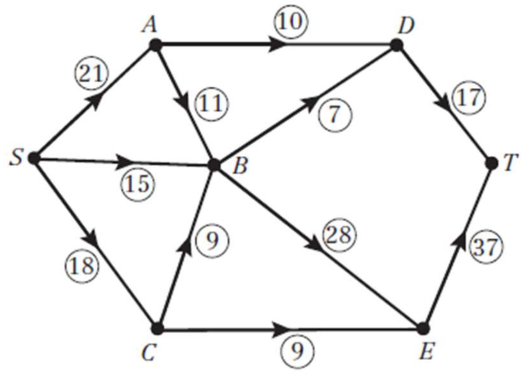


4 c The value of a cut is equal to the sum of all upper capacities of arcs flowing into the cut, minus the sum of all lower capacities of arcs flowing out of the cut. Thus:
$C = AE_{max} + AD_{max} + BD_{max} + BG_{max} - BC_{min} = 6 + 5 + 25 + 20 - 10 = 46$
Hence, by the maximum flow – minimum cut theorem, the maximal flow of this network cannot exceed 46.

d Assume node F is shut down for maintenance. Now consider vertex D. The minimal flow into D equals $AD_{min} + BD_{min}$ which is equal to 23. If F is shut down, no flow can be directed towards it, hence all flow from D must go to G. But the maximum capacity of arc DG is 10, so there is no feasible flow in this network.

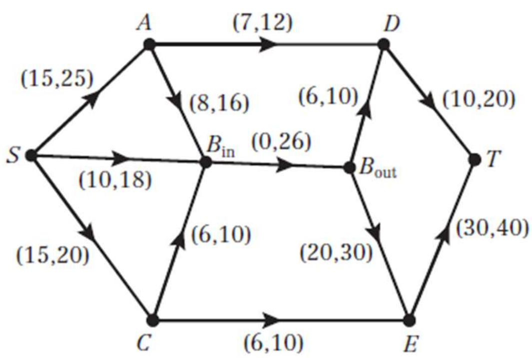
e Following from part **d**, we know that the minimum flow into D is 23. Arc DG can handle at most 10, so the minimum capacity of the new pipe, ED, must be at least 13.

5 a To solve this question we will use flow conservation at various nodes. First, consider node A. the flow out of A is equal to 21, so the flow into A must also be 21. Thus $SA = 21$. Next, consider node C. The flow into C is 18, so the flow out of C must also be 18. Currently the flow through CE is 9. So it must be that $CB = 9$. Now, consider vertex B. The flow into B equals 35. So the flow out of B must be the same. Hence, $BE = 7$. Lastly, consider node E. Flow into E is 37, so $ET = 37$.
The value of this flow is 54.

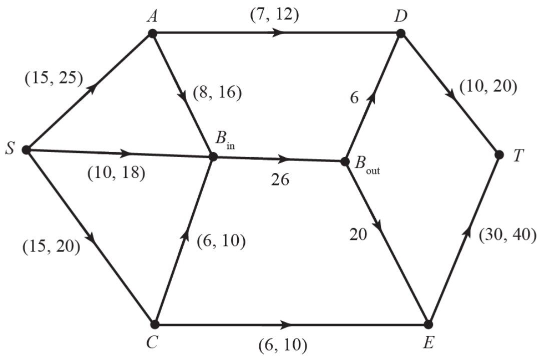


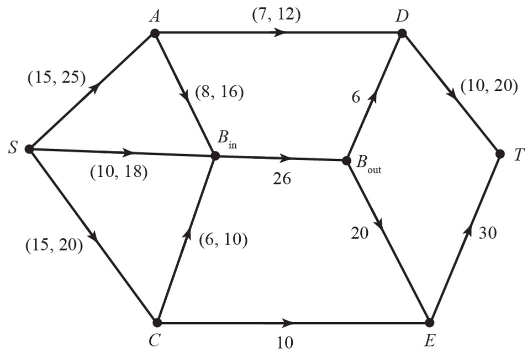
b Value of the flow through node B is equal to:
$B = CB + SB + AB = 9 + 11 + 15 = 35.$

c To show the reduced capacity on the diagram, we split node B into B_{in} and B_{out}. We connect all arcs flowing into B to B_{in} and all arcs flowing out of B to B_{out}. The arc going from B_{in} to B_{out} will have lower capacity of 0 (the question does not specify any restrictions on the lower capacity) and upper capacity of 26.

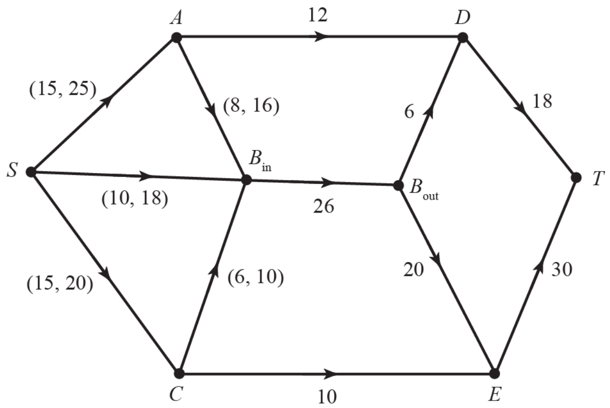


5 d In order to keep the flow as high as possible, we will keep the value of flow at B at its maximum. Note that this forces both arcs $B_{out}D$ and $B_{out}E$ to flow at their minimum. So we have:
Now, consider vertex E. To satisfy its minimal flow, we need CE to flow at its maximum capacity. Hence:



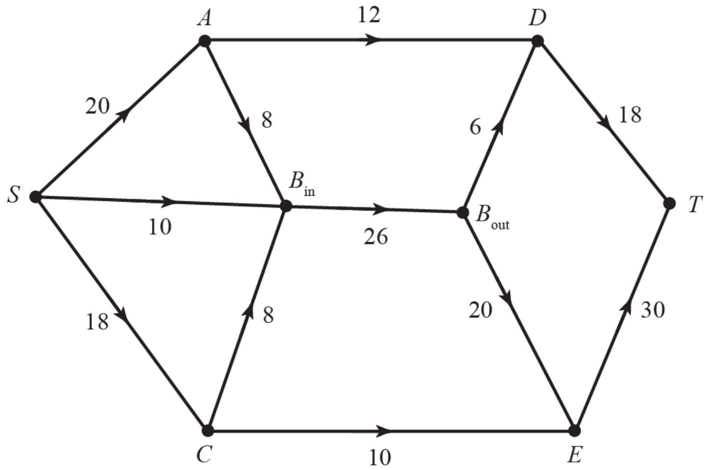


These are the only arcs with fixed flows. Note that all of them are saturated or flowing at their minimal flow. Hence, if we can have AD to be saturated, we can create a cut through AD, $B_{in}B_{out}$, CE which would prove that this flow is maximal. Setting AD to 12 forces DT to be 18

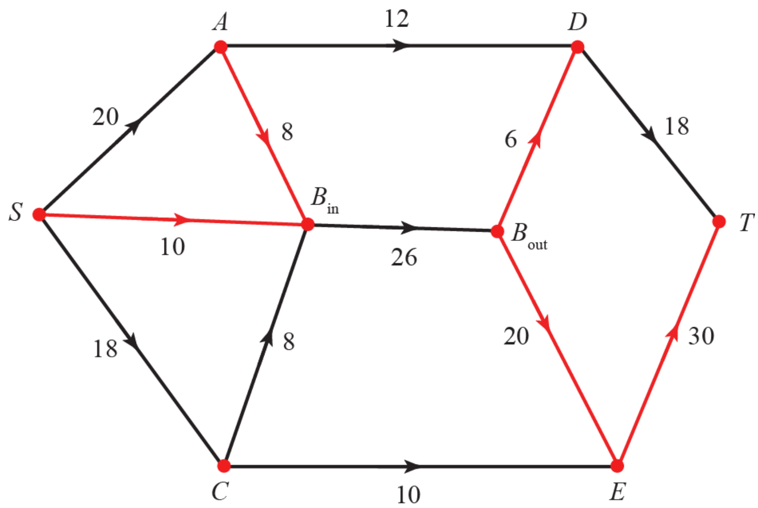


To fill in the remaining arcs we need to ensure that the flow into B_{in} is 26. We can achieve that by putting AB_{in} = 8, SB_{in} = 10 and CB_{in} = 8. This forces SA = 20 and SC = 18. Final flow of 48:

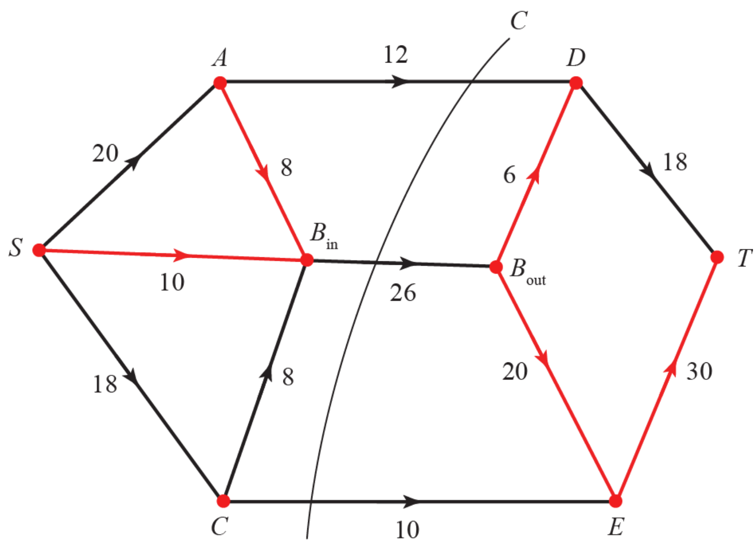
5 e



To show that this flow is indeed maximal, first identify all saturated arcs (purple) and all arcs flowing at their lower capacity (red).



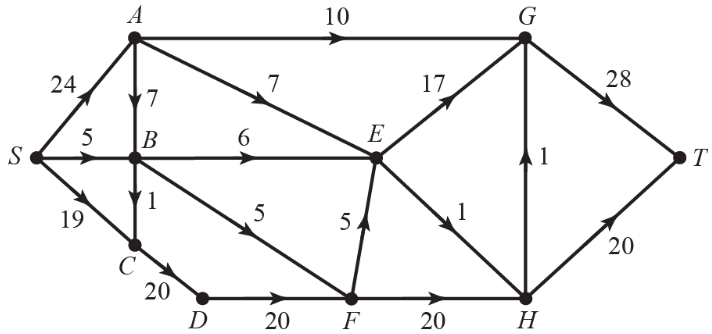
Next, draw a cut through (for example) AD, $B_{in}B_{out}$ and CE. The value of this cut is:
$C = AC + B_{in}B_{out} + CE = 48$



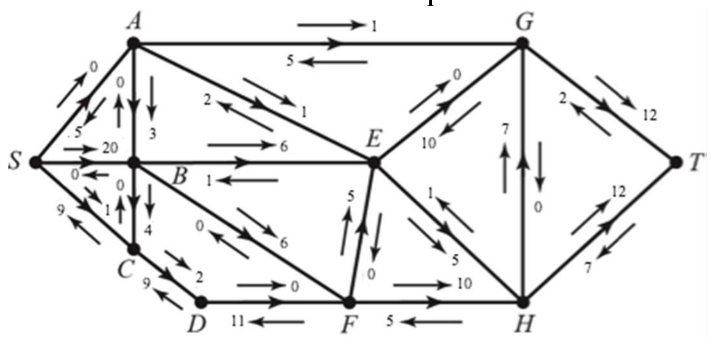
Which is the same as the flow calculated in part **d**. Hence, by the maximum flow – minimum cut principle this flow is maximal.

Challenge

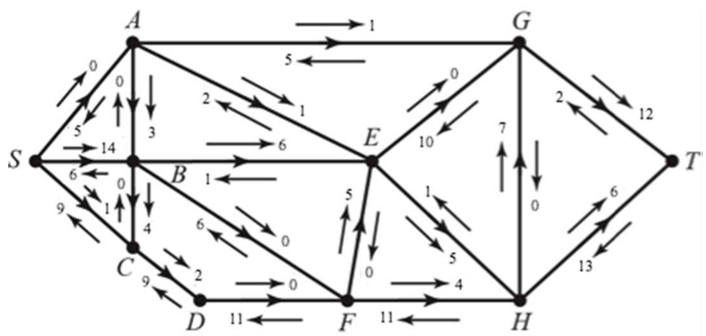
a To find the maximum flow through this network we wish to use flow augmentation method. To that end, we first need to find a feasible flow (any flow will do, we just need a starting point). For example:



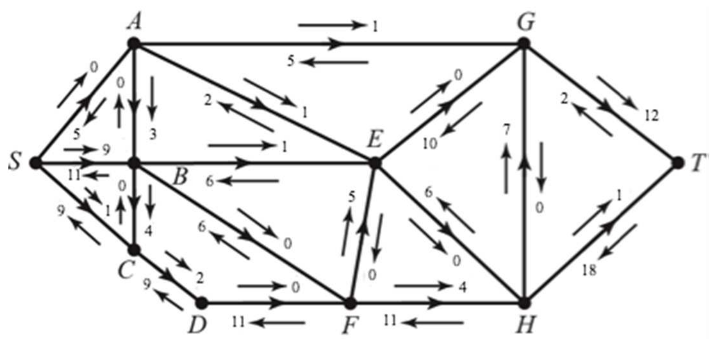
Remember to make sure that flow conservation, as well as the upper and lower capacities are satisfied! This network can be represented as follows:



Next, *SA* is already saturated. *SC* is not saturated, but there is no spare capacity at *DF*. So the only possibility is to follow *SB*. We notice a spare capacity of 6 along *SBFHT*. Updated network:

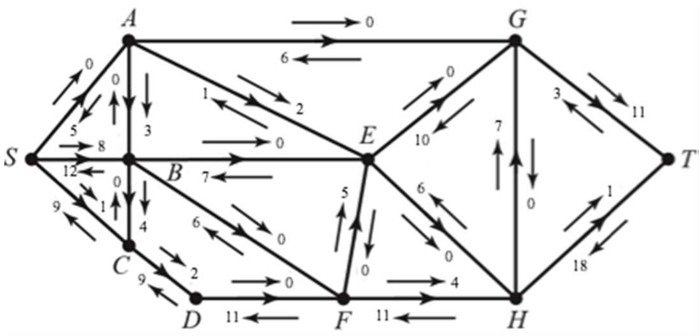


Now, if we analyse the vertex *B* we see that we could go do *C* and then *D* but we already know that there is no spare capacity there. So the only option is to follow to *E*. There is a capacity of 5 along *SBEHT*. Updated network:

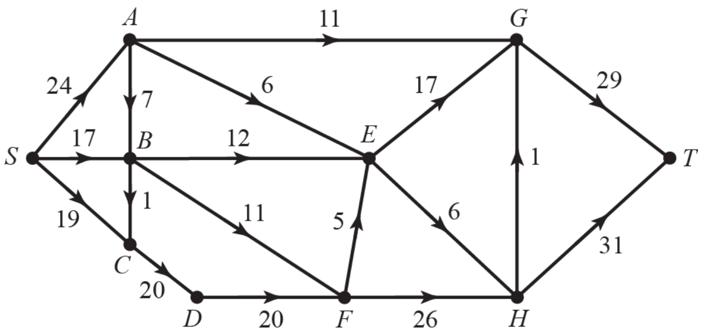


Challenge
a (continued)

Finally, we can increase the flow by 1 along *SBEAGT*. Updated network:

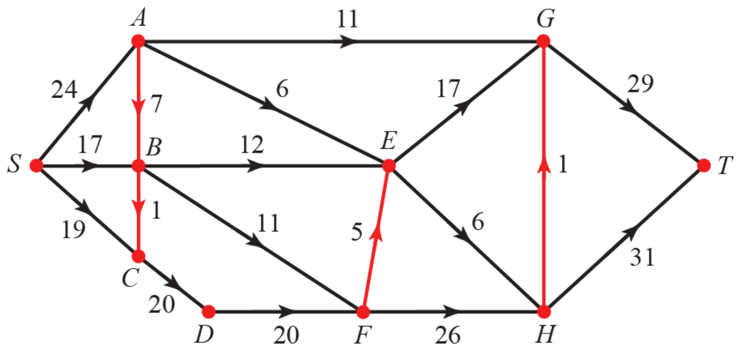


Further inspection shows that there are no more routes with spare capacities. Hence the final flow of this network is equal to 60:



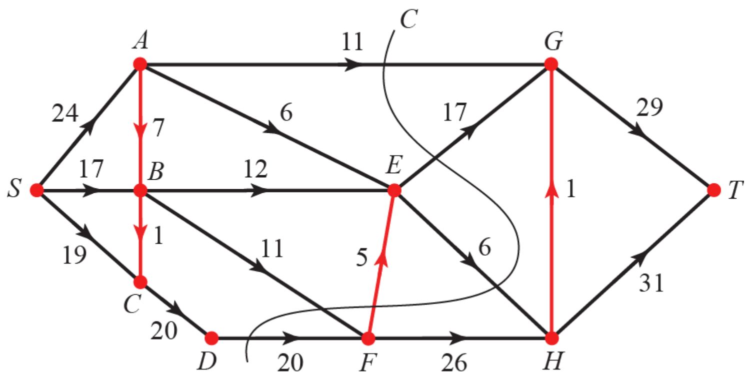
Challenge

b To show that the flow is maximal, we begin with identifying all saturated arcs (purple) an arcs flowing at their lowest capacity (red).



Now we see that we can draw a cut through *DF, BF, FE, EH, EG* and *AG*.
The value of this cut is equal to the sum of of arcs flowing into the cut, minus the sum of arcs flowing out of the cut.
Hence:

$C = DF + BF - EF + EH + EG + AG$
$\quad = 20 + 11 - 5 + 6 + 17 + 11 = 60$



This is the same as the flow of this network, hence, by the maximum flow – minimum cut theorem, the flow is maximal.

c The blocked node reduced the flow through the network by 21. If node *A* was blocked, the flow through the network would be reduced by 24, which is more than the question states. If we blocked *B*, we would reduce the flow by 24 (7 coming from *AB* and 17 coming from *SB*), but we could send an extra 1 through *SC* and extra 2 through *AE*, so the total reduction would be 21. Hence this is the blocked node. Example flow of 39 through the network: