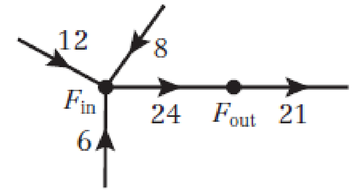
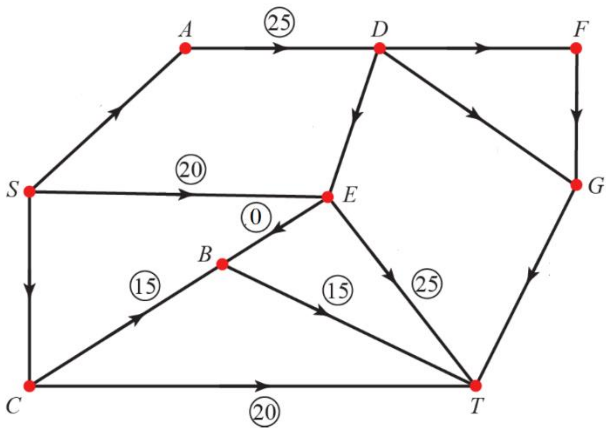
Flows in networks 4C

1 This information can be shown by replacing node F with two nodes, F_{in} and F_{out} and by adding an arc with capacity 24 between the two nodes.

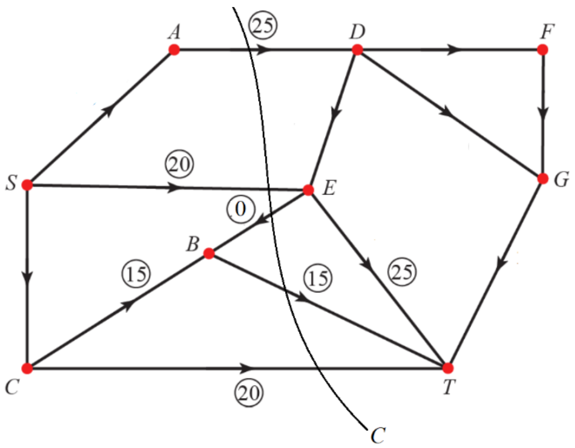


2 a The value of the flow through a system is just a sum of the flows of arcs coming out of the source, or arcs coming into the sink. In our case
$Flow = SA + SE + SC = 25 + 20 + 35 = 80$

To show that the flow is maximal, first identify all saturated arcs.

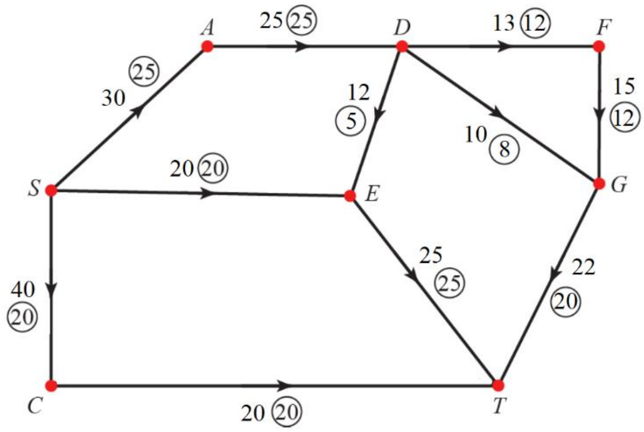


Now we see that we can draw a cut through *CT*, *BT*, *EB*, *SE* and *AD*.

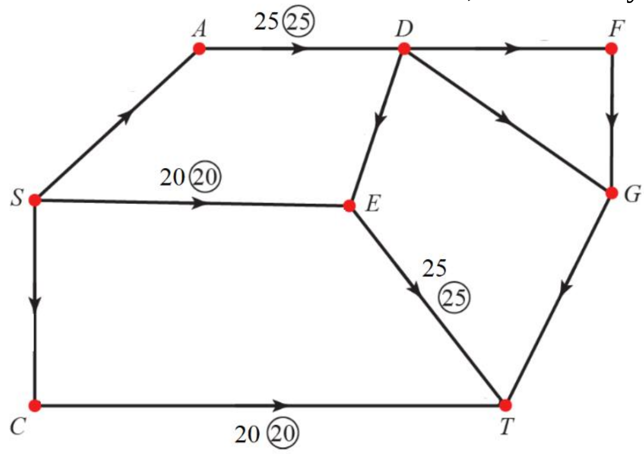


The value of this cut is equal to the sum of flows coming into the cut.
$C = AD + SE + EB + BT + CT + EB = 25 + 20 + 0 + 15 + 20 = 80$
So by the maximum flow – minimum cut theorem, this flow is maximal.

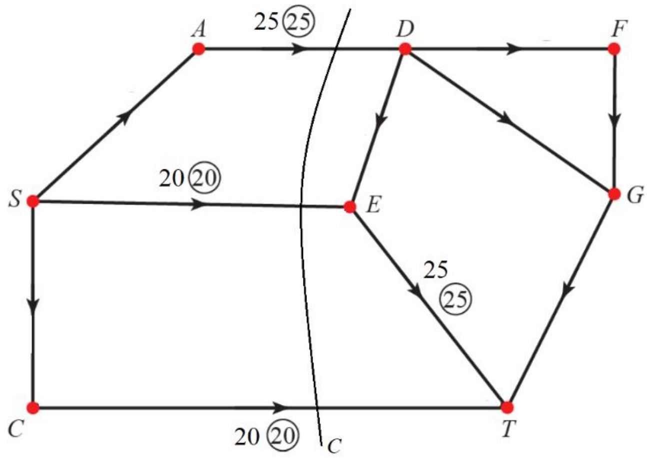
2 b If a vertex becomes blocked, we can just remove it from the diagram, together with all arcs connected to it. Remember to adjust the flows at all other arcs, making sure that the flow conservation is satisfied! The new network looks like this:



c New value of flow pattern = $SA + SE + SC = 20 + 20 + 25 = 65$
To show that the flow is maximal, first identify all saturated and empty arcs.

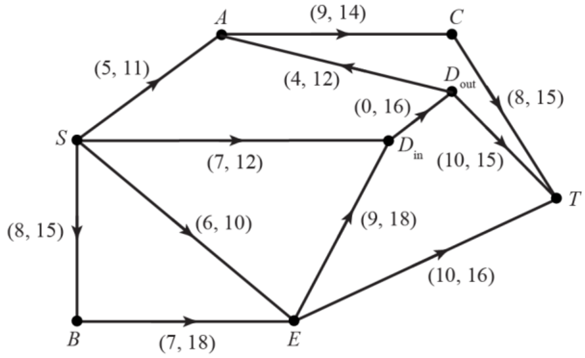


Now we see that we can draw a cut through *CT*,*SE* and *AD.*

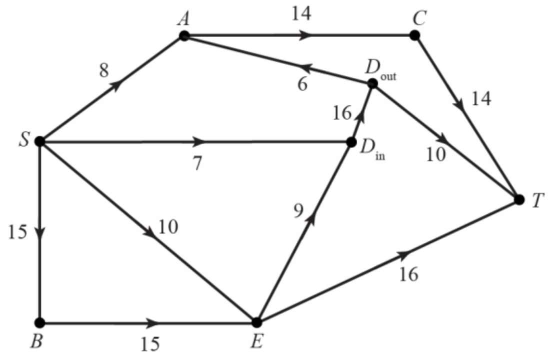


The value of this cut is equal to the sum of flows coming into the cut.
$C = AD + SE + CT = 25 + 20 + 20 = 65$
So by the maximum flow – minimum cut theorem, this flow is maximal

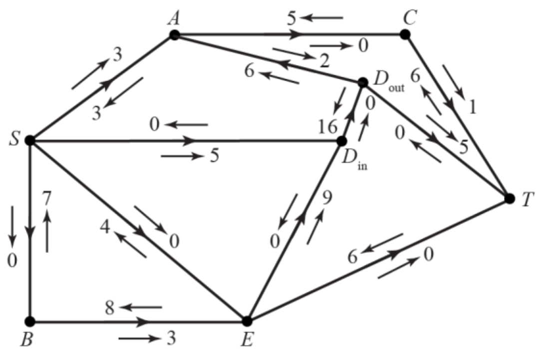
3 a To show this information on the diagram, we split the vertex D into D_{in} and D_{out}. We connect all arcs flowing into D to D_{in} and all arcs flowing out of D to D_{out}. The arc connecting D_{in} and D_{out} has no restriction on lower capacity, so we assign 0 there. The upper capacity of this arc is 16:



b To find the new feasible flow, we firstly need to decrease the flow into D_{in}. Since ED_{in} is already at its minimum, we need to decrease the flow through SD_{in} to 7. Since we didn't change anything at vertex E, the bottom half of the network remains unchanged. Next, we need to decrease the flow out of D_{out} to satisfy flow conservation. To that end, we decrease the flow through $D_{out}T$ to its minimum, i.e. 10. Now the flow conservation is satisfied and the flow into T sums to 40, as required.



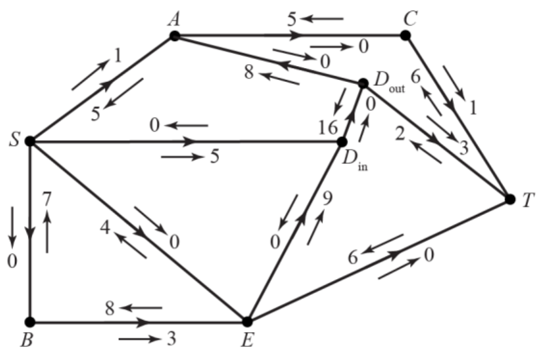
c To use flow augmentation, first draw the initial state of the network

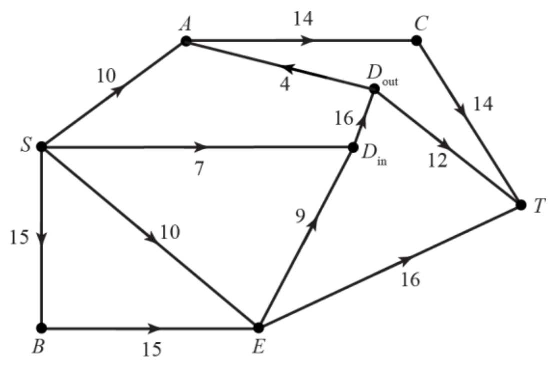


3 c continued

All arcs coming out of *S*, except for *SA*, are saturated. Hence, we have to follow *SA*. However, there is only a capacity of 2 from *A* (along AD_{out}). Thus we can increase the flow by 2 and the route is $SAD_{out}T$. Updated network:

We now see that there are no more routes with spare capacities from vertex *A*. Hence this is the final flow:



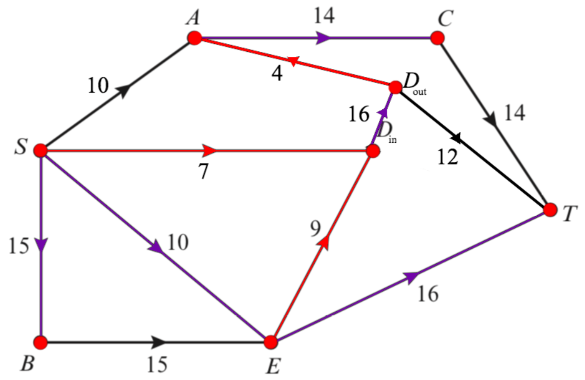


The value of this flow equals to the sum of flows of arcs coming out of *S* or into *T*. Thus:

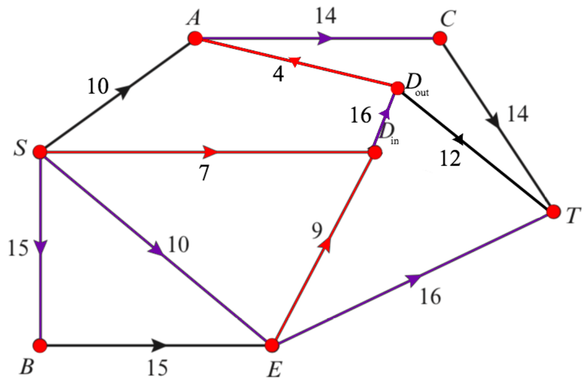
$Flow = SB + SE + SD_{in} + SA$

$= 15 + 10 + 7 + 10 = 42$

3 d First, identify saturated arcs (purple) and arcs at their lowest capacity (red)



Next, draw a cut through ET, $D_{in}D_{out}$, $D_{out}A$ and AC.



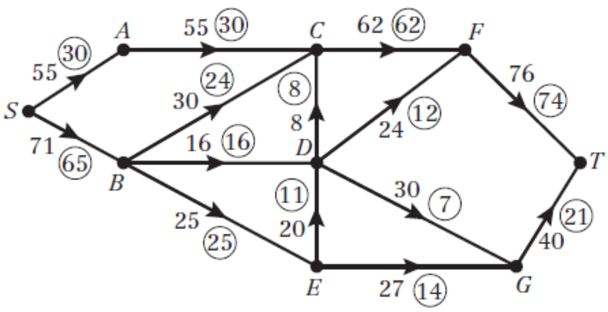
The value of this cut equals to the sum of arcs flowing into it minus the sum of cuts flowing out of it. Thus:

$C = ET + D_{in}D_{out} + AC - D_{out}A$

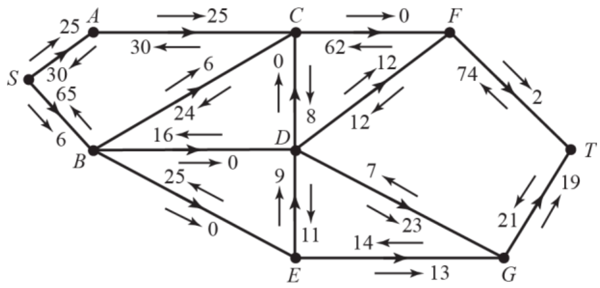
$= 16 + 16 + 14 – 4 = 42$

Hence, by the maximum flow – minimum cut theorem, this is the maximal flow.

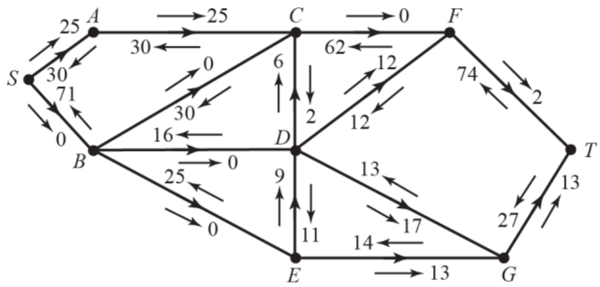
4 a When adding the super source remember to sum the capacities and the flows of all arcs coming out of the sources connected to that super source.



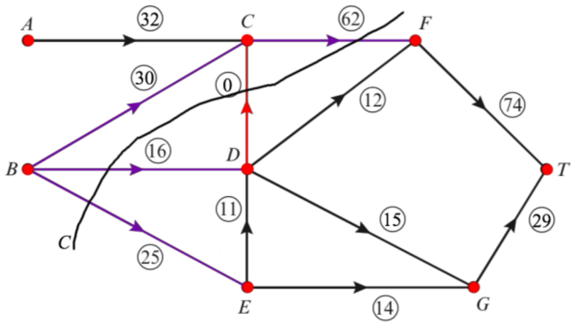
4 b To find the maximum flow we will use flow augmentation. Initial network:
First, note that the only route with spare capacity from vertex *B* is *BC*. Further inspection shows that we can increase the flow of students by 6 along *SBCDGT*. Updated network:



There is no more capacity in the direction of *SB*. Thus we need to follow *SA*. Once we get to vertex *C*, we notice that we could wither go to *B*, where there is no capacity left, or to *D*, where there is a spare capacity of 2. Hence, we can add 2 along *SACDGT*. Updated network:

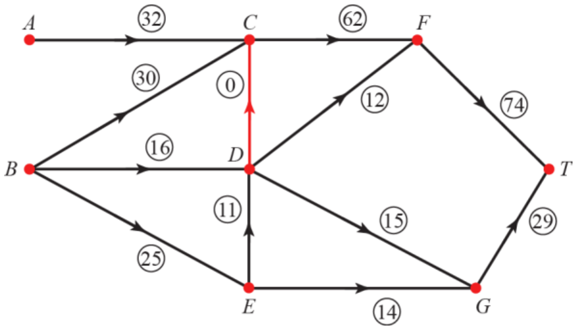


Further inspection shows that there are no more routes with space capacities. Hence this is the final network:



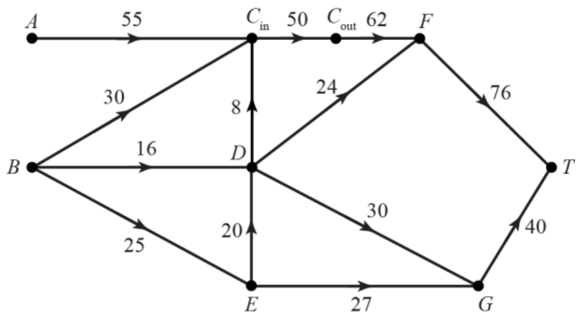
C = *BE* + *BD* + *CF* = 103
Maximal flow = 103

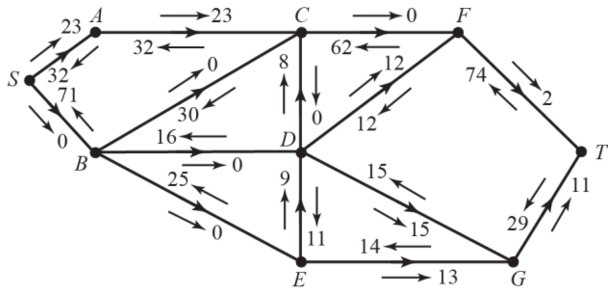
4 c To prove that the flow is maximal, first identify all saturated arcs (purple) and all empty arcs (red)
Next, draw a cut through *BE*, *BD*, *DC* and *CF*:
The value of this cut is equal to the sum of arcs flowing into it, i.e.



C = *BE* + *BD* + *CF* = 103.
This is equal to the sum of arcs flowing into *T*:
T = *GT* + *FT* = 74 + 29 = 103.
Hence, but the maximum flow – minimum cut theorem, this flow is maximal.

d To add the new limitation to the existing diagram, we split the vertex *C* into C_{in} and C_{out}. We connect all arcs flowing into *C* to C_{in} and all arcs flowing out of *C* to C_{out}. The arc between C_{in} and C_{out} will have the maximum capacity of 50.





e We need to adjust the flow at all arcs to satisfy the flow conservation. The easiest way to do it is to decrease the flow through AC_{in} to 20 and the flow through *FT* to 62.
So the new maximal flow is equal to:
Flow = *FT* + *GT* = 62 + 29 = 91

f