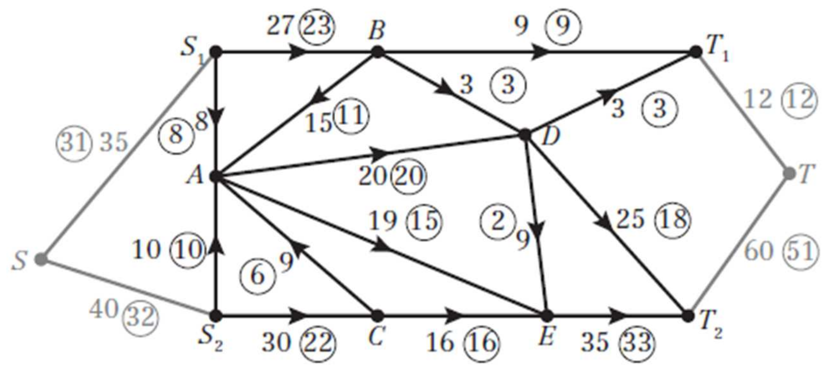
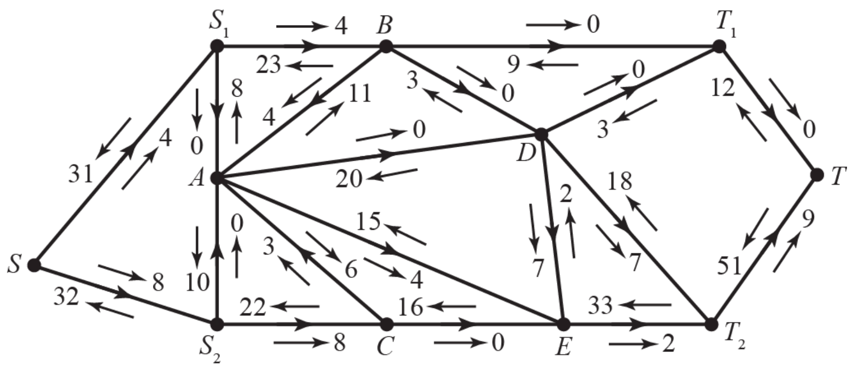
Flows in networks 2 4B

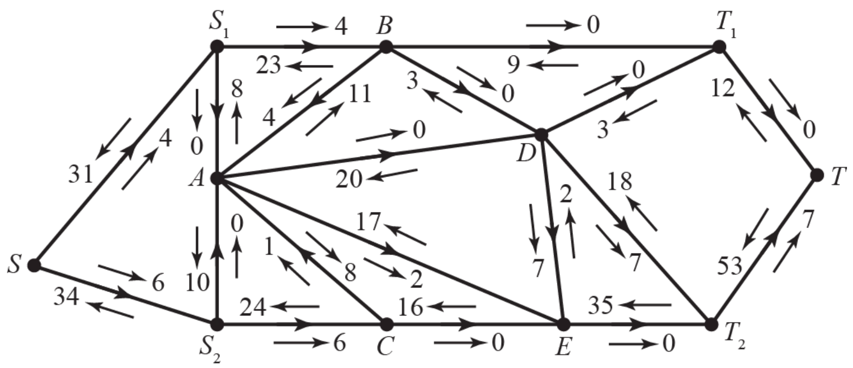
1 a Network with added super source and super sink, together with their corresponding flows and capacities:



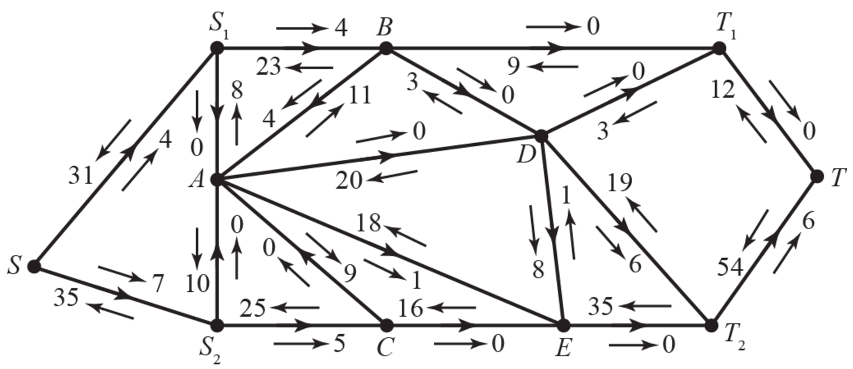
b The initial state of the network can be represented as follows:



We see a spare capacity of 2 along SS_2CAET_2T. Updated diagram:

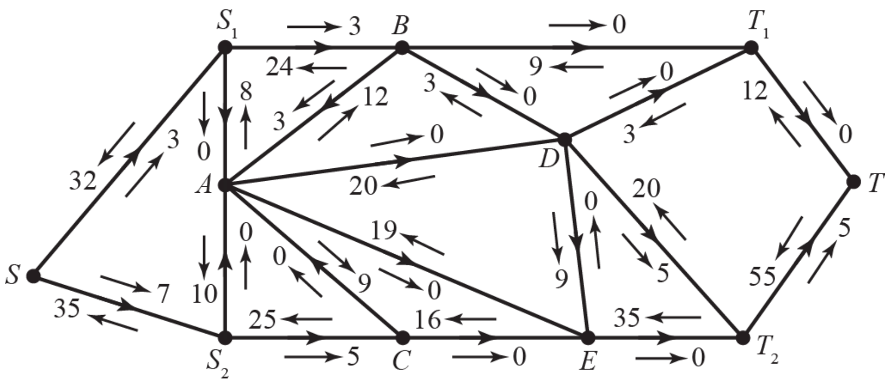


Now we can send additional 1 along SS_2CAEDT_2T:

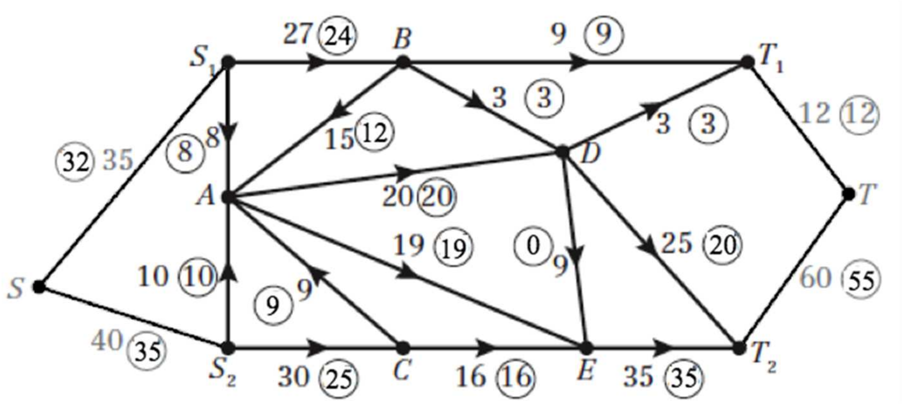


1 b (continued)

We can no longer send anything along SS_2 because there is no spare capacity in the direction of A, and if we went to C, we would also get stuck. But there's still a capacity of 1 along SS_1BAEDT_2T:

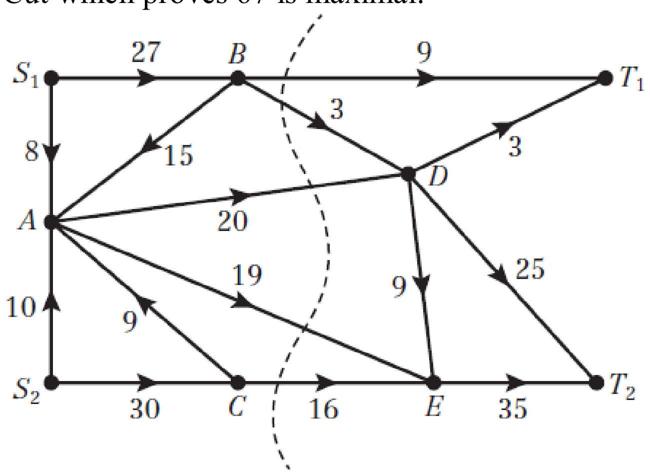


Further inspection shows that there are no more possibilities. The final network:

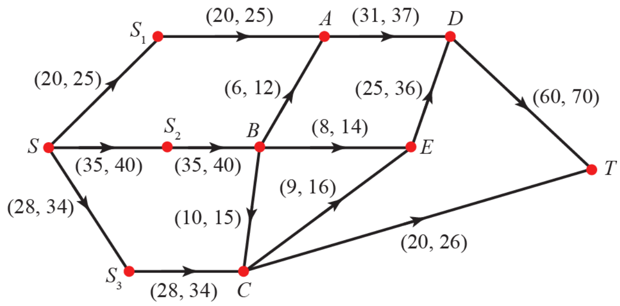


The current flow is $FLOW = SS_1 + SS_2 = 32 + 35 = 67$

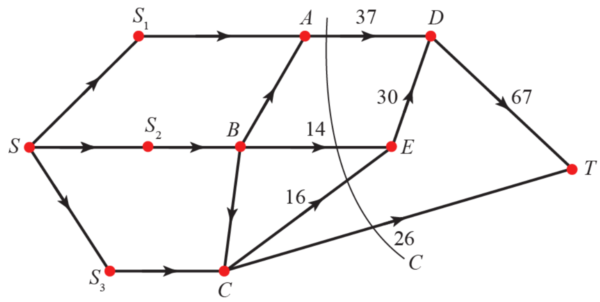
c Cut which proves 67 is maximal:



2 a The super source can be added as follows

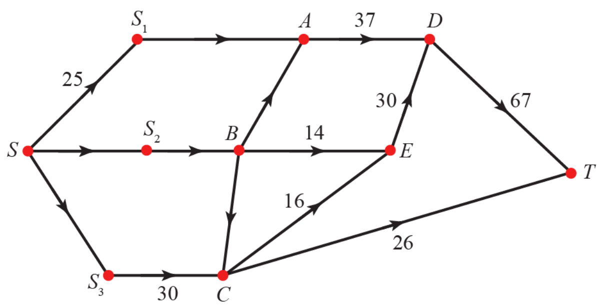


b Arcs *AD*, *BE, CE* and *CT* are saturated, i.e. $AD = 37$, $BE = 14$, $CE = 16$ and $CT = 26$.
Now, consider vertex *E*. By flow conservation, *ED* must be equal to $BE + CE$. So $ED = 30$. Next, consider vertex *D*. Again, using flow conservation, we must have *DT* equal to $AD + ED$. So $DT = 67$. The flow into *T* is now 93. Note that we can draw a cut through the saturated arcs *AD*, *BE, CE* and *CT* (see below).

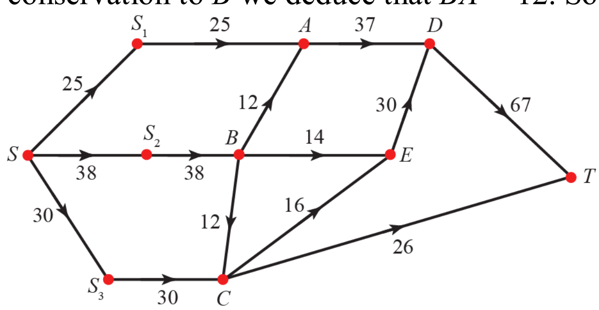


The value of this cut is equal to 93, so this flow is maximal by the maximum flow – minimum cut theorem.

c Let's start with adding in the flows calculated in part **b**

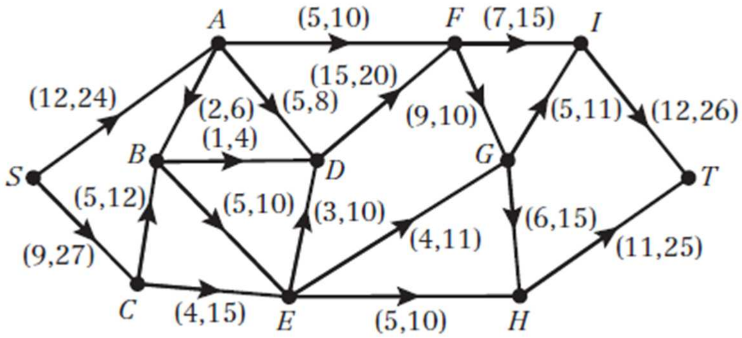


Next, consider vertex S_3. By flow conservation we must have $SS_3 = 30$. Similarly, by considering S_1, we find that $S_1A = 25$. Finally, because the flow through this network is 93, we must have that flow out of *S* is 93 as well. This means that SS_2 and S_2B are both equal to 38.
Next, by considering the flow conservation at *C* we deduce that $BC = 12$. Finally, applying flow conservation to *B* we deduce that $BA = 12$. So the flow is as follows:

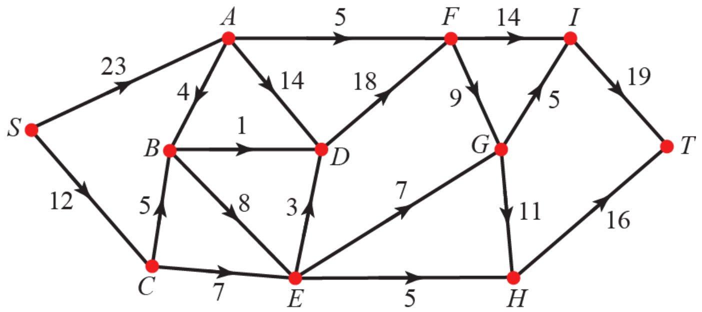


3 a To identify the sources we need to look for vertices which have no arcs flowing into them. These are *A* and *C*. Similarly, to identify sinks, we need vertices with no arcs flowing out of them. These are *H* and *I*.

b The super source needs to be able to supply at least as much as the lower capacities of sources it connects to and at most as much as their upper capacities. Similarly, the super sink will receive at least as much as the lower capacities of the sinks it connects to, and at most as much as their upper capacities. Thus:

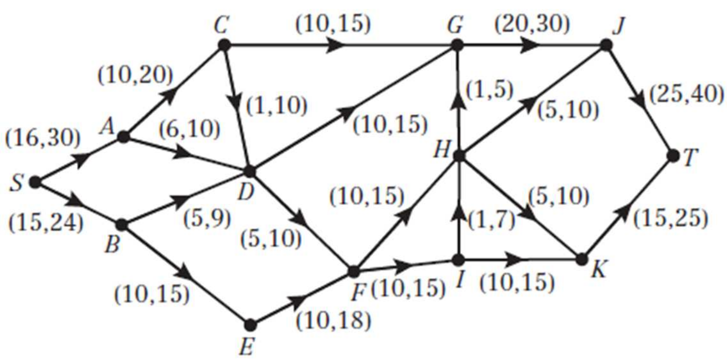


c An example of a flow of 35:

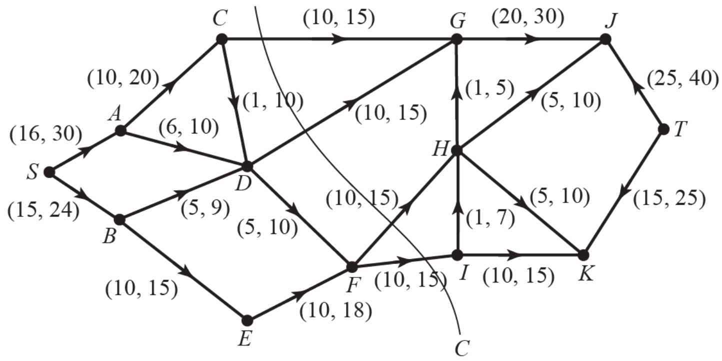


Note that there are other possibilities.

4 a The super source needs to be able to supply at least as much as the lower capacities of sources it connects to and at most as much as their upper capacities. Similarly, the super sink will receive at least as much as the lower capacities of the sinks it connects to, and at most as much as their upper capacities. Thus:



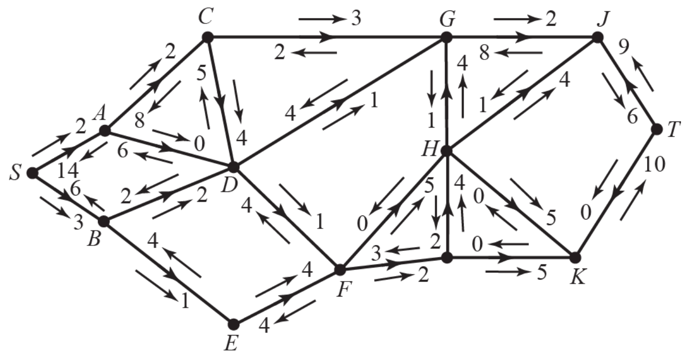
4 b The required cut divides the network as follows:



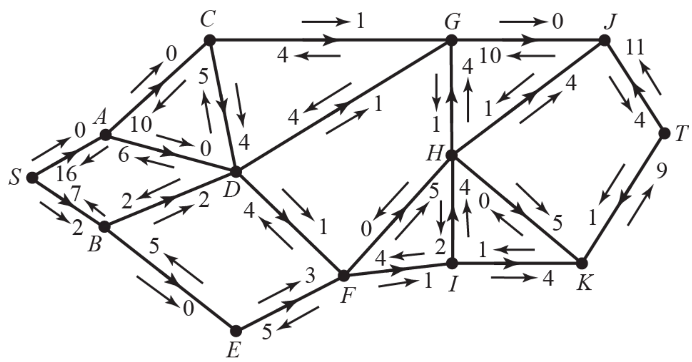
Its capacity is equal to the sum of upper capacities of arcs flowing **into** the cut, minus the sum of lower capacities of arcs flowing **out of** the cut. Thus:
$C = CG_{max} + DG_{max} + FH_{max} + FI_{max} = 60$

c Using the maximum flow – minimum cut theorem we deduce that the maximum flow is at most 60 (there might exist a smaller cut somewhere).

d Consider this network with added super source and super sink
There's a capacity of 2 along *SACGJT*. Updated network:

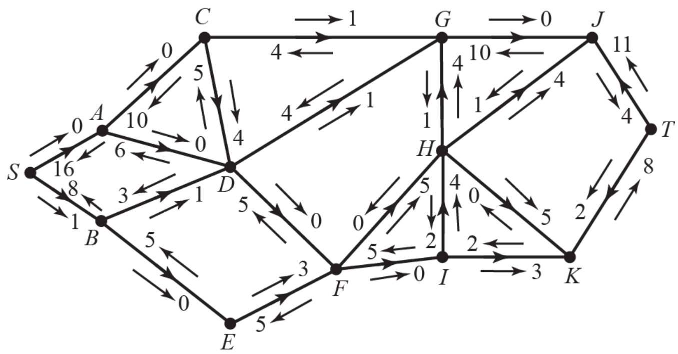


Now, there's no more capacity left through *SA*, so we have to follow *SB*. There's a capacity of 1 along *SBEFIKT*. Updated network:

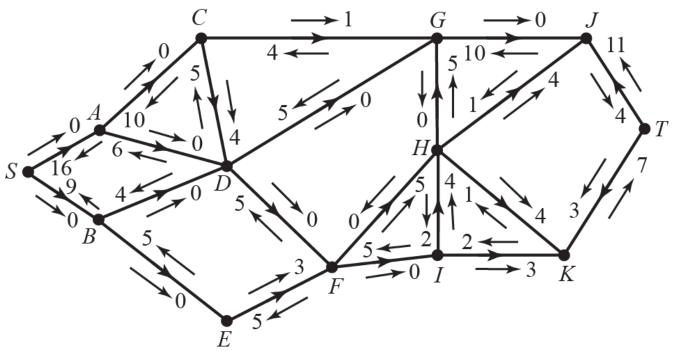


4 d (continued)

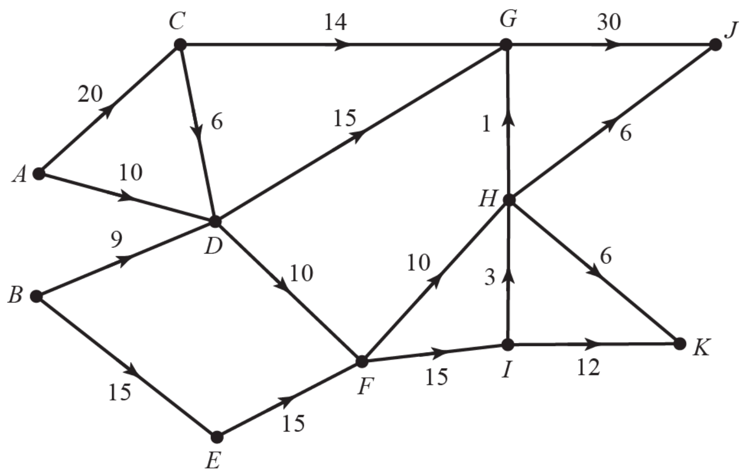
BE is now saturated, so we have to follow *BD*. From there, there are several routes of capacity 1 that could be chosen, we follow along *SBDFIKT*. Updated network:



There is still a capacity of 1 remaining along *SB*. There are several routes which could be chosen here, we follow *SBDGHKT*. Updated network:



Now we see that all arcs leaving the super source *S* (and thus all arcs leaving the original sources *A* and *B*) are saturated, so we cannot send more fuel through this network. Final network has flow 54:



e As mentioned in part **d**, all arcs leaving the sources are saturated, so this must be the maximal flow. This can also be seen be drawing a cut through *AC*, *AD*, *BD*, and *BE*, so the maximal flow is given by 20 + 10 + 9 + 15 = 54.