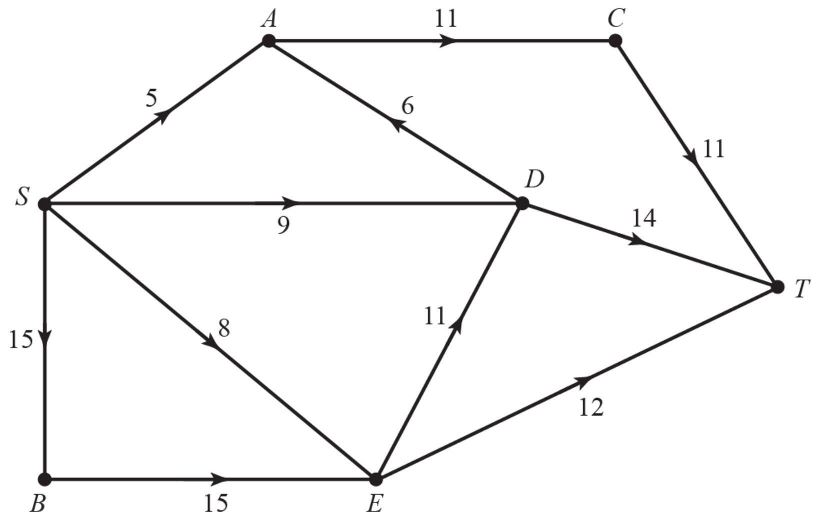
Flows in networks 2 4A

1 **a** The maximum flow out of B is equal to the maximum flow through BC plus the maximum flow through BD. So the maximum flow out of B is 7. It is also the minimum flow into B. Thus, by the conservation condition, the flows must me as follows:
$AB = 7$, $BC = 4$, $BD = 3$

b The minimum flow out of Q is equal to the minimum flow through QR, i.e. 15. The maximum flow into Q is equal to the sum of maximum flows through SQ and PQ, i.e. 15. Thus, by the conservation condition, the flows must be as follows:
$PQ = 7$, $SQ = 8$, $QR = 15$

c The maximum flow into B is 10, which is equal to the sum of minimum flows out of B.
Thus $AB = 10$, $BD = 4$, $BE = 6$. Now, the minimum flow out of D is 8. Since $BD = 4$ and the maximum flow of $CD = 4$, it must be that $CD = 4$ and $DF = 8$. The flow out of E = the flow into E, so $EF = 6$. The flow out of F = the flow into F, so $FG = 6 + 8 = 14$.

2



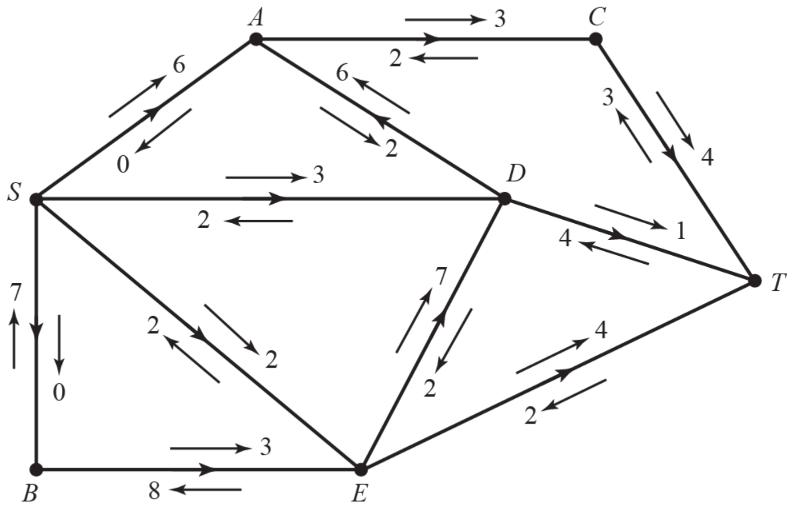
a The value of a cut is equal to the sum of maximum capacities flowing **into** the cut minus the sum of minimum capacities flowing **out of** the cut. Hence:

$C_1 = AD_{max} + BD_{max} + ET_{max} - DE_{min}$
$= 20 + 16 + 12 - 6 = 42$

$C_2 = SA_{max} + SB_{max} + CE_{max}$
$= 10 + 12 + 17 = 39$

b By the maximum flow – minimum cut principle, we deduce that the maximum flow is at most 39.

2 c The maximum flow into B is equal to the sum of maximum flows through EB and SB, i.e. 20. The minimum flow out of B is equal to the sum of minimum flows through BA and BD, which is also 20. Since the maximum flow into B equals the minimum flow out of B, it must be:
$BA = 9$, $BD = 11$, $SB = 12$, $BE = 8$.



3 a Consider vertex C. The minimum flow into this vertex is equal to $SC_{min} + BC_{min} = 35$, whereas the maximum flow out of C is equal to 27. Hence there is no feasible flow.

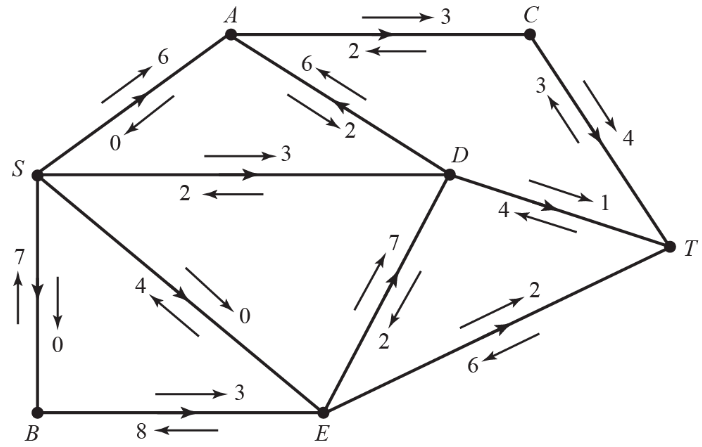
b i Based on part **a** we know that the capacity of arc CE would have to be increased to make the flow through this network possible.

ii Again, referring to part **a** we see that the minimum required value of the upper capacity of CE is 35.

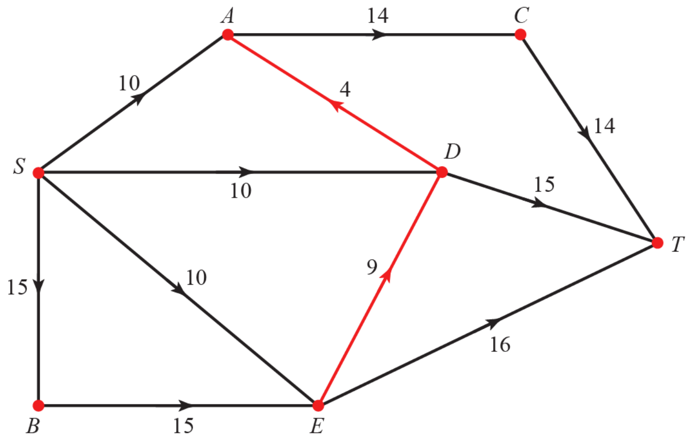
4 a Consider vertex D. Currently, the flow into D is equal to 20 whereas the flow out of D is equal to 6. Thus, to satisfy the conservation condition it must be $DT = 14$. Next, consider vertex A. Flow into A is 11, so flow out of A must also be 11. Thus $AC = 11$. Now, consider vertex C. Flow into C is equal to 11, so the flow out of C must also be 11. Thus $CT = 11$. Now, consider vertex E. Flow out of E equals 23, current flow into E is 8. So, to satisfy the conservation condition, it must be that $BE = 15$. Lastly, by considering vertex B we see that to satisfy the conservation condition we must have $SB = 15$.

4 b The initial network can be represented as follows.

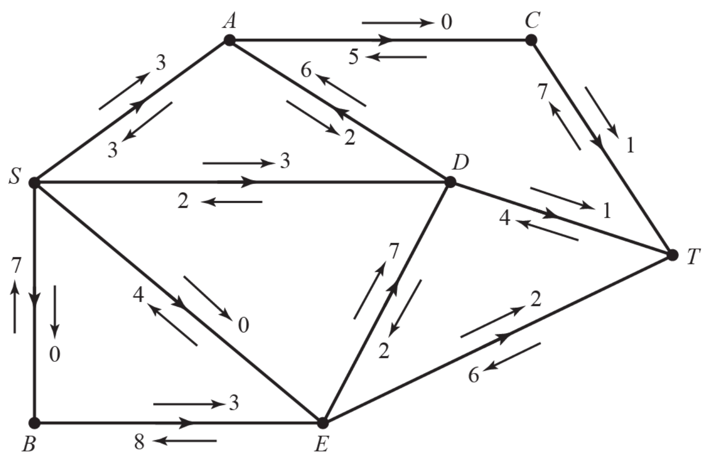
We see that there is no capacity through *SB*, but there is a capacity of 2 at *SET*. The updated network looks as follows:



Next, we see a spare capacity of 3 through *SACT*. Updated network:

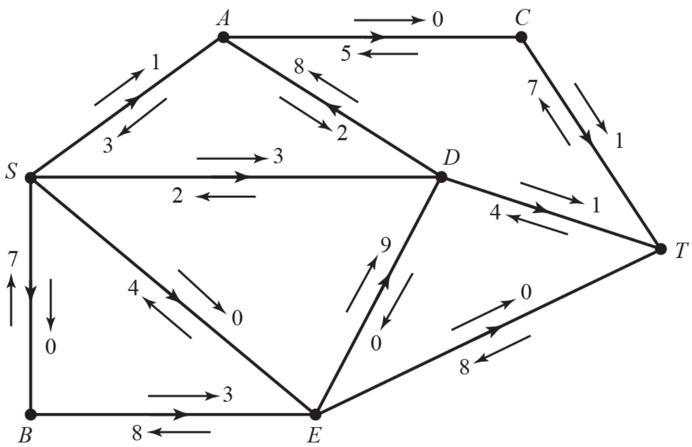


Since *SA* is not yet saturated, we find a spare capacity of 2 through *SADET*. Updated network:

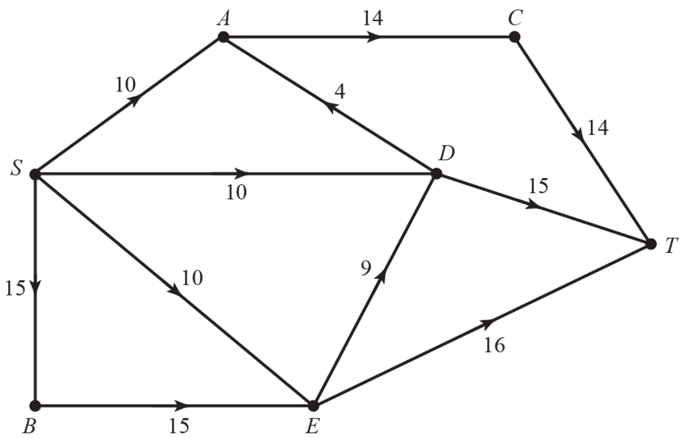


4 b (continued)

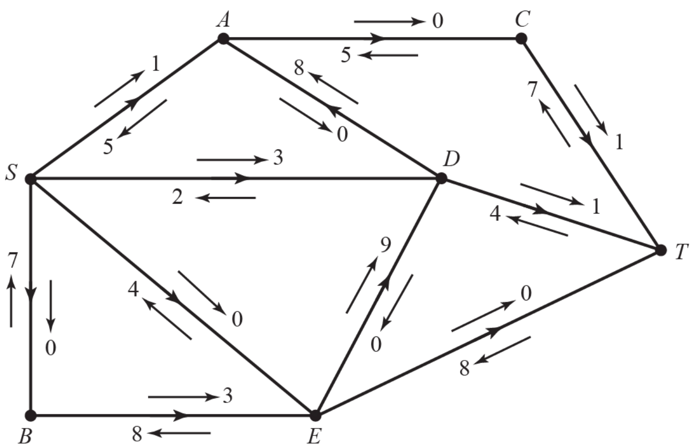
Now, *SA* is still not saturated, but both arcs flowing out of *A* are, so there is no more spare capacity in this direction. It remains to examine arc *SD*. We see spare capacity of 1 through *SDT*. Updated network:



Further examination shows that there are no more routes with spare capacities remaining.

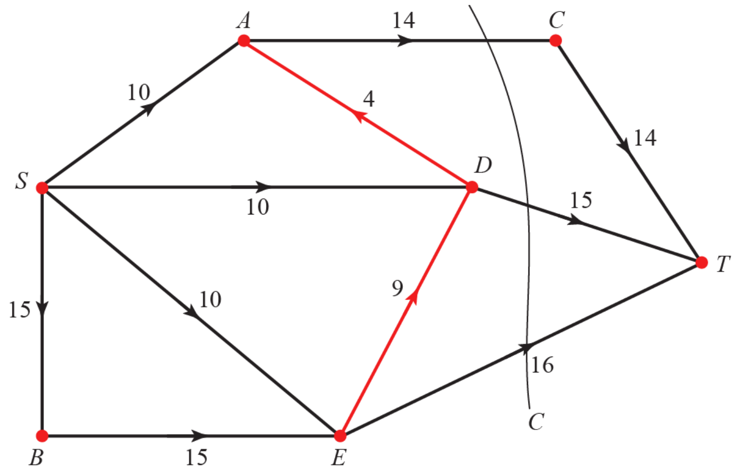


Thus this is the maximum flow through this network. Final flow looks as follows:



The maximum flow is equal to the sum of flows out of *S* or into *T*, which are both equal to 45.

4 c To prove that the flow is maximal, start with identifying all saturated arcs (purple) and arcs at minimum flow (red).
Next, draw a cut through AC, DT, ET.



The value of this cut is equal to 45, which is the same as the flow through this network. Thus, by the maximum flow – minimum cut theorem, this flow is maximal.

