Allocation problems 2E

1 The initial cost matrix is shown below (with the numbers representing minutes):

	Task *C*	**Task *D***	**Task *E***
Worker *L*	37	15	12
Worker *M*	25	13	16
Worker *N*	32	41	35

The following linear programming problem can be formulated to minimise the total time taken.

Decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{L, M, N\}$ and $j \in \{C, D, E\}$

Objective function:
To minimise the total time (in minutes) taken P.

$$\begin{aligned} \text{Minimise } P = 37x_{LC} &+ 15x_{LD} + 12x_{LE} \\ + 25x_{MC} &+ 13x_{MD} + 16x_{ME} \\ + 32x_{NC} &+ 41x_{ND} + 35x_{NE} \end{aligned}$$

Subject to the following constraints:

Each worker can be assigned to at most one task:

$x_{LC} + x_{LD} + x_{LE} = 1$ or $\sum x_{Lj} = 1$

$x_{MC} + x_{MD} + x_{ME} = 1$ or $\sum x_{Mj} = 1$

$x_{NC} + x_{ND} + x_{NE} = 1$ or $\sum x_{Nj} = 1$

Each task must be done by just one worker:

$x_{LC} + x_{MC} + x_{NC} = 1$ or $\sum x_{iC} = 1$

$x_{LD} + x_{MD} + x_{ND} = 1$ or $\sum x_{iD} = 1$

$x_{LE} + x_{ME} + x_{NE} = 1$ or $\sum x_{iE} = 1$

2 The initial cost matrix is shown below (with the numbers representing hundreds of pounds):

	Task *S*	**Task *T***	**Task *U***	**Task *V***
Worker *C*	36	34	32	35
Worker *D*	37	32	34	33
Worker *E*	42	35	37	36
Worker *F*	39	34	35	35

The following linear programming problem can be formulated to minimise the total training cost.

Decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ is trained in task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{C, D, E, F\}$ and $j \in \{S, T, U, V\}$

Objective function:
To minimise the total training cost (in hundreds of pounds) P.

$$\text{Minimise } P = 36x_{CS} + 34x_{CT} + 32x_{CU} + 35x_{CV} + 37x_{DS} + 32x_{DT} + 34x_{DU} + 33x_{DV} + 42x_{ES} + 35x_{ET} + 37x_{EU} + 36x_{EV} + 39x_{FS} + 34x_{FT} + 35x_{FU} + 35x_{FV}$$

Subject to the following constraints:

Each worker is to be trained in exactly one task:

$x_{CS} + x_{CT} + x_{CU} + x_{CV} = 1$ or $\sum x_{Cj} = 1$

$x_{DS} + x_{DT} + x_{DU} + x_{DV} = 1$ or $\sum x_{Dj} = 1$

$x_{ES} + x_{ET} + x_{EU} + x_{EV} = 1$ or $\sum x_{Ej} = 1$

$x_{FS} + x_{FT} + x_{FU} + x_{FV} = 1$ or $\sum x_{Fj} = 1$

Each task must have one worker trained to carry it out:

$x_{CS} + x_{DS} + x_{ES} + x_{FS} = 1$ or $\sum x_{iS} = 1$

$x_{CT} + x_{DT} + x_{ET} + x_{FT} = 1$ or $\sum x_{iT} = 1$

$x_{CU} + x_{DU} + x_{EU} + x_{FU} = 1$ or $\sum x_{iU} = 1$

$x_{CV} + x_{DV} + x_{EV} + x_{FV} = 1$ or $\sum x_{iV} = 1$

3 The initial cost matrix is shown below (with the figures representing the number of names and addresses):

	1	**2**	**3**
A	11	15	–
B	14	18	17
C	16	13	23
D	15	14	22

Replacing the '–' entry with a large value of 1000 gives:

	1	**2**	**3**
A	11	15	1000
B	14	18	17
C	16	13	23
D	15	14	22

The following linear programming problem can be formulated to maximise the number of names and addresses collected.
Decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ is assigned to site } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{A, B, C, D\}$ and $j \in \{1, 2, 3\}$

Objective function:
To maximise the number of names and addresses collected P.

$$\begin{aligned} \text{Maximise } P = 11x_{A1} &+ 15x_{A2} \\ &+ 14x_{B1} + 18x_{B2} + 17x_{B3} \\ &+ 16x_{C1} + 13x_{C2} + 23x_{C3} \\ &+ 15x_{D1} + 14x_{D2} + 22x_{D3} \end{aligned}$$

Subject to the following constraints:

Each worker must be assigned to just one site:

$x_{A1} + x_{A2} + x_{A3} = 1$ or $\sum x_{Aj} = 1$

$x_{B1} + x_{B2} + x_{B3} = 1$ or $\sum x_{Bj} = 1$

$x_{C1} + x_{C2} + x_{C3} = 1$ or $\sum x_{Cj} = 1$

$x_{D1} + x_{D2} + x_{D3} = 1$ or $\sum x_{Dj} = 1$

Each site must be assigned to just one worker:

$x_{A1} + x_{B1} + x_{C1} + x_{D1} = 1$ or $\sum x_{i1} = 1$

$x_{A2} + x_{B2} + x_{C2} + x_{D2} = 1$ or $\sum x_{i2} = 1$

$x_{A3} + x_{B3} + x_{C3} + x_{D3} = 1$ or $\sum x_{i3} = 1$

4 a The initial cost matrix is shown below (with the numbers representing £100s):

	W	***X***	***Y***	***Z***
A	12	8	11	9
B	14	10	9	13
C	11	9	12	10
D	13	11	10	12

As the table above is an n by n matrix (with $n = 4$), we do not need to add any additional dummy rows or columns.
To maximise the profit, subtract every number from the largest value in the table.

b The largest value in the table is 14. Subtracting every value from 14 the table becomes:

	W	***X***	***Y***	***Z***
A	2	6	3	5
B	0	4	5	1
C	3	5	2	4
D	1	3	4	2

The smallest numbers in rows 1, 2, 3 and 4 are 2, 0, 2 and 1. We reduce rows first by subtracting these numbers from each element in the row. The table becomes:

	W	***X***	***Y***	***Z***
A	0	4	1	3
B	0	4	5	1
C	1	3	0	2
D	0	2	3	1

The smallest numbers in columns 1, 2, 3 and 4 are 0, 2, 0 and 1. We reduce columns by subtracting these numbers from each element in the column. Therefore, the reduced cost matrix is:

	W	***X***	***Y***	***Z***
A	0	2	1	2
B	0	2	5	0
C	1	1	0	1
D	0	0	3	0

We can cover all the zeros with four lines, so we have found our optimal solution of:

$A - W$ (12)
$B - Z$ (13)
$C - Y$ (12)
$D - X$ (11)

Maximum profit = $100 \times (12 + 13 + 12 + 11) = 100 \times 48 = £4800$

4 c Using the initial cost matrix below the following linear programming problem can be formulated to maximise the amount of profit made.

	W	***X***	***Y***	***Z***
A	12	8	11	9
B	14	10	9	13
C	11	9	12	10
D	13	11	10	12

Decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{A, B, C, D\}$ and $j \in \{W, X, Y, Z\}$

Objective function:
To maximise the total profit (in £100s) P.

$$\begin{aligned} \text{Maximise } P = & 12x_{AW} + 8x_{AX} + 11x_{AY} + 9x_{AZ} \\ & + 14x_{BW} + 10x_{BX} + 9x_{BY} + 13x_{BZ} \\ & + 11x_{CW} + 9x_{CX} + 12x_{CY} + 10x_{CZ} \\ & + 13x_{DW} + 11x_{DX} + 10x_{DY} + 12x_{DZ} \end{aligned}$$

Subject to the following constraints:

Each worker can be assigned to at most one task:

$x_{AW} + x_{AX} + x_{AY} + x_{AZ} = 1$ or $\sum x_{Aj} = 1$

$x_{BW} + x_{BX} + x_{BY} + x_{BZ} = 1$ or $\sum x_{Bj} = 1$

$x_{CW} + x_{CX} + x_{CY} + x_{CZ} = 1$ or $\sum x_{Cj} = 1$

$x_{DW} + x_{DX} + x_{DY} + x_{DZ} = 1$ or $\sum x_{Dj} = 1$

Each task must be done by just one worker:

$x_{AW} + x_{BW} + x_{CW} + x_{DW} = 1$ or $\sum x_{iW} = 1$

$x_{AX} + x_{BX} + x_{CX} + x_{DX} = 1$ or $\sum x_{iX} = 1$

$x_{AY} + x_{BY} + x_{CY} + x_{DY} = 1$ or $\sum x_{iY} = 1$

$x_{AZ} + x_{BZ} + x_{CZ} + x_{DZ} = 1$ or $\sum x_{iZ} = 1$

Alternatively, the linear programming problem on the next page shows the problem formulated as a minimisation problem using the modified cost matrix found in part b.

4 c (continued)

Modified cost matrix from part b:

	W	**X**	**Y**	**Z**
A	2	6	3	5
B	0	4	5	1
C	3	5	2	4
D	1	3	4	2

Decision variables:

Let $x_{ij} = \begin{cases} 1 \text{ if worker } i \text{ does task } j \\ 0 \text{ otherwise} \end{cases}$

where $i \in \{A, B, C, D\}$ and $j \in \{W, X, Y, Z\}$

Objective:
To minimise the objective function P.

Minimise $P = 2x_{AW} + 6x_{AX} + 3x_{AY} + 5x_{AZ}$
$+ 4x_{BX} + 5x_{BY} + x_{BZ}$
$+ 3x_{CW} + 5x_{CX} + 2x_{CY} + 4x_{CZ}$
$+ x_{DW} + 3x_{DX} + 4x_{DY} + 2x_{DZ}$

Subject to the following constraints:

Each worker can be assigned to at most one task:

$x_{AW} + x_{AX} + x_{AY} + x_{AZ} = 1$ or $\sum x_{Aj} = 1$
$x_{BW} + x_{BX} + x_{BY} + x_{BZ} = 1$ or $\sum x_{Bj} = 1$
$x_{CW} + x_{CX} + x_{CY} + x_{CZ} = 1$ or $\sum x_{Cj} = 1$
$x_{DW} + x_{DX} + x_{DY} + x_{DZ} = 1$ or $\sum x_{Dj} = 1$

Each task must be done by just one worker:

$x_{AW} + x_{BW} + x_{CW} + x_{DW} = 1$ or $\sum x_{iW} = 1$
$x_{AX} + x_{BX} + x_{CX} + x_{DX} = 1$ or $\sum x_{iX} = 1$
$x_{AY} + x_{BY} + x_{CY} + x_{DY} = 1$ or $\sum x_{iY} = 1$
$x_{AZ} + x_{BZ} + x_{CZ} + x_{DZ} = 1$ or $\sum x_{iZ} = 1$