Allocation problems 2C

1 $\begin{pmatrix} 37 & 15 & 12 \\ 25 & 13 & 16 \\ 32 & 41 & 35 \end{pmatrix}$ Subtracting all terms from 41 $\begin{pmatrix} 4 & 26 & 29 \\ 16 & 28 & 25 \\ 9 & 0 & 6 \end{pmatrix}$

reducing rows $\begin{pmatrix} 0 & 22 & 25 \\ 0 & 12 & 9 \\ 9 & 0 & 6 \end{pmatrix}$ reducing columns $\begin{pmatrix} 0 & 22 & 19 \\ 0 & 12 & 3 \\ 9 & 0 & 0 \end{pmatrix}$

Minimum uncovered element is 3 $\begin{pmatrix} 0 & 19 & 16 \\ 0 & 9 & 0 \\ 3 & 0 & 0 \end{pmatrix}$

Solution L – C (37)
M – E (16) profit £94
N – D (41)

2 $\begin{pmatrix} 36 & 34 & 32 & 35 \\ 37 & 32 & 34 & 33 \\ 42 & 35 & 37 & 36 \\ 39 & 34 & 35 & 35 \end{pmatrix}$ Subtracting all terms from 42 $\begin{pmatrix} 6 & 8 & 10 & 7 \\ 5 & 10 & 8 & 9 \\ 0 & 7 & 5 & 6 \\ 3 & 8 & 7 & 7 \end{pmatrix}$

reducing rows $\begin{pmatrix} 0 & 2 & 4 & 1 \\ 0 & 5 & 3 & 4 \\ 0 & 7 & 5 & 6 \\ 0 & 5 & 4 & 4 \end{pmatrix}$ reducing columns $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 5 & 2 & 5 \\ 0 & 3 & 1 & 3 \end{pmatrix}$

Minimum uncovered element is 3 $\begin{pmatrix} 3 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

There are two solution

C – T (34)		C – V (35)
D – U (34)	or	D – U (34)
E – S (42)		E – S (42) profit £145
F – V (35)		F – T (34)

3 $\begin{pmatrix} 20 & 22 & 14 & 24 \\ 20 & 19 & 12 & 20 \\ 13 & 10 & 18 & 16 \\ 22 & 23 & 9 & 28 \end{pmatrix}$ Subtracting all terms from 28 $\begin{pmatrix} 8 & 6 & 14 & 4 \\ 8 & 9 & 16 & 8 \\ 15 & 18 & 10 & 12 \\ 6 & 5 & 19 & 0 \end{pmatrix}$

reducing rows $\begin{pmatrix} 4 & 2 & 10 & 0 \\ 0 & 1 & 8 & 0 \\ 5 & 8 & 0 & 2 \\ 6 & 5 & 19 & 0 \end{pmatrix}$ reducing columns $\begin{pmatrix} 4 & 1 & 10 & 0 \\ 0 & 0 & 8 & 0 \\ 5 & 7 & 0 & 2 \\ 6 & 4 & 19 & 0 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 3 & 0 & 9 & 0 \\ 0 & 0 & 8 & 1 \\ 5 & 7 & 0 & 3 \\ 5 & 3 & 18 & 0 \end{pmatrix}$

Solution R – F (22)
S – E (20) profit £88
T – G (18)
U – H (28)

4 a The initial cost matrix is shown below:

	Farm	Research	Build	Mine	Explore
Unit *A*	85	95	86	87	97
Unit *B*	110	111	95	115	100
Unit *C*	90	95	86	93	105
Unit *D*	85	87	84	85	87
Unit *E*	100	100	105	120	95

As the table above is an n by n matrix (with $n = 5$), we do not need to add any additional dummy rows or columns.
To maximise the total number of points scored, subtract every number from the largest value in the table.

b The largest value in the table is 120. Subtracting every value from 120 the table becomes:

	Farm	Research	Build	Mine	Explore
Unit *A*	35	25	34	33	23
Unit *B*	10	9	25	5	20
Unit *C*	30	25	34	27	15
Unit *D*	35	33	36	35	33
Unit *E*	20	20	15	0	25

The smallest numbers in rows 1, 2, 3, 4 and 5 are 23, 5, 15, 33 and 0. We reduce rows first by subtracting these numbers from each element in the row. The table becomes:

4 b (continued)

	Farm	Research	Build	Mine	Explore
Unit *A*	12	2	11	10	0
Unit *B*	5	4	20	0	15
Unit *C*	15	10	19	12	0
Unit *D*	2	0	3	2	0
Unit *E*	20	20	15	0	25

The smallest numbers in columns 1, 2, 3, 4 and 5 are 2, 0, 3, 0 and 0. We reduce columns by subtracting these numbers from each element in the column. Therefore, the reduced cost matrix is:

	Farm	Research	Build	Mine	Explore
Unit *A*	10	2	8	10	0
Unit *B*	3	4	17	0	15
Unit *C*	13	10	16	12	0
Unit *D*	0	0	0	2	0
Unit *E*	18	20	12	0	25

We now apply the Hungarian algorithm.

We can cover all the zeros in three lines, so the solution is not optimal:

	Farm	Research	Build	Mine	Explore
Unit *A*	10	2	8	10	0
Unit *B*	3	4	17	0	15
Unit *C*	13	10	16	12	0
Unit *D*	0	0	0	2	0
Unit *E*	18	20	12	0	25

	Farm	Research	Build	Mine	Explore
Unit *A*	10	2	8	10	0
Unit *B*	3	4	17	0	15
Unit *C*	13	10	16	12	0
Unit *D*	0	0	0	2	0
Unit *E*	18	20	12	0	25

The smallest uncovered element is 2, so we augment the matrix as follows:

- Add 2 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 2 from the uncovered elements

This gives the following matrix:

4 b (continued)

	Farm	Research	Build	Mine	Explore
Unit *A*	8	0	6	10	0
Unit *B*	1	2	15	0	15
Unit *C*	11	8	14	12	0
Unit *D*	0	0	0	4	2
Unit *E*	16	18	10	0	25

We can cover all the zeros in four lines, so the solution is not optimal:

	Farm	Research	Build	Mine	Explore
Unit *A*	8	0	6	10	0
Unit *B*	1	2	15	0	15
Unit *C*	11	8	14	12	0
Unit *D*	0	0	0	4	2
Unit *E*	16	18	10	0	25

	Farm	Research	Build	Mine	Explore
Unit *A*	8	0	6	10	0
Unit *B*	1	2	15	0	15
Unit *C*	11	8	14	12	0
Unit *D*	0	0	0	4	2
Unit *E*	16	18	10	0	25

The smallest uncovered element is 1, so we augment the matrix as follows:

- Add 1 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 1 from the uncovered elements

This gives the following matrix:

	Farm	Research	Build	Mine	Explore
Unit *A*	8	0	6	11	1
Unit *B*	0	1	14	0	15
Unit *C*	10	7	13	12	0
Unit *D*	0	0	0	5	3
Unit *E*	15	17	9	0	25

We can cover all the zeros with five lines, so we have found our optimal solution of:

Unit *A* – Research (95)
Unit *B* – Farm (110)
Unit *C* – Explore (105)
Unit *D* – Build (84)
Unit *E* – Mine (120)

Maximum number of points scored $= 95 + 110 + 105 + 84 + 120 = 514$.

5 a The initial cost matrix is shown below:

	V	*W*	*X*	*Y*	*Z*
A	74	68	123	86	95
B	79	75	111	92	93
C	68	72	134	96	99
D	78	67	109	88	92
E	70	78	130	81	88

As the table above is an n by n matrix (with $n = 5$), we do not need to add any additional dummy rows or columns.
To maximise the total profit, subtract every number from the largest value in the table.
The largest value in the table is 134. Subtracting every value from 134 the table becomes:

	V	*W*	*X*	*Y*	*Z*
A	60	66	11	48	39
B	55	59	23	42	41
C	66	62	0	38	35
D	56	67	25	46	42
E	64	56	4	53	46

The smallest numbers in rows 1, 2, 3, 4 and 5 are 11, 23, 0, 25 and 4. We reduce rows first by subtracting these numbers from each element in the row. The table becomes:

	V	*W*	*X*	*Y*	*Z*
A	49	55	0	37	28
B	32	36	0	19	18
C	66	62	0	38	35
D	31	42	0	21	17
E	60	52	0	49	42

The smallest numbers in columns 1, 2, 3, 4 and 5 are 31, 36, 0, 19 and 17. We reduce columns by subtracting these numbers from each element in the column. Therefore, the reduced cost matrix is:

	V	*W*	*X*	*Y*	*Z*
A	18	19	0	18	11
B	1	0	0	0	1
C	35	26	0	19	18
D	0	6	0	2	0
E	29	16	0	30	25

We now apply the Hungarian algorithm.

5 a (continued)

We can cover all the zeros in three lines, so the solution is not optimal:

	V	*W*	*X*	*Y*	*Z*
A	18	19	0	18	11
B	1	0	0	0	1
C	35	26	0	19	18
D	0	6	0	2	0
E	29	16	0	30	25

	V	*W*	*X*	*Y*	*Z*
A	18	19	0	18	11
B	1	0	0	0	1
C	35	26	0	19	18
D	0	6	0	2	0
E	29	16	0	30	25

The smallest uncovered element is 11, so we augment the matrix as follows:

- Add 11 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 11 from the uncovered elements

This gives the following matrix:

	V	*W*	*X*	*Y*	*Z*
A	7	8	0	7	0
B	1	0	11	0	1
C	24	15	0	8	7
D	0	6	11	2	0
E	18	5	0	19	14

We can cover all the zeros in four lines, so the solution is not optimal:

	V	*W*	*X*	*Y*	*Z*
A	7	8	0	7	0
B	1	0	11	0	1
C	24	15	0	8	7
D	0	6	11	2	0
E	18	5	0	19	14

	V	*W*	*X*	*Y*	*Z*
A	7	8	0	7	0
B	1	0	11	0	1
C	24	15	0	8	7
D	0	6	11	2	0
E	18	5	0	19	14

5 a (continued)

The smallest uncovered element is 5, so we augment the matrix as follows:
- Add 5 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 5 from the uncovered elements

This gives the following matrix:

	V	***W***	***X***	***Y***	***Z***
A	2	3	0	2	0
B	1	0	16	0	6
C	19	10	0	3	7
D	0	6	16	2	5
E	13	0	0	14	14

We can cover all the zeros with five lines, so we have found our optimal solution of:

$A - Z$ (95)
$B - Y$ (92)
$C - X$ (134)
$D - V$ (78)
$E - W$ (78)

b Maximum profit = 95 + 92 + 134 + 78 + 78 = £477.