Allocation problems 2B

1 $\begin{pmatrix} 23 & 26 & 0 \\ 26 & 30 & 0 \\ 29 & 28 & 0 \end{pmatrix}$ reducing columns $\begin{pmatrix} 0 & 0 & 0 \\ 3 & 4 & 0 \\ 6 & 2 & 0 \end{pmatrix}$

Minimum uncovered element is 2 $\begin{pmatrix} 0 & 0 & 2 \\ 1 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix}$

Solution: J – M (23)
K – dummy cost: £51
L – N (28)

2 $\begin{pmatrix} 31 & 43 & 19 & 35 \\ 28 & 46 & 10 & 34 \\ 24 & 42 & 13 & 33 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow$ reducing rows $\begin{pmatrix} 12 & 24 & 0 & 16 \\ 18 & 36 & 0 & 24 \\ 11 & 29 & 0 & 20 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Minimum uncovered element is 11 $\begin{pmatrix} 1 & 13 & 0 & 5 \\ 7 & 25 & 0 & 13 \\ 0 & 18 & 0 & 9 \\ 0 & 0 & 11 & 0 \end{pmatrix}$

Minimum uncovered element is 5 $\begin{pmatrix} 1 & 8 & 0 & 0 \\ 7 & 20 & 0 & 8 \\ 0 & 13 & 0 & 4 \\ 5 & 0 & 16 & 0 \end{pmatrix}$

Solution:
A – Z (35)
B – Y (10)
C – W (24) cost: £69
dummy – X (0)

Alternative solution

Minimum uncovered element is 1 $\begin{pmatrix} 0 & 12 & 0 & 4 \\ 6 & 24 & 0 & 12 \\ 0 & 18 & 1 & 9 \\ 0 & 0 & 12 & 0 \end{pmatrix}$

then

Minimum uncovered element is 4 $\begin{pmatrix} 0 & 8 & 0 & 0 \\ 6 & 20 & 0 & 8 \\ 0 & 14 & 1 & 5 \\ 4 & 0 & 16 & 0 \end{pmatrix}$

3 $\begin{pmatrix} 81 & 45 & 55 & 0 \\ 67 & 32 & 48 & 0 \\ 87 & 38 & 58 & 0 \\ 73 & 37 & 60 & 0 \end{pmatrix}$ reducing columns $\begin{pmatrix} 14 & 13 & 7 & 0 \\ 0 & 0 & 0 & 0 \\ 20 & 6 & 10 & 0 \\ 6 & 5 & 12 & 0 \end{pmatrix}$

Minimum uncovered element is 5 $\begin{pmatrix} 9 & 8 & 2 & 0 \\ 0 & 0 & 0 & 5 \\ 15 & 1 & 5 & 0 \\ 1 & 0 & 7 & 0 \end{pmatrix}$

3 (continued)

Minimum uncovered element is 1 $\begin{pmatrix} 8 & 8 & 1 & 0 \\ 0 & 1 & 0 & 6 \\ 14 & 1 & 4 & 9 \\ 0 & 0 & 6 & 0 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 7 & 7 & 0 & 0 \\ 0 & 2 & 0 & 7 \\ 13 & 0 & 3 & 0 \\ 0 & 0 & 6 & 1 \end{pmatrix}$

There are two solutions

W – dummy	W – T (55)	
X – T (48)	X – R (67)	cost £159
Y – S (38)	Y – dummy	
Z – R (73)	Z – S (37)	

4 $\begin{pmatrix} 24 & 42 & 32 & 31 & 0 \\ 22 & 39 & 30 & 35 & 0 \\ 13 & 34 & 22 & 25 & 0 \\ 19 & 41 & 27 & 29 & 0 \\ 18 & 40 & 31 & 33 & 0 \end{pmatrix}$ reducing columns $\begin{pmatrix} 11 & 8 & 10 & 6 & 0 \\ 9 & 5 & 8 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 6 & 7 & 5 & 4 & 0 \\ 5 & 6 & 9 & 8 & 0 \end{pmatrix}$

Minimum uncovered element is 4 $\begin{pmatrix} 7 & 4 & 6 & 2 & 0 \\ 5 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 2 & 3 & 1 & 0 & 0 \\ 1 & 2 & 5 & 4 & 0 \end{pmatrix}$

<u>Either</u>

Minimum uncovered element is 1 $\begin{pmatrix} 6 & 3 & 5 & 1 & 0 \\ 4 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 5 \\ 2 & 3 & 1 & 0 & 1 \\ 0 & 1 & 4 & 3 & 0 \end{pmatrix}$ or $\begin{pmatrix} 6 & 3 & 5 & 2 & 0 \\ 4 & 0 & 3 & 6 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 4 & 4 & 0 \end{pmatrix}$

Solution:

P – dummy
Q – F (39)
R – G (22) cost £108
S – H (29)
T – E (18)

5 a The initial cost matrix is shown below:

	1	**2**	**3**	**4**	**5**	**6**
A	53	75	58	60	75	78
B	44	81	64	55	78	77
C	51	72	51	61	81	72
D	60	77	60	55	76	73
E	51	72	58	51	75	78

As the table above is not an n by n matrix we need to add an additional (dummy) row to make the problem balanced. Therefore, by adding an extra worker (F) to the table the following n by n matrix is formed:

	1	**2**	**3**	**4**	**5**	**6**
A	53	75	58	60	75	78
B	44	81	64	55	78	77
C	51	72	51	61	81	72
D	60	77	60	55	76	73
E	51	72	58	51	75	78
F	0	0	0	0	0	0

b The smallest numbers in rows 1, 2, 3, 4, 5 and 6 are 53, 44, 51, 55, 51 and 0. We reduce rows first by subtracting these numbers from each element in the row. The table becomes:

	1	**2**	**3**	**4**	**5**	**6**
A	0	22	5	7	22	25
B	0	37	20	11	34	33
C	0	21	0	10	30	21
D	5	22	5	0	21	18
E	0	21	7	0	24	27
F	0	0	0	0	0	0

We cannot reduce the columns, since there is a zero already in each column. Therefore, the reduced cost matrix is the one above. We now apply the Hungarian algorithm.
We can cover all the zeros in four lines as follows, so the solution is not optimal.

	1	**2**	**3**	**4**	**5**	**6**
A	0	22	5	7	22	25
B	0	37	20	11	34	33
C	0	21	0	10	30	21
D	5	22	5	0	21	18
E	0	21	7	0	24	27
F	0	0	0	0	0	0

5 b (continued)

	1	**2**	**3**	**4**	**5**	**6**
A	0	22	5	7	22	25
B	0	37	20	11	34	33
C	0	21	0	10	30	21
D	5	22	5	0	21	18
E	0	21	7	0	24	27
F	0	0	0	0	0	0

The smallest uncovered element is 18, so we augment the matrix as follows:

- Add 18 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 18 from the uncovered elements

This gives the following matrix:

	1	**2**	**3**	**4**	**5**	**6**
A	0	4	5	7	4	7
B	0	19	20	11	16	15
C	0	3	0	10	12	3
D	5	4	5	0	3	0
E	0	3	7	0	6	9
F	18	0	18	18	0	0

We can cover all the zeros in five lines as follows, so the solution is not optimal.

	1	**2**	**3**	**4**	**5**	**6**
A	0	4	5	7	4	7
B	0	19	20	11	16	15
C	0	3	0	10	12	3
D	5	4	5	0	3	0
E	0	3	7	0	6	9
F	18	0	18	18	0	0

	1	**2**	**3**	**4**	**5**	**6**
A	0	4	5	7	4	7
B	0	19	20	11	16	15
C	0	3	0	10	12	3
D	5	4	5	0	3	0
E	0	3	7	0	6	9
F	18	0	18	18	0	0

5 b (continued)

The smallest uncovered element is 4, so we augment the matrix as follows:

- Add 4 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 4 from the uncovered elements

This gives the following matrix:

	1	**2**	**3**	**4**	**5**	**6**
A	0	0	1	3	0	3
B	0	15	16	7	12	11
C	4	3	0	10	12	3
D	9	4	5	0	3	0
E	4	3	7	0	6	9
F	22	0	18	18	0	0

We can cover all the zeros with six lines, therefore the solution is optimal.
There are two optimal solutions of:

A – 2 (75)
B – 1 (44)
C – 3 (51)
D – 6 (73)
E – 4 (51)
F – 5 (0)

or

A – 5 (75)
B – 1 (44)
C – 3 (51)
D – 6 (73)
E – 4 (51)
F – 2 (0)

Both solutions give a minimum cost of £294.
Minimum cost = 75 + 44 + 51 + 73 + 51 + 0 = £294, or
Minimum cost = 75 + 44 + 51 + 73 + 51 + 0 = £294.

c For the first optimal solution task 5 will not be completed as it is assigned to a dummy worker. For the second optimal solution task 2 will not be completed for the same reason.