Allocation problems 2A

1 $\begin{pmatrix} 34 & 35 & 31 \\ 26 & 31 & 27 \\ 30 & 37 & 32 \end{pmatrix} \rightarrow \text{reducing rows} \rightarrow \begin{pmatrix} 3 & 4 & 0 \\ 0 & 5 & 1 \\ 0 & 7 & 2 \end{pmatrix} \rightarrow$

$\text{reducing columns} \rightarrow \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{pmatrix}$

$\text{Minimum uncovered is 1} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

Two solutions:

A – Y (35) or A – Z (31)
B – Z (27) or B – Y (31) cost £92
C – X (30) or C – X (30)

2 $\begin{pmatrix} 34 & 37 & 32 & 32 \\ 35 & 32 & 34 & 37 \\ 42 & 35 & 37 & 36 \\ 38 & 34 & 35 & 39 \end{pmatrix} \rightarrow \text{reducing rows} \rightarrow \begin{pmatrix} 2 & 5 & 0 & 0 \\ 3 & 0 & 2 & 5 \\ 7 & 0 & 2 & 1 \\ 4 & 0 & 1 & 5 \end{pmatrix} \rightarrow$

$\rightarrow \text{reducing columns} \begin{pmatrix} 0 & 5 & 0 & 0 \\ 1 & 0 & 2 & 5 \\ 5 & 0 & 2 & 1 \\ 2 & 0 & 1 & 5 \end{pmatrix}$

$\text{Minimum uncovered is 1} \begin{pmatrix} 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 4 & 0 & 1 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}$

Three solutions

P – A (34) or P – D (32) or P – C (32)
Q – B (32) or Q – A (35) or Q – A (35)
R – D (36) or R – B (35) or R – D (36) cost £137
S – C (35) or S – C (35) or S – B (34)

3 $\begin{pmatrix} 20 & 22 & 14 & 24 \\ 20 & 19 & 12 & 20 \\ 13 & 10 & 18 & 16 \\ 22 & 23 & 9 & 28 \end{pmatrix} \rightarrow \text{reducing rows} \rightarrow \begin{pmatrix} 6 & 8 & 0 & 10 \\ 8 & 7 & 0 & 8 \\ 3 & 0 & 8 & 6 \\ 13 & 14 & 0 & 19 \end{pmatrix}$

reducing columns $\begin{pmatrix} 3 & 8 & 0 & 4 \\ 5 & 7 & 0 & 2 \\ 0 & 0 & 8 & 0 \\ 10 & 14 & 0 & 13 \end{pmatrix}$

Minimum uncovered element is 2 $\begin{pmatrix} 1 & 6 & 0 & 2 \\ 3 & 5 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 8 & 12 & 0 & 11 \end{pmatrix}$

Minimum uncovered element is 1 $\begin{pmatrix} 0 & 5 & 0 & 1 \\ 3 & 5 & 1 & 0 \\ 0 & 0 & 11 & 0 \\ 7 & 11 & 0 & 10 \end{pmatrix}$ or Minimum uncovered element is 1 $\begin{pmatrix} 0 & 5 & 0 & 2 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 11 & 0 \\ 7 & 11 & 0 & 10 \end{pmatrix}$

Solution

J – R (20)
K – U (20)
L – S (10) cost £59
M – T (9)

4 $\begin{pmatrix} 85 & 95 & 97 & 87 & 80 \\ 110 & 115 & 95 & 105 & 100 \\ 90 & 95 & 86 & 93 & 105 \\ 85 & 83 & 84 & 85 & 87 \\ 100 & 100 & 105 & 120 & 95 \end{pmatrix} \rightarrow \text{reducing rows} \rightarrow \begin{pmatrix} 5 & 15 & 17 & 7 & 0 \\ 15 & 20 & 0 & 10 & 5 \\ 4 & 9 & 0 & 7 & 19 \\ 2 & 0 & 1 & 2 & 4 \\ 5 & 5 & 10 & 25 & 0 \end{pmatrix}$

reducing columns $\begin{pmatrix} 3 & 15 & 17 & 5 & 0 \\ 13 & 20 & 0 & 8 & 5 \\ 2 & 9 & 0 & 5 & 19 \\ 0 & 0 & 1 & 0 & 4 \\ 3 & 5 & 10 & 23 & 0 \end{pmatrix}$

Minimum uncovered element is 2 $\begin{pmatrix} 1 & 13 & 17 & 3 & 0 \\ 11 & 18 & 0 & 6 & 5 \\ 0 & 7 & 0 & 3 & 19 \\ 0 & 0 & 3 & 0 & 6 \\ 1 & 3 & 10 & 21 & 0 \end{pmatrix}$

4 (continued)

Minimum uncovered element is 3 $\begin{pmatrix} 1 & 10 & 17 & 0 & 0 \\ 11 & 15 & 0 & 3 & 5 \\ 0 & 4 & 0 & 0 & 19 \\ 3 & 0 & 6 & 0 & 9 \\ 1 & 0 & 10 & 18 & 0 \end{pmatrix}$

There are two solutions

D – Z (80)	D – Y (87)	
E – X (95)	E – X (95)	
F – V (90)	F – V (90)	
G – Y (85)	G – W (83)	cost £450
H – W (100)	H – Z (95)	

5 a The initial cost matrix is shown below:

	100 m	**Hurdles**	**200 m**	**400 m**
Ahmed	14	21	37	64
Ben	13	22	40	68
Chang	12	20	38	70
Davina	13	21	39	74

As the table above is an n by n matrix (with $n = 4$), we do not need to add any additional dummy rows or columns.
The smallest numbers in rows 1, 2, 3 and 4 are 14, 13, 12 and 13. We reduce rows first by subtracting these numbers from each element in the row. The table becomes:

	100 m	**Hurdles**	**200 m**	**400 m**
Ahmed	0	7	23	50
Ben	0	9	27	55
Chang	0	8	26	58
Davina	0	8	26	61

The smallest numbers in columns 1, 2, 3 and 4 are 0, 7, 23 and 50. We reduce columns by subtracting these numbers from each element in the column. Therefore, the reduced cost matrix is:

	100 m	**Hurdles**	**200 m**	**400 m**
Ahmed	0	0	0	0
Ben	0	2	4	5
Chang	0	1	3	8
Davina	0	1	3	11

5 b We now apply the Hungarian algorithm.
We can cover all the zeros in two lines as follows, so the solution is not optimal.

	100 m	Hurdles	200 m	400 m
Ahmed	0	0	0	0
Ben	0	2	4	5
Chang	0	1	3	8
Davina	0	1	3	11

	100 m	Hurdles	200 m	400 m
Ahmed	0	0	0	0
Ben	0	2	4	5
Chang	0	1	3	8
Davina	0	1	3	11

The smallest uncovered element is 1, so we augment the matrix as follows:

- Add 1 to the elements covered by two lines
- Leave the elements covered by just one line unchanged
- Subtract 1 from the uncovered elements

This gives the following matrix:

	100 m	Hurdles	200 m	400 m
Ahmed	1	0	0	0
Ben	0	1	3	4
Chang	0	0	2	7
Davina	0	0	2	10

We can cover all the zeros in three lines, so the solution is not optimal:

	100 m	Hurdles	200 m	400 m
Ahmed	1	0	0	0
Ben	0	1	3	4
Chang	0	0	2	7
Davina	0	0	2	10

	100 m	Hurdles	200 m	400 m
Ahmed	1	0	0	0
Ben	0	1	3	4
Chang	0	0	2	7
Davina	0	0	2	10

The smallest uncovered element is 2, so we augment the matrix as follows:

- Add 2 to the elements covered by two lines
- Leave the elements covered by just one line unchanged
- Subtract 2 from the uncovered elements

This gives the following matrix:

5 b (continued)

	100 m	**Hurdles**	**200 m**	**400 m**
Ahmed	3	2	0	0
Ben	0	1	1	2
Chang	0	0	0	5
Davina	0	0	0	8

We can cover all the zeros with four lines, therefore the solution is optimal.
There are two optimal solutions of:

Ahmed – 400 m (64)
Ben – 100 m (13)
Chang – Hurdles (20)
Davina – 200 m (39)

or

Ahmed – 400 m (64)
Ben – 100 m (13)
Chang – 200 m (38)
Davina – Hurdles (21)

Both solutions give a minimum time of 136 seconds.
Minimum time $= 64 + 13 + 20 + 39 = 136$ seconds, or
Minimum time $= 64 + 13 + 38 + 21 = 136$ seconds.

c For an n by n matrix with $n = 4$, exactly 4 lines are needed to cover all the zeros.

6 a The initial cost matrix is shown below:

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	153	87	62	144	76	68
B	162	105	87	152	88	72
C	159	84	75	165	79	77
D	145	98	63	170	85	81
E	149	94	70	138	82	89
F	160	92	82	147	80	85

As the table above is an n by n matrix (with $n = 6$), we do not need to add any additional dummy rows or columns.
The smallest numbers in rows 1, 2, 3, 4, 5 and 6 are 62, 72, 75, 63, 70 and 80. We reduce rows first by subtracting these numbers from each element in the row. The table becomes:

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	91	25	0	82	14	6
B	90	33	15	80	16	0
C	84	9	0	90	4	2
D	82	35	0	107	22	18
E	79	24	0	68	12	19
F	80	12	2	67	0	5

The smallest numbers in columns 1, 2, 3, 4, 5 and 6 are 79, 9, 0, 67, 0 and 0. We reduce columns by subtracting these numbers from each element in the column. Therefore, the reduced cost matrix is:

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	12	16	0	15	14	6
B	11	24	15	13	16	0
C	5	0	0	23	4	2
D	3	26	0	40	22	18
E	0	15	0	1	12	19
F	1	3	2	0	0	5

We now apply the Hungarian algorithm.
We can cover all the zeros in five lines as follows, so the solution is not optimal.

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	12	16	0	15	14	6
B	11	24	15	13	16	0
C	5	0	0	23	4	2
D	3	26	0	40	22	18
E	0	15	0	1	12	19
F	1	3	2	0	0	5

6 a (continued)

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	12	16	0	15	14	6
B	11	24	15	13	16	0
C	5	0	0	23	4	2
D	3	26	0	40	22	18
E	0	15	0	1	12	19
F	1	3	2	0	0	5

The smallest uncovered element is 3, so we augment the matrix as follows:

- Add 3 to the elements covered by two lines circled in red.
- Leave the elements covered by just one line unchanged
- Subtract 3 from the uncovered elements

This gives the following matrix:

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	9	13	0	12	11	6
B	8	21	15	10	13	0
C	5	0	3	23	4	5
D	0	23	0	37	19	18
E	0	15	3	1	12	22
F	1	3	5	0	0	8

We can cover all the zeros in five lines as follows, so the solution is not optimal.

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	9	13	0	12	11	6
B	8	21	15	10	13	0
C	5	0	3	23	4	5
D	0	23	0	37	19	18
E	0	15	3	1	12	22
F	1	3	5	0	0	8

	Beech	Elm	Eucalyptus	Oak	Olive	Willow
A	9	13	0	12	11	6
B	8	21	15	10	13	0
C	5	0	3	23	4	5
D	0	23	0	37	19	18
E	0	15	3	1	12	22
F	1	3	5	0	0	8

6 a (continued)

The smallest uncovered element is 1, so we augment the matrix as follows:

- Add 1 to the elements covered by two lines circled in red
- Leave the elements covered by just one line unchanged
- Subtract 1 from the uncovered elements

This gives the following matrix:

	Beech	**Elm**	**Eucalyptus**	**Oak**	**Olive**	**Willow**
A	9	12	0	11	10	5
B	9	21	16	10	13	0
C	6	0	4	23	4	5
D	0	22	0	36	18	17
E	0	14	3	0	11	21
F	2	3	6	0	0	8

We can cover all the zeros with six lines, so we have found our optimal solution of:

A – Eucalyptus (£62)
B – Willow (£72)
C – Elm (£84)
D – Beech (£145)
E – Oak (£138)
F – Olive (£80)

b Minimum cost = 62 + 72 + 84 + 145 + 138 + 80 = £581.