Transportation problems Mixed exercise

1 a Applying the north-west corner method gives:

	L	*M*	**Supply**
A	15		15
B	1	4	5
C		8	8
Demand	16	12	28

b The shadow costs of the initial solution are:

		20	10
		L	*M*
0	***A***	20	
20	***B***	40	30
80	***C***		90

Improvement indices for cell:
$AM = 70 - 0 - 10 = 60$ $\quad CL = 60 - 80 - 20 = -40$

The solution may not be optimal, since there is a cell, CL, with a negative improvement index.

First iteration
The entering cell is CL. Entering θ into cell CL and applying the stepping-stone method:

	L	*M*	**Supply**
A	15		15
B	$1-\theta$	$4+\theta$	5
C	θ	$8-\theta$	8
Demand	16	12	28

The maximum value of θ is 1, making BL the exiting cell. This is the improved solution.

	L	*M*	**Supply**
A	15		15
B		5	5
C	1	7	8
Demand	16	12	28

1 b (continued)

The shadow costs of the new solution are:

		20	50
		L	***M***
0	***A***	20	
–20	***B***		30
40	***C***	60	90

Improvement indices for cells:

$AM = 70 - 0 - 50 = 20$ $BL = 40 - (-20) - 20 = 40$

c There are no negative improvement indices, so the solution produced after the first iteration (in part b is optimal. This solution is:

15 units from A to L
5 units from B to M Cost: 1140
1 unit from C to L
7 units from C to M

Cost = $15 \times 20 + 5 \times 30 + 1 \times 60 + 7 \times 90 = £1140$

d Let x_{ij} be the number of cars transported from i to j where

$i \in \{A, B, C\}$

$j \in \{L, M\}$

$x_{ij} \geqslant 0$

Minimise:

$C = 20x_{AL} + 70x_{AM} + 40x_{BL} + 30x_{BM} + 60x_{CL} + 90x_{CM}$

Subject to:

$x_{AL} + x_{AM} \leqslant 15$

$x_{BL} + x_{BM} \leqslant 5$

$x_{CL} + x_{CM} \leqslant 8$

$x_{AL} + x_{BL} + x_{CL} \geqslant 16$

$x_{AM} + x_{BM} + x_{CM} \geqslant 12$

2 a Applying the north-west corner method gives:

	P	***Q***	***R***	**Supply**
F	10	5		15
G		25	10	35
H			10	10
Demand	10	30	20	60

2 b The shadow costs of the initial solution are:

		23	21	22
		P	***Q***	***R***
0	***F***	23	21	
2	***G***		23	24
1	***H***			23

Improvement indices for cells:

$FR = 22 - 0 - 22 = 0$ $\quad$ $HP = 22 - 1 - 23 = -2$

$GP = 21 - 2 - 23 = -4$ $\quad$ $HQ = 21 - 1 - 21 = -1$

The solution may not be optimal, since there are cells with negative improvement indices.

First iteration

The entering cell is *GP*. Entering θ into cell *GP* and applying the stepping-stone method:

	P	***Q***	***R***	**Supply**
F	$10-\theta$	$5+\theta$		15
G	θ	$25-\theta$	10	35
H			10	10
Demand	10	30	20	60

The maximum value of θ is 10, making *FP* the exiting cell. This is the improved solution.

	P	***Q***	***R***	**Supply**
F		15		15
G	10	15	10	35
H			10	10
Demand	10	30	20	60

The shadow costs of the new solution are:

		19	21	22
		P	***Q***	***R***
0	***F***		21	
2	***G***	21	23	24
1	***H***			23

Improvement indices for cells:

$FP = 23 - 0 - 19 = 4$ $\quad$ $HP = 22 - 1 - 19 = 2$

$FR = 22 - 0 - 22 = 0$ $\quad$ $HQ = 21 - 1 - 21 = -1$

2 b (continued)

Second iteration
The entering cell is *HQ*. Entering θ into cell *HQ* and applying the stepping-stone method:

	P	***Q***	***R***	**Supply**
F		15		15
G	10	$15-\theta$	$10+\theta$	35
H		θ	$10-\theta$	10
Demand	10	30	20	60

The maximum value of θ is 10, making *HR* the exiting cell. This is the improved solution.

	P	***Q***	***R***	**Supply**
F		15		15
G	10	5	20	35
H		10		10
Demand	10	30	20	60

The shadow costs of the new solution are:

		19	21	22
		P	***Q***	***R***
0	***F***		21	
2	***G***	21	23	24
0	***H***		21	

Improvement indices for cells:

$FP = 23 - 0 - 19 = 4$ $\quad$ $HP = 22 - 0 - 19 = 3$

$FR = 22 - 0 - 22 = 0$ $\quad$ $HR = 23 - 0 - 22 = 1$

c There are no negative improvement indices, so the solution is optimal.

d Cost = $10 \times 21 + 15 \times 21 + 5 \times 23 + 10 \times 21 + 20 \times 24 = 1330$

2 e ***Third iteration***

The entering cell is FR, the cell with a zero improvement index (see part **b**). Entering θ into cell FR and applying the stepping-stone method:

	P	***Q***	***R***	**Supply**
F		$15-\theta$	θ	15
G	10	$5+\theta$	$20-\theta$	35
H		10		10
Demand	10	30	20	60

The maximum value of θ is 15, making FQ the exiting cell. This is the new solution.

	P	***Q***	***R***	**Supply**
F			15	15
G	10	20	5	35
H		10		10
Demand	10	30	20	60

The shadow costs of the new solution are:

		19	21	22
		P	***Q***	***R***
0	***F***			22
2	***G***	21	23	24
0	***H***		21	

Improvement indices for cells:

$FP = 23 - 0 - 19 = 4$ $\quad$ $HP = 22 - 0 - 19 = 3$

$FQ = 21 - 0 - 21 = 0$ $\quad$ $HR = 23 - 0 - 22 = 1$

Cost = $10 \times 21 + 20 \times 23 + 10 \times 21 + 15 \times 22 + 5 \times 24 = 1330$

The cost of this second optimal solution is the same as that found for the solution in part **b**.

3 a Adding a zero to cell LY ensures that the solution is non-degenerate. The problem has 4 rows and 3 columns so a non-degenerate solution must have 4 + 3 – 1 = 6 non-blank cells. If cell LY is left blank the solution would be degenerate.

b Cost = $25 \times 8 + 5 \times 5 + 40 \times 5 + 50 \times 10 + 50 \times 15 = £1675$

3 c Entering θ into cell MX and applying the stepping-stone method:

	X	***Y***	***Z***	**Supply**
J	$25-\theta$	$5+\theta$		30
K		40		40
L		$0-\theta$	$50+\theta$	50
M	θ		$50-\theta$	50
Demand	25	45	100	170

The maximum value of θ is 0, making LY the exiting cell. This is the new solution.

	X	***Y***	***Z***	**Supply**
J	25	5		30
K		40		40
L			50	50
M	0		50	50
Demand	25	45	100	170

Cost = $25\times8+5\times5+40\times5+50\times10+50\times15=£1675$

d The shadow costs of the solution in part **c** are:

		8	5	17
		X	***Y***	***Z***
0	***J***	8	5	
0	***K***		5	
–7	***L***			10
–2	***M***	6		15

Improvement indices for cells:

$JZ=7-0-17=-10$ $\quad$ $LX=7-(-7)-8=6$

$KX=5-0-8=-3$ $\quad$ $LY=2-(-7)-5=4$

$KZ=9-0-17=-8$ $\quad$ $MY=3-(-2)-5=0$

This solution is not optimal since there are negative improvement indices.

3 e The shadow costs of the new solution found after two further iterations are:

		6	3	7
		X	***Y***	***Z***
0	***J***			7
2	***K***		5	9
3	***L***			10
0	***M***	6	3	

Improvement indices for cells:

$JX = 8 - 0 - 6 = 2$ $\quad$ $LX = 7 - 3 - 6 = -2$

$JY = 5 - 0 - 3 = 2$ $\quad$ $LY = 2 - 3 - 3 = -4$

$KX = 5 - 2 - 6 = -3$ $\quad$ $MZ = 15 - 0 - 7 = 8$

The entering cell is LY. Entering θ into cell LY and applying the stepping-stone method:

	X	***Y***	***Z***	**Supply**
J			30	30
K		$20 - \theta$	$20 + \theta$	40
L		θ	$50 - \theta$	50
M	25	25		50
Demand	25	45	100	170

The maximum value of θ is 20, making KY the exiting cell. This is the new solution.

	X	***Y***	***Z***	**Supply**
J			30	30
K			40	40
L		20	30	50
M	25	25		50
Demand	25	45	100	170

The shadow costs of this solution are:

		2	–1	7
		X	***Y***	***Z***
0	***J***			7
2	***K***			9
3	***L***		2	10
4	***M***	6	3	

3 e (continued)

Improvement indices for cells:

$JX = 8 - 0 - 2 = 6$ $KY = 5 - 2 - (-1) = 4$

$JY = 5 - 0 - (-1) = 6$ $LX = 7 - 3 - 2 = 2$

$KX = 5 - 2 - 2 = 1$ $MZ = 15 - 4 - 7 = 4$

All improvement indices are non-negative, so this solution is optimal.

Cost = $25 \times 6 + 20 \times 2 + 25 \times 3 + 30 \times 7 + 40 \times 9 + 30 \times 10 = £1135$

3 f Cell JY is not part of the optimum route and increasing the cost will not change this situation, so the cost of the optimal solution will not be affected.

4 a The total demand is 150, the total stock is 170, so demand < stock.
A dummy demand point, V, is to absorb the surplus stock.
The problem becomes:

	S	***T***	***U***	***V***	**Supply**
A	6	10	7	0	50
B	7	5	8	0	70
C	6	7	7	0	50
Demand	100	30	20	20	170

b Applying the north-west corner method to get an initial solution gives:

	S	***T***	***U***	***V***	**Supply**
A	50				50
B	50	20			70
C		10	20	20	50
Demand	100	30	20	20	170

c The shadow costs of the initial solution in part **b** are:

		6	4	4	–3
		S	***T***	***U***	***V***
0	***A***	6			
1	***B***	7	5		
3	***C***		7	7	0

Improvement indices for cells:

$AT = 10 - 0 - 4 = 6$ $BU = 8 - 1 - 4 = 3$

$AU = 7 - 0 - 4 = 3$ $BV = 0 - 1 - (-3) = 2$

$AV = 0 - 0 - (-3) = 3$ $CS = 6 - 3 - 6 = -3$

4 c (continued)

First iteration

The entering cell is *CS*. Entering θ into cell *CS* and applying the stepping-stone method:

	S	***T***	***U***	***V***	**Supply**
A	50				50
B	$50-\theta$	$20+\theta$			70
C	θ	$10-\theta$	20	20	50
Demand	100	30	20	20	170

The maximum value of θ is 10, making *CT* the exiting cell. This is the new solution.

	S	***T***	***U***	***V***	**Supply**
A	50				50
B	40	30			70
C	10		20	20	50
Demand	100	30	20	20	170

The shadow costs of the new solution are:

		6	4	7	0
		S	***T***	***U***	***V***
0	***A***	6			
1	***B***	7	5		
0	***C***	6		7	0

Improvement indices for cells:

$AT = 10 - 0 - 4 = 6$ $BU = 8 - 1 - 7 = 0$

$AU = 7 - 0 - 7 = 0$ $BV = 0 - 1 - 0 = -1$

$AV = 0 - 0 - 0 = 0$ $CT = 7 - 0 - 4 = 3$

Second iteration

The entering cell is *BV*. Entering θ into cell *BV* and applying the stepping-stone method:

	S	***T***	***U***	***V***	**Supply**
A	50				50
B	$40-\theta$	30		θ	70
C	$10+\theta$		20	$20-\theta$	50
Demand	100	30	20	20	170

4 c (continued)

The maximum value of θ is 20, making CV the exiting cell. This is the new solution.

	S	***T***	***U***	***V***	**Supply**
A	50				50
B	20	30		20	70
C	30		20		50
Demand	100	30	20	20	170

The shadow costs of the new solution are:

		6	4	7	–1
		S	***T***	***U***	***V***
0	***A***	6			
1	***B***	7	5		0
0	***C***	6		7	

Improvement indices for cells:

$AT = 10 - 0 - 4 = 6$ $\quad$ $BU = 8 - 1 - 7 = 0$

$AU = 7 - 0 - 7 = 0$ $\quad$ $CT = 7 - 0 - 4 = 3$

$AV = 0 - 0 - (-1) = 1$ $\quad$ $CV = 0 - 0 - (-1) = 1$

All improvement indices are non-negative, so this solution is optimal.
Note that the two zero improvement indices indicate that there are two further optimal solutions but the question does not require that these are found.

d Cost = $50 \times 6 + 20 \times 7 + 30 \times 6 + 30 \times 5 + 20 \times 7 + 20 \times 0 = £910$

e Let x_{ij} be the number of van load of fruit-tree seedlings transported from *i* to *j* where

$i \in \{A, B, C\}$

$j \in \{S, T, U, V\}$ $\quad$ V is a dummy demand point

$x_{ij} \geqslant 0$

Minimise:

$C = 6x_{AS} + 10x_{AT} + 7x_{AU} + 7x_{BS} + 5x_{BT} + 8x_{BU} + 6x_{CS} + 7x_{CT} + 7x_{CU}$

Subject to:

$x_{AS} + x_{AT} + x_{AU} + x_{AV} \leqslant 50$

$x_{BS} + x_{BT} + x_{BU} + x_{BV} \leqslant 70$

$x_{CS} + x_{CT} + x_{CU} + x_{CV} \leqslant 50$

$x_{AS} + x_{BS} + x_{CS} \geqslant 100$

$x_{AT} + x_{BT} + x_{CT} \geqslant 30$

$x_{AU} + x_{BU} + x_{CU} \geqslant 20$

$x_{AV} + x_{BV} + x_{CV} \geqslant 20$

5 Total demand = 16 + 20 + 26 + 17 = 79
Total supply = 28 + 33 + 18 = 79
Total demand = Total supply
So the problem is balanced and a dummy demand point or dummy supply point is not required.

Let x_{ij} be the number of rolls of carpet transported from i to j where

$i \in \{A, B, C\}$
$j \in \{P, Q, R, S\}$
$x_{ij} \geqslant 0$

Minimise:

$$C = 28x_{AP} + 12x_{AQ} + 19x_{AR} + 16x_{AS} + 31x_{BP} + 28x_{BQ} + 23x_{BR} + 19x_{BS} + 18x_{CP} + 21x_{CQ} + 22x_{CR} + 28x_{CS}$$

Subject to:

$x_{AP} + x_{AQ} + x_{AR} + x_{AS} \leqslant 28$
$x_{BP} + x_{BQ} + x_{BR} + x_{BS} \leqslant 33$
$x_{CP} + x_{CQ} + x_{CR} + x_{CS} \leqslant 18$

$x_{AP} + x_{BP} + x_{CP} \geqslant 16$
$x_{AQ} + x_{BQ} + x_{CQ} \geqslant 20$
$x_{AR} + x_{BR} + x_{CR} \geqslant 26$
$x_{AS} + x_{BS} + x_{CS} \geqslant 17$

Challenge

a Total demand = 15 + 9 + 11 = 35
Total supply = 14 + 12 + 16 = 42
Total demand ≠ Total supply

The problem is unbalanced, total supply is greater than total demand so a dummy demand point, *S*, needs to be set up to absorb the excess supply. Stock in cells *AS*, *BS*, and *CS* remains in warehouses *A*, *B* and *C* respectively, but incurs weekly storage charges. These charges are £3, £5 and £4 respectively. The problem therefore can be represented as:

	P	***Q***	***R***	***S***	**Stock**
A	7	8	6	3	14
B	5	7	9	5	12
C	6	8	8	4	16
Demand	15	9	11	7	42

Challenge

b Let x_{ij} be the units of stock transported from *i* to *j* where

$i \in \{A, B, C\}$

$j \in \{P, Q, R, S\}$

$x_{ij} \geqslant 0$

Minimise:

$C = 7x_{AP} + 8x_{AQ} + 6x_{AR} + 3x_{AS} + 5x_{BP} + 7x_{BQ} + 9x_{BR} + 5x_{BS} + 6x_{CP} + 8x_{CQ} + 8x_{CR} + 4x_{CS}$

Subject to:

$x_{AP} + x_{AQ} + x_{AR} + x_{AS} \leqslant 14$

$x_{BP} + x_{BQ} + x_{BR} + x_{BS} \leqslant 12$

$x_{CP} + x_{CQ} + x_{CR} + x_{CS} \leqslant 16$

$x_{AP} + x_{BP} + x_{CP} \geqslant 15$

$x_{AQ} + x_{BQ} + x_{CQ} \geqslant 9$

$x_{AR} + x_{BR} + x_{CR} \geqslant 11$

$x_{AS} + x_{BS} + x_{CS} \geqslant 7$