Transportation problems 1D

1 The initial solution to this transportation problem in Exercise **1C** is found in question **1a** of Exercise **1A**. This solution is:

	P	***Q***	***R***	**Supply**
A	28	4		32
B		41	3	44
C			34	34
Demand	28	45	37	110

First iteration
In question **1c** of Exercise **1C** it is established that the entering cell is *CQ*, as this has the most negative improvement index. So enter θ into cell *CQ*.

	P	***Q***	***R***	**Supply**
A	28	4		32
B		41	3	44
C		θ	34	34
Demand	28	45	37	110

In order to keep stock at *C* correct, decrease the entry at *CR*.

	P	***Q***	***R***	**Supply**
A	28	4		32
B		41	3	44
C		θ	$34-\theta$	34
Demand	28	45	37	110

In order to keep demand at *R* correct, increase the entry at *BR*.

	P	***Q***	***R***	**Supply**
A	28	4		32
B		41	$3+\theta$	44
C		θ	$34-\theta$	34
Demand	28	45	37	110

1 (continued)

In order to keep supply at B correct, decrease the entry at BQ.

	P	***Q***	***R***	**Supply**
A	28	4		32
B		$41-\theta$	$3+\theta$	44
C		θ	$34-\theta$	34
Demand	28	45	37	110

No more adjustments can be made without breaking the rule that there can only be one increasing cell and one decreasing cell in any row or column.

The largest possible value of θ is 34, making CR the exiting cell, and giving this improved solution.

	P	***Q***	***R***	**Supply**
A	28	4		32
B		7	37	44
C		34		34
Demand	28	45	37	110

Cost $= 28\times150+4\times213+7\times204+34\times198+37\times218=21278$

This is an improvement on the costs of 22 434 found in Exercise **1A**, question **1c**. It may not be optimal, so find the shadow costs:

		150	213	227
		P	***Q***	***R***
0	***A***	150	213	
–9	***B***		204	218
–15	***C***		198	

Improvement indices for cells:

$BP = 175-(-9)-150 = 34$

$CP = 188-(-15)-150 = 53$

$AR = 222-0-227 = -5$

$CR = 246-(-15)-227 = 34$

The solution may not be optimal, since there is a cell, AR, with a negative improvement index.

1 (continued)

Second iteration

The new entering cell is AR. Entering θ into cell AR and applying the stepping-stone method:

	P	***Q***	***R***	**Supply**
A	28	$4-\theta$	θ	32
B		$7+\theta$	$37-\theta$	44
C		34		34
Demand	28	45	37	110

The maximum value of θ is 4, making AQ the exiting cell. This is the improved solution.

	P	***Q***	***R***	**Supply**
A	28		4	32
B		11	33	44
C		34		34
Demand	28	45	37	110

Cost $= 28\times150+11\times204+34\times198+4\times222+33\times218=21258$

This is an improvement on the costs of 21 278 found in the first iteration. It may not be optimal, so find the new shadow costs:

		150	208	222
		P	***Q***	***R***
0	***A***	150		222
–4	***B***		204	218
–10	***C***		198	

Improvement indices for cells:

$AQ = 213 - 0 - 208 = 5$

$BP = 175 - (-4) - 150 = 29$

$CP = 188 - (-10) - 150 = 48$

$CR = 246 - (-10) - 222 = 34$

Since all improvement indices are non-negative, the solution is optimal.

The optimal solution is:

28 units from A to P

4 units from A to R

11 units from B to Q

33 units from B to R

34 units from C to Q

Cost: 21 258

2 The initial solution to this transportation problem in Exercise **1C** is found in question **2a** of Exercise **1A**. This solution is:

	P	***Q***	***R***	***S***	**Supply**
A	21	32	1		54
B			50	17	67
C				29	29
Demand	21	32	51	46	150

First iteration

In question **2c** of Exercise **1C** it is established that the entering cell is *CR*, as this has the most negative improvement index. Entering θ into cell *CR* and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A	21	32	1		54
B			$50-\theta$	$17+\theta$	67
C			θ	$29-\theta$	29
Demand	21	32	51	46	150

The largest possible value of θ is 29, making *CS* the exiting cell. This is the improved solution:

	P	***Q***	***R***	***S***	**Supply**
A	21	32	1		54
B			21	46	67
C			29		29
Demand	21	32	51	46	150

Cost $= 21\times27+32\times33+1\times34+21\times37+29\times28+46\times30=4626$

This is an improvement on the costs of 5 032 found in Exercise **1A**, question **2c**. It may not be optimal, so find the shadow costs:

		27	33	34	27
		P	***Q***	***R***	***S***
0	***A***	27	33	34	
3	***B***			37	30
–6	***C***			28	

2 (continued)

Improvement indices for cells:

$AS = 41 - 0 - 27 = 14$ $\quad CP = 40 - (-6) - 27 = 19$

$BP = 31 - 3 - 27 = 1$ $\quad CQ = 32 - (-6) - 33 = 5$

$BQ = 29 - 3 - 33 = -7$ $\quad CS = 35 - (-6) - 27 = 14$

The solution may not be optimal, since there is a cell, BQ, with a negative improvement index.

Second iteration
The new entering cell is BQ. Entering θ into cell BQ and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A	21	$32-\theta$	$1+\theta$		54
B		θ	$21-\theta$	46	67
C			29		29
Demand	21	32	51	46	150

The largest possible value of θ is 21, making BR the exiting cell, and giving this improved solution.

	P	***Q***	***R***	***S***	**Supply**
A	21	11	22		54
B		21		46	67
C			29		29
Demand	21	32	51	46	150

Cost $= 21 \times 27 + 11 \times 33 + 22 \times 34 + 21 \times 29 + 29 \times 28 + 46 \times 30 = 4479$
This is an improvement on the costs of 4626 found in the first iteration. It may not be optimal, so find the new shadow costs:

		27	33	34	34
		P	***Q***	***R***	***S***
0	***A***	27	33	34	
–4	***B***		29		30
–6	***C***			28	

Improvements indices are:

$AS = 41 - 0 - 34 = 7$ $\quad CP = 40 - (-6) - 27 = 19$

$BP = 31 - (-4) - 27 = 8$ $\quad CQ = 32 - (-6) - 33 = 5$

$BR = 37 - (-4) - 34 = 7$ $\quad CS = 35 - (-6) - 34 = 7$

All improvement indices are non-negative, so this solution is optimal.

3 The initial solution to this transportation problem in Exercise **1C** is found in question **4a** of Exercise **1A**. This solution is:

	P	***Q***	***R***	***S***	**Supply**
A	134				134
B	41	162			203
C		13	163		176
D			12	175	187
Demand	175	175	175	175	700

First iteration
In question **4c** of Exercise **1C** it is established that the entering cell is *DQ*, as this has the most negative improvement index. Entering θ into cell *DQ* and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A	134				134
B	41	162			203
C		$13-\theta$	$163+\theta$		176
D		θ	$12-\theta$	175	187
Demand	175	175	175	175	700

The largest possible value of θ is 12, making *DR* the exiting cell, and giving this improved solution.

	P	***Q***	***R***	***S***	**Supply**
A	134				134
B	41	162			203
C		1	175		176
D		12		175	187
Demand	175	175	175	175	700

Cost $= 134\times56+41\times59+162\times76+1\times70+12\times68+175\times57+175\times71=45521$
This is an improvement on the costs of 45 761 found in Exercise **1A**, question **4c**. It may not be optimal, so find the shadow costs:

		56	73	60	76
		P	***Q***	***R***	***S***
0	***A***	56			
3	***B***	59	76		
–3	***C***		70	57	
–5	***D***		68		71

3 (continued)

Improvement indices for cells:

$AQ = 86 - 0 - 73 = 13$
$AR = 80 - 0 - 60 = 20$
$AS = 61 - 0 - 76 = -15$
$BR = 78 - 3 - 60 = 15$
$BS = 65 - 3 - 76 = -14$
$CP = 62 - (-3) - 56 = 9$
$CS = 67 - (-3) - 76 = -6$
$DP = 60 - (-5) - 56 = 9$
$DR = 75 - (-5) - 60 = 20$

The solution may not be optimal, since there are cells with negative improvement indices.

Second iteration
The new entering cell is AS. Entering θ into cell AS and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A	$134-\theta$			θ	134
B	$41+\theta$	$162-\theta$			203
C		1	175		176
D		$12+\theta$		$175-\theta$	187
Demand	175	175	175	175	700

The largest possible value of θ is 134, making AP the exiting cell, and giving this improved solution.

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	175	28			203
C		1	175		176
D		146		41	187
Demand	175	175	175	175	700

Cost $= 175 \times 59 + 28 \times 76 + 1 \times 70 + 146 \times 68 + 175 \times 57 + 134 \times 61 + 41 \times 71 = 43511$
This is an improvement on the costs of 45 521 found in the first iteration. It may not be optimal, so find the shadow costs:

		41	58	45	61
		P	***Q***	***R***	***S***
0	***A***				61
18	***B***	59	76		
12	***C***		70	57	
10	***D***		68		71

3 (continued)

Improvement indices for cells:

$AP = 56 - 0 - 41 = 15$
$AQ = 86 - 0 - 58 = 28$
$AR = 80 - 0 - 45 = 35$
$BR = 78 - 18 - 45 = 15$
$BS = 65 - 18 - 61 = -14$
$CP = 62 - 12 - 41 = 9$
$CS = 67 - 12 - 61 = -6$
$DP = 60 - 10 - 41 = 9$
$DR = 75 - 10 - 45 = 20$

The solution may not be optimal, since there are cells with negative improvement indices.

Third iteration

The new entering cell is *BS*. Entering θ into cell *BS* and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	175	$28-\theta$		θ	203
C		1	175		176
D		$146+\theta$		$41-\theta$	187
Demand	175	175	175	175	700

The largest possible value of θ is 28, making *BQ* the exiting cell, and giving this improved solution.

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	175			28	203
C		1	175		176
D		174		13	187
Demand	175	175	175	175	700

Cost $= 175 \times 59 + 1 \times 70 + 174 \times 68 + 175 \times 57 + 134 \times 61 + 28 \times 65 + 13 \times 71 = 43119$

This is an improvement on the costs of 43 511 found in the second iteration. It may not be optimal, so find the shadow costs:

		55	58	45	61
		P	***Q***	***R***	***S***
0	***A***				61
4	***B***	59			65
12	***C***		70	57	
10	***D***		68		71

3 (continued)

Improvement indices for cells:

$AP = 56 - 0 - 55 = 1$

$AQ = 86 - 0 - 58 = 28$

$AR = 80 - 0 - 45 = 35$

$BQ = 76 - 4 - 58 = 14$

$BR = 78 - 4 - 45 = 29$

$CP = 62 - 12 - 55 = -5$

$CS = 67 - 12 - 61 = -6$

$DP = 60 - 10 - 55 = -5$

$DR = 75 - 10 - 45 = 20$

The solution may not be optimal, since there are cells with negative improvement indices.

Fourth iteration

The new entering cell is *CS*. Entering θ into cell *CS* and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	175			28	203
C		$1-\theta$	175	θ	176
D		$174+\theta$		$13-\theta$	187
Demand	175	175	175	175	700

The largest possible value of θ is 1, making *CQ* the exiting cell, and giving this improved solution.

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	175			28	203
C			175	1	176
D		175		12	187
Demand	175	175	175	175	700

Cost $= 175 \times 59 + 175 \times 68 + 175 \times 57 + 134 \times 61 + 28 \times 65 + 1 \times 67 + 12 \times 71 = 43113$

This is an improvement on the costs of 43 119 found in the third iteration. It may not be optimal, so find the shadow costs:

		55	58	51	61
		P	***Q***	***R***	***S***
0	***A***				61
4	***B***	59			65
6	***C***			57	67
10	***D***		68		71

3 (continued)

Improvement indices for cells:

$AP = 56 - 0 - 55 = 1$
$AQ = 86 - 0 - 58 = 28$
$AR = 80 - 0 - 51 = 29$
$BQ = 76 - 4 - 58 = 14$
$BR = 78 - 4 - 51 = 23$
$CP = 62 - 6 - 55 = 1$
$CQ = 70 - 6 - 58 = 6$
$DP = 60 - 10 - 55 = -5$
$DR = 75 - 10 - 51 = 14$

The solution may not be optimal, since cell *DP* has a negative improvement index.

Fifth iteration

The new entering cell is *DP*. Entering θ into cell *CS* and applying the stepping-stone method gives:

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	$175-\theta$			$28+\theta$	203
C			175	1	176
D	θ	175		$12-\theta$	187
Demand	175	175	175	175	700

The largest possible value of θ is 12, making *DS* the exiting cell, and giving this improved solution.

	P	***Q***	***R***	***S***	**Supply**
A				134	134
B	163			40	203
C			175	1	176
D	12	175			187
Demand	175	175	175	175	700

Cost $= 163 \times 59 + 12 \times 60 + 175 \times 68 + 175 \times 57 + 134 \times 61 + 40 \times 65 + 1 \times 67 = 43053$

This is an improvement on the costs of 43 113 found in the fourth iteration. It may not be optimal, so find the shadow costs:

		55	63	51	61
		P	***Q***	***R***	***S***
0	***A***				61
4	***B***	59			65
6	***C***			57	67
5	***D***	60	68		

3 (continued)

Improvement indices for cells:

$AP = 56 - 0 - 55 = 1$

$AQ = 86 - 0 - 63 = 23$

$AR = 80 - 0 - 51 = 29$

$BQ = 76 - 4 - 63 = 9$

$BR = 78 - 4 - 51 = 23$

$CP = 62 - 6 - 55 = 1$

$CQ = 70 - 6 - 63 = 1$

$DR = 75 - 5 - 51 = 19$

$DS = 71 - 5 - 61 = 5$

There are no negative improvement indices, so the solution is optimal.
This is the optimal solution:

134 units from A to S
163 units from B to P
40 units from B to S
175 units from C to R
1 unit from C to S
12 units from D to P
175 units from D to Q

Cost: 43 053

4 The initial solution using the north-west corner method is:

	P	***Q***	**Supply**
A	3		3
B	3	2	5
C		2	2
Demand	6	4	10

Cost $= 3\times 2 + 3\times 2 + 2\times 7 + 2\times 9 = £44$
This may not be an optimal solution so find the shadow costs:

		2	7
		P	***Q***
0	***A***	2	
0	***B***	2	7
2	***C***		9

Improvement indices for cells:
$AQ = 6 - 0 - 7 = -1$ $\quad$ $CP = 6 - 2 - 2 = 2$

First iteration
The entering cell is AQ. Entering θ into cell AQ and applying the stepping-stone method gives:

	P	***Q***	**Supply**
A	$3-\theta$	θ	3
B	$3+\theta$	$2-\theta$	5
C		2	2
Demand	6	4	10

The largest possible value of θ is 2, making BQ the exiting cell, and giving this improved solution.

	P	***Q***	**Supply**
A	1	2	3
B	5		5
C		2	2
Demand	6	4	10

4 (continued)

Cost $= 1 \times 2 + 5 \times 2 + 2 \times 6 + 2 \times 9 = £42$

This is an improvement on the costs of £44 found in the initial solution. It may not be optimal, so find the new shadow costs:

		2	6
		P	***Q***
0	***A***	2	6
0	***B***	2	
3	***C***		9

Improvement indices for cells:

$BQ = 7 - 0 - 6 = 1$ $\quad\quad$ $CP = 6 - 3 - 2 = 1$

There are no negative improvement indices, so the solution is optimal.

This is the optimal solution:

1 unit from A to P
2 units from A to Q
5 units from B to P
2 units from C to Q

Cost: £42

5 **a** Total supply = 17 + 14 + 14 = 45
Total demand = 12 + 15 + 18 = 45
Total supply = Total demand, so the problem is balanced.

b The initial solution using the north-west corner method is:

	A	***B***	***C***	**Supply**
X	12	5		17
Y		10	4	14
Z			14	14
Demand	12	15	18	45

Cost $= 12 \times 16 + 5 \times 21 + 10 \times 22 + 4 \times 19 + 14 \times 16 = £817$

5 c The values in the table are the costs corresponding to the non-empty cells in the initial solution. Choosing 0 as the shadow cost for X, the other shadow costs can be calculated to ensure the cost associated with the supply point plus the cost associated with the demand point sums to the cost of transporting a unit.

		16	21	18
		A	***B***	***C***
0	***X***	16	21	
1	***Y***		22	19
–2	***Z***			16

Improvement indices for cells:

$XC = 15 - 0 - 18 = -3$

$YA = 23 - 1 - 16 = 6$

$ZA = 18 - (-2) - 16 = 4$

$ZB = 24 - (-2) - 21 = 5$

The solution may not be optimal, since cell XC has a negative improvement index.

d The entering cell is XC. Entering θ into cell XC and applying the stepping-stone method gives:

	A	***B***	***C***	**Supply**
X	12	$5-\theta$	θ	17
Y		$10+\theta$	$4-\theta$	14
Z			14	14
Demand	12	15	18	10

The largest possible value of θ is 4, making YC the exiting cell, and giving this improved solution.

	A	***B***	***C***	**Supply**
X	12	1	4	17
Y		14		14
Z			14	14
Demand	12	15	18	

Cost $= 12 \times 16 + 1 \times 21 + 14 \times 22 + 4 \times 15 + 14 \times 16 = £805$

This is an improvement on the costs of £817 found in the initial solution.

5 e The solution found in part **d** may not be optimal, so find the shadow costs:

		16	21	15
		A	***B***	***C***
0	***X***	16	21	15
1	***Y***		22	
1	***Z***			16

Improvement indices for cells:

$YA = 23 - 1 - 16 = 6$

$YC = 19 - 1 - 15 = 3$

$ZA = 18 - 1 - 16 = 1$

$ZB = 24 - 1 - 21 = 2$

There are no negative improvement indices, so this solution is optimal.
The cost is £805.

6 a Total supply (26 + 23 + 24 = 73) ≠ Total demand (22 + 16 + 18 = 56)
So the problem is not balanced.
A dummy demand point must be created as supply is greater than demand.

b Introducing a dummy demand point *G* with demand 17 (73 – 56 = 17), the initial solution using the north-west corner method is:

	D	***E***	***F***	***G***	**Supply**
A	22	4			26
B		12	11		23
C			7	17	24
Demand	22	16	18	17	73

c The shadow costs for the initial solution are:

		24	32	39	9
		D	***E***	***F***	***G***
0	***A***	24	32		
–11	***B***		21	28	
–9	***C***			30	0

Improvement indices for cells:

$AF = 25 - 0 - 39 = -14$

$AG = 0 - 0 - 9 = -9$

$BD = 27 - (-11) - 24 = 14$

$BG = 0 - (-11) - 9 = 2$

$CD = 19 - (-9) - 24 = 4$

$CE = 26 - (-9) - 32 = 3$

6 d The entering cell is AF, as this cell has the most negative improvement index. Entering θ into cell AF and applying the stepping-stone method gives:

	D	*E*	*F*	*G*	**Supply**
A	22	$4-\theta$	θ		26
B		$12+\theta$	$11-\theta$		23
C			7	17	24
Demand	22	16	18	17	73

The largest possible value of θ is 4, making AE the exiting cell, and giving this improved solution.

	D	*E*	*F*	*G*	**Supply**
A	22		4		26
B		16	7		23
C			7	17	24
Demand	22	16	18	17	

7 a Total supply (30 + 34 + 35 = 99) ≠ Total demand (35 + 37 + 42 = 114)
So the problem is not balanced.
A dummy supply point must be created as demand is greater than supply.

7 b Introducing a dummy supply point D with supply 15 (114 – 99 = 15), the initial solution using the north-west corner method is:

	X	Y	Z	Supply
A	30			30
B	5	29		34
C		8	27	35
D			15	15
Demand	35	37	42	

The shadow costs for the initial solution are:

		18	11	7
		X	Y	Z
0	A	18		
8	B	26	19	
17	C		28	24
–7	D			0

Improvement indices for cells:

$AY = 25 - 0 - 11 = 14$ $\quad$ $CX = 20 - 17 - 18 = -15$

$AZ = 21 - 0 - 7 = 14$ $\quad$ $DX = 0 - (-7) - 18 = -11$

$BZ = 27 - 8 - 7 = 12$ $\quad$ $DY = 0 - (-7) - 11 = -4$

The solution may not be optimal, since there are cells with negative improvement indices.

First iteration

The entering cell is CX. Entering θ into cell CX and applying the stepping-stone method gives:

	X	Y	Z	Supply
A	30			30
B	$5-\theta$	$29+\theta$		34
C	θ	$8-\theta$	27	35
D			15	15
Demand	35	37	42	

7 b (continued)

The largest possible value of θ is 5, making BX the exiting cell, and giving this improved solution.

	X	**Y**	**Z**	**Supply**
A	30			30
B		34		34
C	5	3	27	35
D			15	15
Demand	35	37	42	

The shadow costs for this solution are:

		18	26	22
		X	**Y**	**Z**
0	**A**	18		
–7	**B**		19	
2	**C**	20	28	24
–22	**D**			0

Improvement indices for cells:

$AY = 25 - 0 - 26 = -1$

$AZ = 21 - 0 - 22 = -1$

$BX = 26 - (-7) - 18 = 15$

$BZ = 27 - (-7) - 22 = 12$

$DX = 0 - (-22) - 18 = 4$

$DY = 0 - (-22) - 26 = -4$

Second iteration

The entering cell is DY. Entering θ into cell DY and applying the stepping-stone method gives:

	X	**Y**	**Z**	**Supply**
A	30			30
B		34		34
C	5	$3-\theta$	$27+\theta$	35
D		θ	$15-\theta$	15
Demand	35	37	42	

The largest possible value of θ is 3, making CY the exiting cell, and giving this improved solution.

	X	**Y**	**Z**	**Supply**
A	30			30
B		34		34
C	5		30	35
D		3	12	15
Demand	35	37	42	

7 b (continued)

The shadow costs for this solution are:

		18	22	22
		X	***Y***	***Z***
0	***A***	18		
–3	***B***		19	
2	***C***	20		24
–22	***D***		0	0

Improvement indices for cells:

$AY = 25 - 0 - 22 = 3$

$AZ = 21 - 0 - 22 = -1$

$BX = 26 - (-3) - 18 = 11$

$BZ = 27 - (-3) - 22 = 8$

$CY = 28 - 2 - 22 = 4$

$DX = 0 - (-22) - 18 = 4$

Third iteration

The entering cell is *AZ*. Entering θ into cell *AZ* and applying the stepping-stone method gives:

	X	***Y***	***Z***	**Supply**
A	$30-\theta$		θ	30
B		34		34
C	$5+\theta$		$30-\theta$	35
D		3	12	15
Demand	35	37	42	

The largest possible value of θ is 30. There is a choice of exiting cell. In this situation, the choice is arbitrary. Choosing *AX* as exiting cell and entering 0 in *CZ*, gives this improved solution.

	X	***Y***	***Z***	**Supply**
A			30	30
B		34		34
C	35		0	35
D		3	12	15
Demand	35	37	42	

7 b (continued)

The shadow costs for this solution are:

		17	21	21
		X	***Y***	***Z***
0	***A***			21
–2	***B***		19	
3	***C***	20		24
–21	***D***		0	0

Improvement indices for cells:

$AX = 18 - 0 - 17 = 1$ $\quad$ $BZ = 27 - (-2) - 21 = 8$

$AY = 25 - 0 - 21 = 4$ $\quad$ $CY = 28 - 3 - 21 = 4$

$BX = 26 - (-2) - 17 = 11$ $\quad$ $DX = 0 - (-21) - 17 = 4$

There are now no negative improvement indices, so this solution is optimal.
The optimal solution is:

A supplies 30 units to *Z*
B supplies 34 units to *Y*

C supplies 35 units to *X*
Y has a shortfall of 3 units
Z has a shortfall of 12 units.

Total cost = 30 × 21 + 34 × 19 + 35 × 20 = 1976