Regression, correlation and hypothesis testing 1A

1 a As noted at the beginning of Section 1.1, the equation $Y = 1.2 + 0.4X$ can be rewritten as $\log y = 1.2 + 0.4\log x$, which is of the form $\log y = \log a + n\log x$ and so $y = ax^n$.

b $Y = 1.2 + 0.4X$

$\Rightarrow \log y = 1.2 + 0.4\log x$

$\Rightarrow y = 10^{1.2+0.4\log x} = 10^{1.2} \times 10^{0.4\log x}$

$\Rightarrow y = 10^{1.2} \times 10^{\log x^{0.4}} = 10^{1.2} \times x^{0.4}$

Therefore $a = 10^{1.2} \approx 15.8$ (3 s.f.) and $n = 0.4$

2 a As noted at the beginning of Section 1.1, the equation $Y = 0.4 + 1.6X$ can be rewritten as $\log y = 0.4 + 1.6x$, which is of the form $\log y = \log k + x\log b$ and so $y = kb^x$.

b $Y = 0.4 + 1.6X$

$\Rightarrow \log y = 0.4 + 1.6x$

$\Rightarrow y = 10^{0.4+1.6x} = 10^{0.4} \times 10^{1.6x}$

$\Rightarrow y = 10^{0.4} \times \left(10^{1.6}\right)^x$

Therefore $k = 10^{0.4} \approx 2.51$ (3 s.f.) and $b = 10^{1.6} \approx 39.8$.

3 In the linear model $Y = mX + c$, where m and c are constants,
$Y = \log y$ and $X = \log x$, so $\log y = m\log x + c$

Therefore $c = \log a$

The point (0, 172) lies on the line, so $c = 172$ and $\log a = 172 \Rightarrow a = 10^{172}$

(23, 109) lies on $Y = mX + 172$:

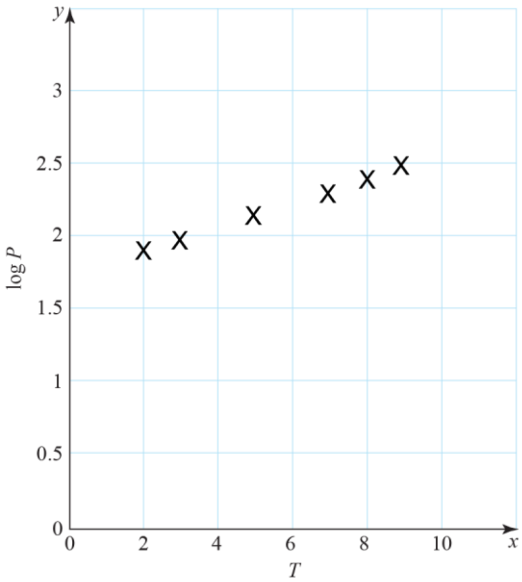
$109 = 23m + 172$

$\Rightarrow 23m = 109 - 172$

$\Rightarrow m = \frac{-63}{23} \approx -2.739$ (3 d.p.).

4 a

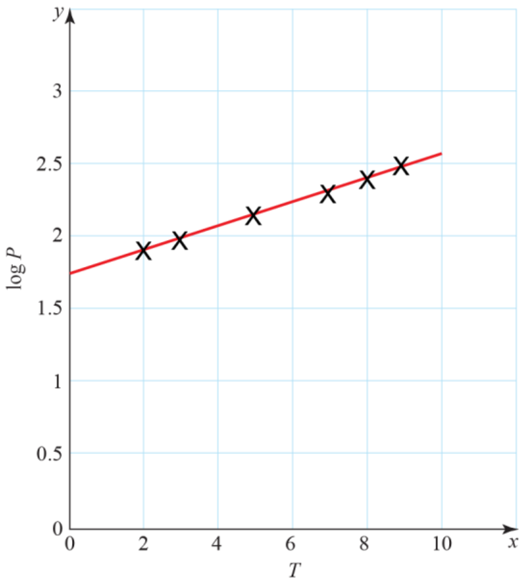
T	2	3	5	7	8	9
log *P*	1.86	1.93	2.10	2.25	2.33	2.41



b The points seem to lie on a straight line with a positive gradient, which suggests a strong positive correlation.

c Yes – the variables show a linear relationship when log P is plotted against T.

d



If $\log P = mT + c$ then $P = 10^c(10^m)^T$. Measuring the gradient and intercept from the line of best fit with computer provides $c = 1.69927$ and $m = 0.07901$. These then give $a = 50.0345502089$ and $b = 1.19949930315$. Allow c between 1.65 and 1.8 so that a can be between 44.7 and 63.1. Since the gradient is small, it is better found using the original data points. Allow m between 0.077 and 0.082 so that b can be between 1.19 and 1.21.

4 e The approximate model is $P = 50.1\times1.2^T$ and so increasing T by 1 gives
$50.1\times1.2^{T+1} = (50.1\times1.2^T)\times1.2$
which means increasing T by 1 corresponds to an increase of the population by 20%. Note that T is recorded in months, and so for every month that passes, the population of moles increases by 20%.

5 a The equation $t = a + bn$ is the equation of a straight line, but the data on the scatter diagram are not close to a straight line.

b $y = -0.301 + 0.6x$
$\Rightarrow \log t = -0.301 + 0.6\log n$
$\Rightarrow t = 10^{-0.301+0.6\log n} = 10^{-0.301}\times10^{0.6\log n}$
$\Rightarrow t = 10^{-0.301}\times10^{\log n^{0.6}}$
$\Rightarrow t = 10^{-0.301}\times n^{0.6}$
Therefore $a = 10^{-0.301} \approx 0.5$ (3 s.f.) and $k = 0.6$.

6 $y = 1.31x - 0.41$
$\Rightarrow \log r = 1.31\log c - 0.41$
$\Rightarrow r = 10^{1.31\log c - 0.41} = 10^{1.31\log c}\times10^{-0.41}$
$\Rightarrow r = 10^{\log c^{1.31}}\times10^{-0.41} = c^{1.31}\times10^{-0.41}$
Therefore $r = 0.389\times c^{1.31}$ (3 s.f.).

7 $y = 0.0023 + 1.8x$
$\Rightarrow \log m = 0.0023 + 1.8\log h$
$\Rightarrow m = 10^{0.0023+1.8\log h} = 10^{0.0023}\times10^{1.8\log h}$
$\Rightarrow m = 10^{0.0023}\times10^{\log h^{1.8}} = 10^{0.0023}\times h^{1.8}$
Therefore $a = 10^{0.0023} \approx 1.0$ (3 s.f.) and $n = 1.8$.

8 a $y = 0.09 + 0.05x$
$\Rightarrow \log g = 0.09 + 0.05t$
$\Rightarrow g = 10^{0.09+0.05t} = 10^{0.09}\times10^{0.05t}$
$\Rightarrow g = 10^{0.09}\times\left(10^{0.05}\right)^t$
Therefore $a = 10^{0.09} \approx 1.23$ and $b \approx 1.12$ (3 s.f.)

b If you increase the temperature by 1 °C, b is the increase in the growth rate g, i.e. b is the rate of change of g per degree.

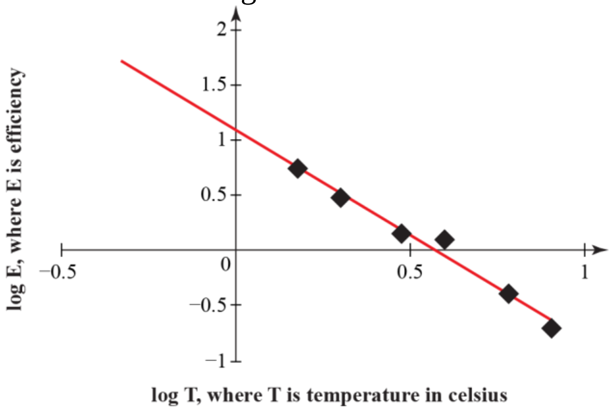
c 35 °C is outside of the range of data (extrapolation).

Challenge

a Construct a table of values for log$T$ and log$E$:

log *T*	0.0792	0.176	0.301	0.477	0.602	0.778	0.903
log *E*	0.954	0.74	0.477	0.146	0.0969	−0.398	−0.699

Plot the scatter diagram and draw a line of best fit:



The fact that the data are fitted by a straight line shows the validity of the relationship.

b The y-intercept of the line of best fit is 1.1 (to 2 s.f.).
So log a = 1.1 (approximately)
$a = 10^{1.1} = 12.58925\ldots = 12.6$
From the graph, the gradient of the line of best fit is approximately −1.90 (to 3 s.f.), so b = −1.90.

c The model is of the form $\log E = \log a + b\log T$, but the expression $\log a + b\log T$ is not defined when $T = 0$ since $\log(0)$ is undefined (it approaches $-\infty$ as $T \to 0^+$).

Regression, correlation and hypothesis testing 1B

1 a $r = 0.9$ is a good approximation, since the points lie roughly, but not exactly, on a straight line. Remember that the value of r tells you how 'close' the data is to having a perfect positive or negative linear relationship.

b Clearly r is negative, and the data is not as close to being linear as in part **a**. $r = -0.7$ is therefore a good approximation.

c The data seems to have some negative correlation, but is rather 'random'. Because so many points would lie far away from a line of best fit, $r = -0.3$ is a good approximation.

2 a The product moment correlation coefficient gives the type (positive or negative) and strength of linear correlation between v and m.

b By inputting the (ordered) data into your calculator, $r = 0.870$ (to 3 s.f.).

3 a $r = -0.854$ (to 3 s.f.)

b There is a negative correlation. The relatively older young people took less time to reach the required level.

4 a The completed table should read:

Time, *t*	1	2	4	5	7
Atoms, *n*	231	41	17	7	2
log *n*	2.36	1.61	1.23	0.845	0.301

b $r = -0.980$ (to 3 s.f.)

c There is an almost perfect negative correlation with the data in the form log n against t, which suggests an exponential decay curve. (This uses knowledge from the previous section.)

d $y = 2.487 - 0.320x$

$\Rightarrow \log n = 2.487 - 0.320t$

$\Rightarrow n = 10^{2.487-0.320t} = 10^{2.487} \times 10^{-0.320t}$

$\Rightarrow n = 10^{2.487} \times \left(10^{-0.320}\right)^t$

Therefore $a = 10^{2.487} = 307$ (3 s.f.) and $b = 10^{-0.320} = 0.479$ (3 s.f.).

5 a

Width, *w*	3	4	6	8	11
Mass, *m*	23	40	80	147	265
log *w*	0.4771	0.6021	0.7782	0.9031	1.041
log *m*	1.362	1.602	1.903	2.167	2.423

b $r = 0.9996$

c A graph of log w against log m is close to a straight line as the value of r is close to 1, therefore $m = kw^n$ is a good model for this data.

d $y = 0.464 + 1.88x$

$\Rightarrow \log m = 0.464 + 1.88 \log w$

$\Rightarrow m = 10^{(0.464+1.88 \log w)}$

$\Rightarrow m = 10^{0.464} \times w^{1.88}$

Therefore $k = 10^{0.464} = 2.91$ (3 s.f.) and $n = 1.88$ (3 s.f.).

6 a $r = -0.833$ (3 s.f.)

b -0.833 is close to -1 so the data values show a strong to moderate negative correlation. A linear regression model is suitable for these data.

7 a ‘tr’ should be interpreted as a trace, which means a small amount.

b $r = -0.473$ (3 s.f.), treating ‘tr’ values as zero.

c The data show a weak negative correlation so a linear model may not be best; there may be other variables affecting the relationship or a different model might be a better fit.

Challenge

Take logs of the data in order to compute all of the required relationships:

x	3.1	5.6	7.1	8.6	9.4	10.7
y	3.2	4.8	5.7	6.5	6.9	7.6
log *x*	0.491	0.748	0.851	0.934	0.973	1.03
log *y*	0.505	0.681	0.756	0.813	0.839	0.881

Compute the PMCC for x and log y: $r = 0.985$ (3 s.f.).
Compute the PMCC for log x and log y: $r = 1.00$ (3 s.f.).
Therefore the data indicate that log x and log y have a strong positive linear relationship. From the previous section, the data indicate a relationship of the form $y = kx^n$.

Regression, correlation and hypothesis testing 1C

1 **a** H_0: $\rho = 0$, H_1: $\rho \neq 0$, critical value = ±0.3120. Reject H_0: there is reason to believe at the 5% level of significance that there is a correlation between the scores.

b H_0: $\rho = 0$, H_1: $\rho \neq 0$, critical value = ±0.3665. Accept H_0: there is no evidence of correlation between the two scores at the 2% level of significance.

2 **a** $r = -0.960$ (3 s.f.)

b H_0: $\rho = 0$, H_1: $\rho \neq 0$, critical value = ±0.8745. Reject H_0: there is reason to believe at the 1% level of significance that there is a correlation between the scores.

3 **a** The product moment correlation coefficient measures the type and strength of linear correlation between two variables.

b $r = 0.935$ (Get this value directly from your calculator.)

c

$$\left.\begin{array}{l} H_0 : \rho = 0 \\ H_1 : \rho > 0 \end{array}\right\} \text{ 1-tail } \alpha = 0.05$$

test statistic = 0.935

critical values = 0.4973

t.s. > c.v., so reject H_0.

Conclude there is positive correlation between theoretical Biology and practical Biology marks – this implies that students who do well in theoretical Biology tests also tend to do well in practical Biology tests.

d There is a probability of 0.05 that the null hypothesis is true.

4 **a** $r = 0.68556...$
so $r = 0.686$ (3 s.f.)
(NB. In the exam get this directly from your calculator. If you set up a table of results you are likely to run out of time.)

b H_0: $\rho = 0$, H_1: $\rho > 0$, critical value = 0.6215. Reject H_0: there is reason to believe that there is a linear correlation between the English and Mathematics marks.

5 $r = 0.793$
(NB. In the exam get this directly from your calculator. If you set up a table of results you are likely to run out of time.)

$$\left.\begin{array}{l} H_0 : \rho = 0 \\ H_1 : \rho > 0 \end{array}\right\} \text{ 1-tail } \alpha = 0.01$$

test statistic = 0.793

critical values = 0.8822

t.s. < c.v. so accept H_0.

Conclude there is insufficient evidence at the 1% significance level to support the company's belief.

6 H_0: $\rho = 0$, H_1: $\rho < 0$, critical value = −0.4409. Accept H_0. There is evidence that the researcher is incorrect to believe that there is negative correlation between the amount of solvent and the rate of the reaction.

7 The safari ranger's test.
Type: 1-tailed
$H_0 : \rho = 0$
$H_1 : \rho > 0$
Sample size: 10
$r = 0.66$
He has sufficient evidence to reject H_0. The corresponding part of the table reads:

0.10	0.05	0.025	0.01	0.005	Sample size
0.4428	494	0.6319	0.715	0.7646	10

Therefore the least possible significance level for the ranger's test is 2.5%.

8 The information from the question is as follows:
Type: 1-tailed
$H_0 : \rho = 0$
$H_1 : \rho > 0$
Sample size: unknown
$r = 0.715$.
He has sufficient evidence to reject H_0. Part of the corresponding column of the table reads:

PMCC at 0.025 level of significance	Sample size
0.9500	4
0.8783	5
0.8114	6
0.7545	7
0.7067	8

Therefore the smallest possible sample size is 8.

9 a $r = -0.846$ (3 s.f.)

b H_0: $\rho = 0$, H_1: $\rho < 0$, critical value = −0.8822. Accept H_0. There is evidence that the employee is incorrect to believe that there is a negative correlation between humidity and visibility.

10 a This is a two-tailed test, so the scientist would need to halve the significance level.

b ± 0.4438

Regression, correlation and hypothesis testing Mixed exercise 1

1 a

Base area, x (cm^2)	1.1	1.3	1.9	2.2	2.5	3.7
Time, t (hours)	0.7	0.9	1.5	1.8	2.2	3.8
log x	0.0414	0.114	0.279	0.342	0.398	0.568
log t	−0.155	−0.0458	0.176	0.255	0.342	0.580

Calculating the PMCC for log x and log t: $r = 0.9998$.

b r is close to 1, so a graph of log t against log x shows a straight line, suggesting that the relationship is in the form $t = ax^n$.

c $\log t = -0.215 + 1.38\log x$

$\Rightarrow t = 10^{-0.215+1.38\log x} = 10^{-0.215} \times 10^{1.38\log x}$

$\Rightarrow t = 10^{-0.215} \times 10^{\log x^{1.38}} = 10^{-0.215} \times x^{1.38}$

Therefore $a = 10^{-0.215} \approx 0.617$ (3 s.f.) and $n = 1.38$

2 a

Temperature, t (°C)	38	51	72	83	89	94
Dry residue, d (grams)	4.3	11.7	58.6	136.7	217.0	318.8
y = log d	0.633	1.07	1.77	2.14	2.34	2.50

$y = -0.635 + 0.0334x$

$\Rightarrow \log d = -0.635 + 0.0334t$

$\Rightarrow d = 10^{(-0.635+0.0334)^t} = 10^{0.635} \times 10^{0.0334t}$

$\Rightarrow d = 10^{-0.635} \times \left(10^{0.0334}\right)^t$

Therefore $a = 10^{-0.635} = 0.232$ (3 s.f.) and $b = 10^{0.0334} = 1.08$ (3 s.f.)

b 151 °C is outside the range of the data (extrapolation).

3 As a person’s age increases their score on a memory test decreases.

4 a Each cow should be given 7 units. The yield levels off at this point. This can be seen even more clearly by drawing a scatter plot.

b $r = 0.952$ (3 s.f.)

c It would be less than 0.952. The yield of the last three cows is no greater than that of the seventh cow.

5 **a** $r = -0.972$ (to 3 s.f.)

b There is strong negative correlation. As c increases, f decreases.

6 **a** $r = 0.340$ (3 s.f.)

b $H_0: \rho = 0$
$H_1: \rho \neq 0$

Sample size = 10

Significance level in each tail = 0.025

From the table, critical values of r for a 2.5% significance level with a sample size of 10 are $r = \pm 0.6319$

So the critical region is $r < -0.6319$ and $r > 0.6319$

$0.340 < 0.6319$ so do not reject H_0.

There is not sufficient evidence, at the 5% level of significance, of correlation between age and salary. This means that an older person in this profession does not necessarily earn more than a younger person.

7 **a** $r = 0.937$ (3 s.f.)

b H_0: $\rho = 0$, H_1: $\rho \neq 0$, critical value = ± 0.6319. Reject H_0. There is evidence that there is a correlation between the age of a machine and its maintenance costs.

8

$$\left.\begin{array}{l} H_0: \rho = 0 \\ H_1: \rho < 0 \end{array}\right\} \text{ 1-tail } \alpha = 0.05$$

Test statistic $= -0.975$

$n = 9$, critical value $= -0.5822$

Lower tail test, t.s. < c.v. since $-0.975 < -0.5822$ reject H_0.

Conclude there is evidence of negative correlation. There is evidence that the greater the height above sea level, the lower the temperature at 7.00 a.m. is likely to be.

9

$$\left.\begin{array}{l} H_0: \rho = 0 \\ H_1: \rho > 0 \end{array}\right\} \text{ 1-tail } \alpha = 0.05$$

Test statistic $= r = 0.972$

$n = 9$, critical value $= 0.5822$

Upper tail test, t.s. > c.v. since $0.972 > 0.5822$ so reject H_0.

Conclude there is evidence of a positive association between age and weight. This means the older a baby is, the heavier it is likely to be.

10 a $r = 0.940$ (to 3 s.f.)

b $H_0: \rho = 0$, $H_1: \rho > 0$, critical value 0.7293. Reject H_0. There is evidence that sunshine hours and ice cream sales are positively correlated.

11 $r = 0.843$ (3 s.f.), $H_0: \rho = 0$, $H_1: \rho > 0$, critical value 0.8054. Reject H_0. There is evidence that mean windspeed and daily maximum gust are positively correlated.

12 $r = -0.793$ (3 s.f.), $H_0: \rho = 0$, $H_1: \rho < 0$, critical value -0.7545. Reject H_0. There is evidence that temperature and pressure are negatively correlated.

Conditional probability 2A

1 a This is the set of anything not in set B but in set A. So the shaded region consists of the part of A which does not intersect with B, i.e. $A \cap B'$.

b The shaded region includes all of B and the region outside of A and B, i.e. $B \cup A'$.

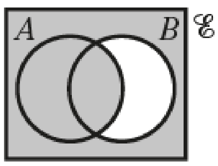
c There are two regions to describe. The first is the intersection of A and B, i.e. $A \cap B$ and the second is everything that is not in either A or B, i.e. $A' \cap B'$. Therefore the shaded region is $(A \cap B) \cup (A' \cap B')$.

d The shaded region is anything that is in A and B and C, i.e. $A \cap B \cap C$.

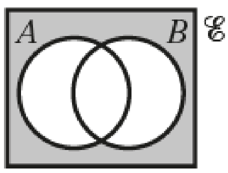
e The shaded region is anything that is either in A or B or C, i.e. $A \cup B \cup C$.

f The shaded region is anything that is either in A or B but is not in C. So the shaded region consists of the part of $A \cup B$ which does not intersect with C, i.e. $(A \cup B) \cap C'$.

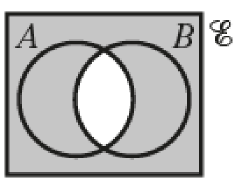
2 a Shade set A. The set B' consists of the region outside of A and B and the region inside A that does not intersect B. Therefore $A \cup B'$ is the region consisting of both these regions.



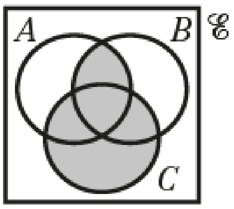
b Since this is an intersection, the region must satisfy both conditions. The first is to be in A'. This consists of two regions: one inside B and not in $A \cap B$; and one outside of A and B. The second condition is to be in B'. Again, this consists of two regions: one inside A and not in $A \cap B$; and one outside of A and B. Therefore $A' \cap B'$ is the region outside of A and B (since this region was in both A' and B'). One way to help picture this is to shade the regions A' and B' differently (either with different colours or using a different pattern for each). The intersection is then the region that includes both colours or patterns.



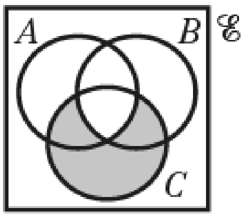
c In order to describe $(A \cap B)'$ it is sensible to first describe $A \cap B$. This is the single region which is included in both A and B. The complement is then everything *except* this region.



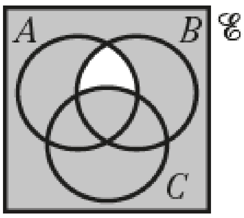
3 a The set $(A \cap B) \cup C$ is the union of the sets $A \cap B$ and C. On the blank diagram, the set $A \cap B$ consists of the two regions that are both contained within A and B. The remaining regions within set C can then be shaded in.



b First describe $A' \cup B'$. The set $A' \cup B'$ is everything apart from $A \cap B$. So the intersection of $A' \cup B'$ and C is everything in C apart from that part of C that intersects $A \cap B$.



3 c First describe $A \cap B \cap C'$. Brackets have not been included since for any sets X, Y and Z $(X \cap Y) \cap Z = X \cap (Y \cap Z)$. The intersection of $A \cap B$ and C' is the region within $A \cap B$ that does not intersect C. Therefore $(A \cap B \cap C')'$ is everything *except* this region.



4 a K is the event 'the card chosen is a king'.

$$\mathrm{P}(K) = \frac{4}{52} = \frac{1}{13}$$

b C is the event 'the card chosen is a club'.

$$\mathrm{P}(C) = \frac{1}{4}$$

c $C \cap K$ is the event 'the card chosen is the king of clubs'.

$$\mathrm{P}(C \cap K) = \frac{1}{52}$$

d $C \cup K$ is the event 'the card chosen is a club or a king or both'.

$$\mathrm{P}(C \cup K) = \frac{16}{52} = \frac{4}{13}$$

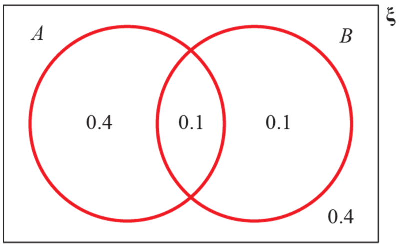
4 e C' is the event 'the card chosen is a not a club'.

$$\mathrm{P}(C') = \frac{3}{4}$$

f $K' \cap C$ is the event 'the card chosen is not a king and is a club'.

$$\mathrm{P}(K' \cap C) = \frac{12}{52} = \frac{3}{13}$$

5 Use the information in the question to draw a Venn diagram that will help in answering each part.



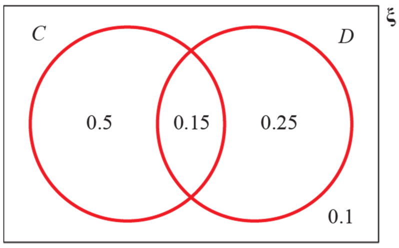
a $A \cup B$ is the region contained by sets A and B. So $\mathrm{P}(A \cup B) = 0.4 + 0.1 + 0.1 = 0.6$

b B' is the region that is not in set B. $\mathrm{P}(B') = 0.8$

c $A \cap B'$ is the region inside set A but outside set B. $\mathrm{P}(A \cap B') = 0.4$

d $A \cup B'$ is the region inside set A and the region outside set B, i.e. everything but the region inside set B that is not also in set A. $\mathrm{P}(A \cup B') = 0.4 + 0.1 + 0.4 = 0.9$

6 Use the information in the question to draw a Venn diagram that will help in answering each part.



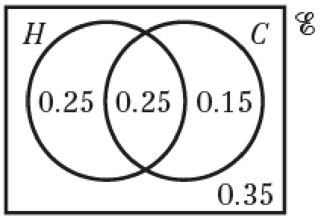
a $C' \cap D$ is the region inside set D but outside set C. $\mathrm{P}(C' \cap D) = 0.25$

b $C \cap D'$ is the region inside set C but outside set D. $\mathrm{P}(C \cap D') = 0.5$

c $\mathrm{P}(C) = 0.65$

d $C' \cup D'$ is the region outside set C and the region outside set D, i.e. everything but the region that is in both sets C and D. $\mathrm{P}(C' \cup D') = 0.85$

7 a

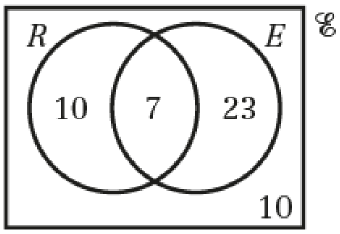


b i $P(H \cup C)$ means that either one of $H \cap C'$, $H \cap C$ or $H' \cap C$ occurs. Alternatively,
$P(H \cup C) = P(H) + P(C) - P(H \cap C) = 0.5 + 0.4 - 0.25 = 0.65$

ii $H' \cap C$ is the region inside set C but outside set H. $P(H' \cap C) = 0.15$

iii $H \cup C'$ is the region inside set H and the region outside set C, i.e. everything but the region inside set C that is not also in set H. $P(H \cup C') = 0.25 + 0.25 + 0.35 = 0.85$

8 a Only the possible outcomes of the two events need to considered, and so the Venn diagram should consist of two circles, one labelled 'R' for red and one labelled 'E' for even. They should intersect.



b i Note that $n(R \cup E) = n(R) + n(E) - n(R \cap E)$
$n(R \cap E) = n(R) + n(E) - n(R \cup E)$
$\Rightarrow n(R \cap E) = 17 + 30 - 40 = 7$

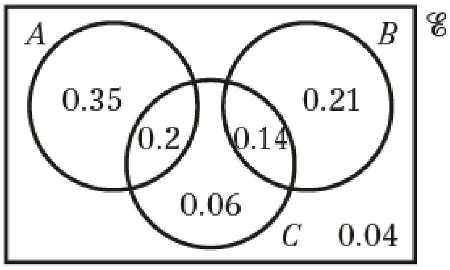
ii The region $R' \cap E'$ lies outside of both R and E.
Since there are 50 counters, $n(R' \cap E') = 50 - n(R \cup E) = 50 - 40 = 10$

So $P(R' \cap E') = \frac{10}{50} = \frac{1}{5} = 0.2$

iii From part **b i** $n(R \cap E) = 7$, so $n(R \cap E)' = 50 - 7 = 43$

So $P((R \cap E)') = \frac{43}{50} = 0.86$

9 a Since A and B are mutually exclusive, $P(A \cap B) = 0$ and they need no intersection on the Venn diagram. From the question, $P(A \cap C) = 0.2$ and so this can immediately be added to the diagram. Since B and C are independent, $P(B \cap C) = P(B) \times P(C) = 0.35 \times 0.4 = 0.14$ and this can also be added to the diagram. The remaining region in B must be $P(B) - P(B \cap C) = 0.35 - 0.14 = 0.21$, the remaining region for A must be $P(A) - P(A \cap C) = 0.55 - 0.2 = 0.35$ and the remaining region for C must be $P(C) - P(A \cap C) - P(B \cap C) = 0.4 - 0.2 - 0.14 = 0.06$. This means that the region outside of A, B and C must be 1 – 0.35 – 0.2 – 0.21 – 0.14 – 0.06 = 0.04.



b i The set $A' \cap B'$ must be outside of A and outside of B. These two regions are labelled 0.06 and 0.04. Therefore $P(A' \cap B') = 0.06 + 0.04 = 0.1$

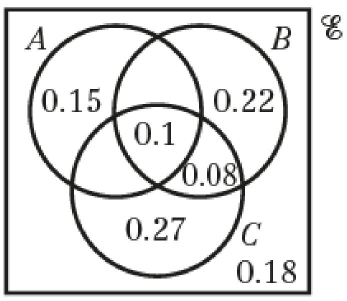
ii The region $B \cap C'$ is the region inside set B but outside set C, it is labelled 0.21 on the Venn diagram and is disjoint from A. Therefore $P(A \cup (B \cap C')) = P(A) + 0.21 = 0.55 + 0.21 = 0.76$

iii Since $A \cap C$ consists of a single region, $(A \cap C)'$ consists of everything in the diagram except for that region. But B' includes the region $A \cap C$ and so $(A \cap C)' \cup B$ includes everything in the diagram, and so $P((A \cap C)' \cup B') = 1$

10 a Start with a Venn diagram with all possible intersections. Then find the region $A \cap B \cap C$, which is at the centre of the diagram, and label it 0.1.

Now, since A and B are independent, $P(A \cap B) = P(A) \times P(B) = 0.25 \times 0.4 = 0.1$, and as B and C are independent $P(B \cap C) = P(B) \times P(C) = 0.4 \times 0.45 = 0.18$. Use these results to find values for the other intersections. $P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C) = 0.1 - 0.1 = 0$;
$P(B \cap C \cap A') = P(B \cap C) - P(A \cap B \cap C) = 0.18 - 0.1 = 0.08$; and $P(A \cap C \cap B') = 0$ is given in the question.

Now find values for the remaining parts of the diagram. For example,
$P(A \cap B' \cap C') = P(A) - P(A \cap B \cap C') - P(A \cap C \cap B') - P(A \cap B \cap C) = 0.25 - 0 - 0 - 0.1 = 0.15$
Similarly, $P(B \cap A' \cap C') = 0.4 - 0.1 - 0.08 = 0.22$ and $P(C \cap A' \cap B') = 0.45 - 0.1 - 0.08 = 0.27$
Finally calculate the region outside sets A, B and C,
$P(A \cup B \cup C)' = 1 - 0.15 - 0.1 - 0.22 - 0.08 - 0.27 = 0.18$

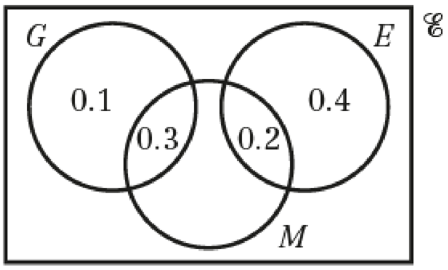


10 b i There are several ways to work out the regions that comprise the set $A' \cap (B' \cup C)$. One way is to determine, for each region, whether it lies in A' and $B' \cup C$. Alternatively, find the regions within A' (there are four) and then note that only one of these does not lie in $B' \cup C$. Summing the three remaining probabilities yields $\mathrm{P}(A' \cap (B' \cup C)) = 0.27 + 0.08 + 0.18 = 0.53$

ii The required region must be contained within *C*. Three of the four regions in C also lie in $A \cup B$, summing the probabilities yields $\mathrm{P}((A \cup B) \cap C) = 0 + 0.1 + 0.08 = 0.18$

c $\mathrm{P}(A') = 1 - P(A) = 0.75$, P(*C*)= 0.45 and, from the Venn diagram, $\mathrm{P}(A' \cap C) = 0.08 + 0.27 = 0.35$. Since $\mathrm{P}(A') \times \mathrm{P}(C) = 0.75 \times 0.45 = 0.3375 \neq 0.35$, the events A' and *C* are not independent.

11 a Since $\mathrm{P}(G \cap E) = 0$, it follows that $\mathrm{P}(M \cap G \cap E) = 0$. So $\mathrm{P}(M \cap G \cap E') = \mathrm{P}(M \cap G) = 0.3$ and $\mathrm{P}(G \cap M') = \mathrm{P}(G) - \mathrm{P}(G \cap M) = 0.4 - 0.3 = 0.1$. This only accounts for 40% of the book club, 60% is unaccounted for, but P(*E*) = 0.6, so this 60% read epic fiction. So all the remaining members who read murder mysteries must also read epic fiction. Therefore $\mathrm{P}(M \cap E' \cap G') = 0$, $\mathrm{P}(M \cap E \cap G') = \mathrm{P}(M) - \mathrm{P}(M \cap G) = 0.5 - 0.3 = 0.2$, and $\mathrm{P}(E \cap M' \cap G') = 0.6 - 0.2 = 0.4$.



b i $\mathrm{P}(M \cup G) = \mathrm{P}(M \cup G \cup E) - \mathrm{P}(E \cap M' \cap G') = 1 - 0.4 = 0.6$

ii In this case $\mathrm{P}((M \cap G) \cup (M \cap E)) = \mathrm{P}((M \cap G \cap E') \cup (M \cap G' \cap E))$ and so the required probability is $\mathrm{P}(M \cap G \cap E') + \mathrm{P}(M \cap G' \cap E) = 0.3 + 0.2 = 0.5$

c $\mathrm{P}(G') = 0.6$, $\mathrm{P}(M) = 0.5$ and so $\mathrm{P}(G') \times \mathrm{P}(M) = 0.6 \times 0.5 = 0.3$. Since $\mathrm{P}(G' \cap M) = 0.2$, the events are not independent.

12 a Since *A* and *B* are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B) = x \times y = xy$

b $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = x + y - xy$

c $\mathrm{P}(A \cup B') = \mathrm{P}(A) + \mathrm{P}(A' \cap B')$ and since
$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B) = 1 - (x + y - xy) = 1 - x - y + xy$ this means
$\mathrm{P}(A \cup B') = \mathrm{P}(A) + 1 - x - y + xy = x + 1 - x - y + xy = 1 - y + xy$

Challenge

a Use that the events are independent.

$$\begin{aligned}P(A \cap B \cap C) &= P((A \cap B) \cap C)\\ &= P(A \cap B) \times P(C)\\ &= P(A) \times P(B) \times P(C)\\ &= xyz\end{aligned}$$

b Using similar logic to the identity $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, build up to the correct expression. First, x represents one circle and its intersections with the other two circles being shaded. Then $x + y - xy$ represents two circles and their intersections with the third being shaded. Finally $x + y - xy + z - xz - yz$ represents all three circles shaded except for where all three intersect. From part **a**, the final expression is therefore $x + y - xy + z - xz - yz + xyz$.

An alternative approach is to start by considering $A \cup B$
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = x + y - xy$

Now find the union of $A \cup B$ and C
$P(A \cup B \cup C) = P(A \cup B) + P(C) - P((A \cup B) \cap C) = x + y + z - xy - P((A \cup B) \cap C)$ **(1)**

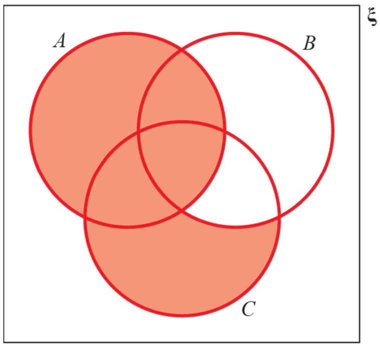
$(A \cup B) \cap C$ consists of the intersections of C with just A, with just B and with both A and B
So $(A \cup B) \cap C = (C \cap A \cap B') + (C \cap B \cap A') + (A \cap B \cap C)$

Consider the probabilities of each of these three regions in turn
$P(A \cap B \cap C) = xyz$ from part **a**
$P(C \cap A \cap B') = P(C \cap A) - P(A \cap B \cap C) = xz - xyz$
$P(C \cap B \cap A') = P(C \cap B) - P(A \cap B \cap C) = yz - xyz$
So $P(A \cup B) \cap C = xz - xyz + yz - xyz + xyz = xz + yz - xyz$ **(2)**

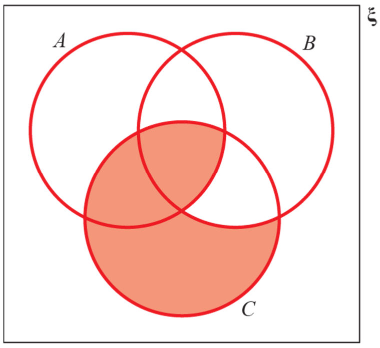
Now substitute the result for $P(A \cup B) \cap C$ from equation (**2**) into equation (**1**). This gives
$P(A \cup B \cup C) = x + y + z - xy - xz - yz + xyz$

Challenge

c First understand the region on a Venn diagram. The set $A \cup B'$ corresponds to the shaded regions:



Therefore the set $(A \cup B') \cap C$ corresponds to the shaded regions:



The unshaded part of C is the region $C \cap B \cap A'$

$\mathrm{P}(C \cap B \cap A') = P(C \cap B) - P(A \cap B \cap C) = yz - xyz$

So $\mathrm{P}((A \cup B') \cap C) = \mathrm{P}(C) - \mathrm{P}(C \cap B \cap A') = z - yz + xyz$

Conditional probability 2B

1 a There are 29 male students out of a total of 60 students.

$$\text{P(Male)} = \frac{29}{60}$$

b Restrict the sample space to the 29 male students; 18 of these prefer curry.

$$\text{P(Curry|Male)} = \frac{18}{29}$$

c Restrict the sample space to the 35 students that prefer curry; 18 of these are male.

$$\text{P(Male|Curry)} = \frac{18}{35}$$

d Restrict the sample space to the 31 female students; 14 of these prefer pizza.

$$\text{P(Pizza|Female)} = \frac{14}{31}$$

2 a By simple subtraction, there are 43 male members of the club (75 – 32 = 43). Of these 21 play badminton (43 – 22 = 21).

	Badminton	**Squash**	**Total**
Male	21	22	43
Female	15	17	32
Total	36	39	75

b i Restrict the sample space to the 39 members that play squash; 22 of these are male.

$$\text{P(Male|Squash)} = \frac{22}{39}$$

ii Restrict the sample space to the 36 members that play badminton; 15 of these are female.

$$\text{P(Female|Badminton)} = \frac{15}{36} = \frac{5}{12}$$

iii Restrict the sample space to the 32 members that are female; 17 of these play squash.

$$\text{P(Squash|Female)} = \frac{17}{32}$$

3 a There are 35 boys (80 – 45 = 35), of which 10 like chocolate (35 – 2 – 23 = 10).
Of the girls, 20 like strawberry (45 – 13 – 12 = 20).

	Girls	**Boys**	**Total**
Vanilla	13	2	15
Chocolate	12	10	22
Strawberry	20	23	43
Total	45	35	80

b i Restrict the sample space to the 43 children that like strawberry; 23 of these are boys.

$\text{P(Boy|Strawberry)} = \frac{23}{43}$

ii Restrict the sample space to the 15 children that like vanilla; 13 of these are girls.

$\text{P(Girl|Vanilla)} = \frac{13}{15}$

iii Restrict the sample space to the 35 boys; 10 of these like chocolate.

$\text{P(Chocolate|Boy)} = \frac{10}{35} = \frac{2}{7}$

4 a

Blue spinner (columns), **Red spinner** (rows)

	1	**2**	**3**	**4**
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

b i There 4 outcomes where $X = 5$, and 16 possible outcomes in total.

$\text{P}(X = 5) = \frac{4}{16} = \frac{1}{4}$

ii There are 4 equally likely outcomes where the red spinner is 2; and for one of these $X = 3$.

$\text{P}(X = 3|\text{Red spinner is 2}) = \frac{1}{4}$

iii There are 4 equally likely outcomes where $X = 5$, and for one of these the blue spinner is 3.

$\text{P}(\text{Blue spinner is 3}|X = 5) = \frac{1}{4}$

5 a

Dice 1 (columns); **Dice 2** (rows)

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

b There are 6 outcomes where Dice 1 shows 5, and for one of these the product is 20.

$$\text{P(Product is 20|Dice 1 shows a 5)} = \frac{1}{6}$$

c There are 4 outcomes where the product is 2, and for one of these Dice 2 shows a 6.

$$\text{P(Dice 2 shows a 6|Product is 12)} = \frac{1}{4}$$

d All outcomes are equally likely.

6 $$\text{P(Ace|Diamond)} = \frac{\text{P(Ace of diamonds)}}{\text{P(Diamond)}} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{1}{13}$$

7 Drawing a sample space diagram can be helpful in answering this question.

Coin 1 (columns); **Coin 2** (rows)

	H	**T**
H	HH	TH
T	HT	TT

a Note there are three outcomes where at least one coin lands on a head.

$$\text{P(HH|H)} = \frac{\text{P(Head and Head)}}{\text{P(Head)}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

7 b $\text{P(Head and Tail|Head)} = \dfrac{\text{P(Head and Tail)}}{\text{P(Head)}}$

$$= \frac{\frac{2}{4}}{\frac{3}{4}} = \frac{2}{3}$$

c Assume that the coins are not biased.

8 a 64 students do not watch sport (120 – 56 = 64).
43 students do not watch drama (120 – 77 = 43).

Use the fact that of those who watch drama, 18 also watch sport to complete the table.
For example, this means that 38 students who watch sport do not watch drama (56 – 18 = 38), and 59 students who watch drama do not watch sport (77 – 18 = 59).

Given that 43 students do not watch drama, but 38 students who do not watch drama watch sport, this means 5 students do not watch drama or sport (43 – 38 = 5).

	Watches drama (D)	**Does not watch drama (D′)**	**Total**
Watches sport (S)	18	38	56
Does not watch sport (S′)	59	5	64
Total	77	43	120

b i The probability that the student does not watch drama.

$$\text{P}(D') = \frac{43}{120}$$

ii The probability that the student does not watch sport ot drama.

$$\text{P}(S' \cap D') = \frac{5}{120} = \frac{1}{24}$$

iii The probability that the student also watches sport if they watch drama.

$$\text{P}(S|D) = \frac{18}{77}$$

iv The probability that the student does not watch drama if they watch sport.

$$\text{P}(D'|S) = \frac{38}{56} = \frac{19}{28}$$

9 a

	Women	**Men**	**Total**
Stick	26	18	44
No stick	37	29	66
Total	63	47	110

b i $\text{P(Uses a stick)} = \frac{44}{110} = \frac{2}{5}$

ii Restrict the sample space to the 63 women; 26 of these use a stick.

$\text{P(Uses a stick|Female)} = \frac{26}{63}$

iii Restrict the sample space to those who use a stick; 18 of these are men.

$\text{P(Male|Uses a stick)} = \frac{18}{44} = \frac{9}{22}$

10 Build up a table to show the options as follows. First note that as there are 450 female owners, so there are 300 male owners (750 – 450 = 300). Consider those who own cats. 320 owners in total own a cat. Since no one owns more than one type of pet, this means that 430 owners do not own a cat (750 – 320 = 430).

175 female owners have a cat. Since there are 450 female owners, this means that 275 female owners do not own a cat (450 – 175 = 275). 145 male owners own a cat (320 – 175 = 145) and so 155 male owners do not own a cat (300 – 145 = 155). This gives this table:

	Owns a cat	**Does not own a cat**	**Total**
Female	175	275	450
Male	145	155	300
Total	320	430	750

Of the 430 owners who do not own a cat, 250 of them own a dog. Therefore 180 of the owners own another type of pet (430 – 250 = 180). Since 25 males own another type of pet, this means that 155 women own another type of pet (180 – 25 = 155).

10 Finally, of the 450 women, 175 own a cat and 155 own something other than a cat or a dog. Therefore 120 women own a dog (450 – 175 –155 = 120) and 130 men own a dog (300 – 145 – 25 = 130). This information is summarised in this table:

	Owns a cat	**Owns a dog**	**Owns another type of pet**	**Total**
Female	175	120	155	450
Male	145	130	25	300
Total	320	250	180	750

a The probability that the owner does not own a dog or a cat.

$$\mathrm{P}(D' \cap C') = \frac{180}{750} = \frac{6}{25}$$

b The probability that a male owner (i.e. not female) owns a dog.

$$\mathrm{P}(D|F') = \frac{130}{300} = \frac{13}{30}$$

c The probability that a cat owner is male (i.e. not female).

$$\mathrm{P}(F'|C) = \frac{145}{320} = \frac{29}{64}$$

d The probability that a female owner does not own a dog or a cat.

$$\mathrm{P}((D' \cap C')|F) = \frac{155}{450} = \frac{31}{90}$$

Conditional probability 2C

1 a The probability $A \cup B$ includes all cases where either event A or event B occurs. So sum the probabilities for these three regions $A \cap B'$, $A \cap B$ and $B \cap A'$.

This gives $P(A \cup B) = 0.3 + 0.12 + 0.28 = 0.7$

b The probability that A occurs given that B occurs means that we are only selecting from those situations where B occurs. So the sample space is restricted to just circle B. The denominator of the fraction is 0.12 + 0.28 = 0.4. The numerator is when A also occurs i.e. when both A and B occur, which is the region $A \cap B$.

Therefore $P(A|B) = \frac{0.12}{0.4} = 0.3$

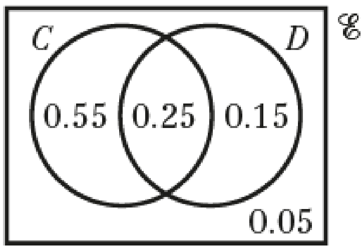
c The sample space is restricted to those instances where A has not occurred i.e. the regions $B \cap A'$ or $B' \cap A'$. This means the denominator will be 0.28 + 0.3 = 0.58. The numerator will consist of the cases where B has occurred i.e. $B \cap A'$.

Therefore $P(B|A') = \frac{0.28}{0.58} = 0.483$ (3 s.f.)

d The sample space is restricted to those instances where A or B has occurred i.e. the region $A \cup B$. From part **a** this has probability 0.7. The numerator will consist of the cases where B has occurred i.e. either $B \cap A'$ or $B \cap A$.

Therefore $P(B|A \cup B) = \frac{0.28 + 0.12}{0.7} = \frac{0.4}{0.7} = 0.571$ (3 s.f.)

2 a Fill in $P(C \cap D) = 0.25$ on the Venn diagram, and then calculate $P(C \cap D') = 0.8 - 0.25 = 0.55$, $P(D \cap C') = 0.4 - 0.25 = 0.15$ and $P(C \cup D)' = 1 - 0.25 - 0.55 - 0.15 = 0.05$



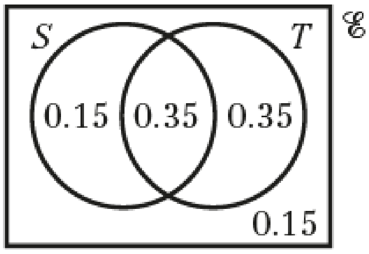
b i $P(C \cup D) = 0.55 + 0.25 + 0.15 = 0.95$

ii $P(C|D) = \frac{P(C \cap D)}{P(D)} = \frac{0.25}{0.4} = 0.625$

iii $P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{0.25}{0.8} = 0.3125$

iv $P(D'|C') = \frac{P(D' \cap C')}{P(C')} = \frac{0.05}{0.15 + 0.05} = 0.25$

3 a Since S and T are independent, $\mathrm{P}(S \cap T) = \mathrm{P}(S) \times \mathrm{P}(T) = 0.5 \times 0.7 = 0.35$, and use this result to fill in the Venn diagram.



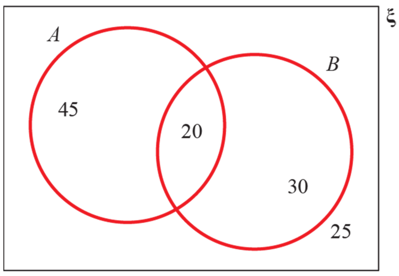
b i This is calculated to complete the Venn diagram in part **a**, $\mathrm{P}(S \cap T) = 0.35$

ii $\mathrm{P}(S|T) = \dfrac{\mathrm{P}(S \cap T)}{\mathrm{P}(T)} = \dfrac{0.35}{0.7} = 0.5$

iii $\mathrm{P}(T|S') = \dfrac{\mathrm{P}(T \cap S')}{\mathrm{P}(S')} = \dfrac{0.35}{0.5} = 0.7$

iv $\mathrm{P}(S|(S' \cup T')) = \dfrac{\mathrm{P}(S \cap (S' \cup T'))}{\mathrm{P}(S' \cup T')} = \dfrac{\mathrm{P}(S \cap T')}{0.15 + 0.35 + 0.15} = \dfrac{0.15}{0.65} = 0.231$ (3 s.f.)

4 a First produce a Venn diagram with the numbers of people in each region.



The Venn diagram can now be used to find the required probabilities. From the diagram,

$\mathrm{P}(A \cap B') = \dfrac{45}{120} = \dfrac{3}{8} = 0.375$

b $\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \dfrac{\frac{20}{120}}{\frac{50}{120}} = \dfrac{20}{50} = \dfrac{2}{5} = 0.4$

c $\mathrm{P}(B|A') = \dfrac{\mathrm{P}(B \cap A')}{\mathrm{P}(A')} = \dfrac{30}{55} = \dfrac{6}{11} = 0.545$ (3 s.f.)

d $\mathrm{P}(A|(A \cup B)) = \dfrac{\mathrm{P}(A \cap (A \cup B))}{\mathrm{P}(A \cup B)} = \dfrac{\mathrm{P}(A)}{\mathrm{P}(A \cup B)} = \dfrac{65}{95} = \dfrac{13}{19} = 0.684$ (3 s.f.)

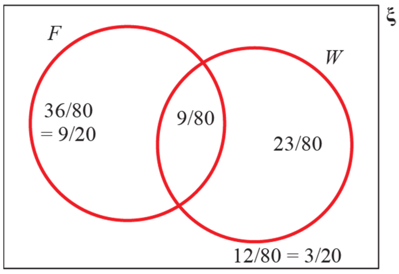
5 a Note that 12 cats like neither brand of food. So 80 – 12 = 68 cats like Feskers or Whilix or both. Use this and the other information in the question to calculate $\mathrm{P}(F \cap W)$ as follows:

$$\mathrm{P}(F \cup W) = \mathrm{P}(F) + \mathrm{P}(W) - \mathrm{P}(F \cap W)$$

$$\Rightarrow \mathrm{P}(F \cap W) = \mathrm{P}(F) + \mathrm{P}(W) - \mathrm{P}(F \cup W)$$

So $\mathrm{P}(F \cap W) = \frac{45}{80} + \frac{32}{80} - \frac{68}{80} = \frac{9}{80}$

This is a Venn diagram showing the result:



b $\mathrm{P}(F|W) = \frac{\mathrm{P}(F \cap W)}{\mathrm{P}(W)} = \frac{\frac{9}{80}}{\frac{32}{80}} = \frac{9}{32} = 0.281$ (3 s.f.)

c $\mathrm{P}(W|F) = \frac{\mathrm{P}(F \cap W)}{\mathrm{P}(F)} = \frac{\frac{9}{80}}{\frac{45}{80}} = \frac{9}{45} = \frac{1}{5} = 0.2$

d $\mathrm{P}(W'|F') = \frac{\mathrm{P}(F' \cap W')}{\mathrm{P}(F')} = \frac{\frac{12}{80}}{\frac{23+12}{80}} = \frac{12}{35} = 0.343$ (3 s.f.)

6 a $\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.2+0.1}{0.2+0.1+0.08+0.12} = \frac{0.3}{0.5} = 0.6$

b $\mathrm{P}(C|A') = \frac{\mathrm{P}(C \cap A')}{\mathrm{P}(A')} = \frac{0.1+0.08}{0.1+0.08+0.12+0.15} = \frac{0.18}{0.45} = 0.4$

c $\mathrm{P}((A \cap B)|C') = \frac{\mathrm{P}(A \cap B \cap C')}{\mathrm{P}(C')} = \frac{0.2}{0.2+0.2+0.12+0.15} = \frac{0.2}{0.67} = 0.299$ (3 s.f.)

d $\mathrm{P}(C|(A' \cup B')) = \frac{\mathrm{P}(C \cap (A' \cup B'))}{\mathrm{P}(A' \cup B')} = \frac{0.05+0.08+0.1}{0.2+0.05+0.08+0.12+0.1+0.15} = \frac{0.23}{0.7} = 0.329$ (3 s.f.)

7 a The fact that the student must watch at least one of the TV programmes means that the student is selected from a region contained in $A \cup B \cup C$. Therefore this question should be interpreted as:

$$\mathrm{P}(C|(A \cup B \cup C)) = \frac{\mathrm{P}(C \cap (A \cup B \cup C))}{\mathrm{P}(A \cup B \cup C)} = \frac{\mathrm{P}(C)}{\left(\frac{23}{29}\right)} = \frac{\left(\frac{9}{29}\right)}{\left(\frac{23}{29}\right)} = \frac{9}{23} = 0.391 \text{ (3 s.f.)}$$

The other way to do this is to note that only 23 students watch at least one of the TV programmes, and of these 9 watch programme *C*.

b The standard method is as follows:

$$\mathrm{P}(\text{exactly two programmes}|A \cup B \cup C) = \frac{\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C) - 3\mathrm{P}(A \cap B \cap C)}{\mathrm{P}(A \cup B \cup C)}$$

$$= \frac{\left(\frac{2}{29} + \frac{0}{29} + \frac{1}{29} - 3\frac{0}{29}\right)}{\left(\frac{23}{29}\right)} = \frac{3}{23} = 0.130 \text{ (3 s.f.)}$$

An alternative method is to note that 2 + 1 = 3 students watch exactly two of the programmes (they watch *A* and *B*, and *B* and *C*, respectively) and so 3 out of the 23 students that watch at least one of the TV programmes watch exactly two of the programmes.

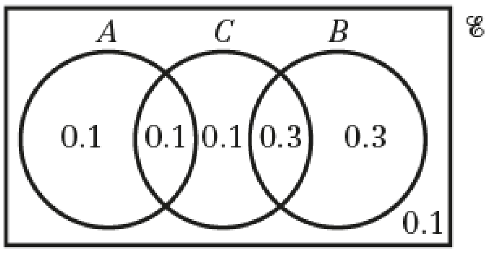
c $\mathrm{P}(B) = \frac{2+7+1}{29} = \frac{10}{29} = 0.345$ (3 s.f.)

$$\mathrm{P}(B|C) = \frac{\mathrm{P}(B \cap C)}{\mathrm{P}(C)} = \frac{\frac{1}{29}}{\frac{9}{29}} = \frac{1}{9} = 0.111 \text{ (3 s.f.)}$$

So $\mathrm{P}(B) \neq \mathrm{P}(B|C)$ and the events are not independent.

8 a $P(A \cap B) = 0$ since A and B are mutually exclusive.
$P(B \cap C) = P(B) \times P(C) = 0.6 \times 0.5 = 0.3$ since B and C are independent.
$P(B \cap C') = P(B) - P(B \cap C) = 0.6 - 0.3 = 0.3$
As $P(A \cap B) = 0$, $P(A \cup C) = 1 - P(B \cap C') - P(A' \cup B' \cup C') = 1 - 0.3 - 0.1 = 0.6$
$P(A \cup C) = P(A) + P(C) - P(A \cap C)$
$\Rightarrow P(A \cap C) = P(A) + P(C) - P(A \cup C) = 0.2 + 0.5 - 0.6 = 0.1$

Now it is straightforward to work out remaining regions for the Venn diagram
$P(A \cap C') = 0.2 - P(A \cap C) = 0.2 - 0.1 = 0.1$
$P(C \cap A' \cap B') = 0.5 - P(A \cap C) - P(B \cap C) = 0.5 - 0.1 - 0.3 = 0.1$

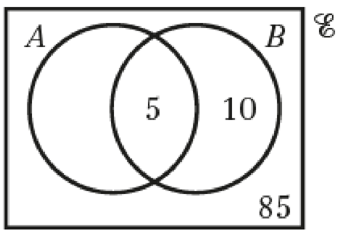


b i $P(A|C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{0.1}{0.5} = 0.2$

ii $P(B|C') = \dfrac{P(B \cap C')}{P(C')} = \dfrac{0.3}{0.1 + 0.3 + 0.1} = \dfrac{0.3}{0.5} = 0.6$

iii $P(C|(A \cup B)) = \dfrac{P(C \cap (A \cup B))}{P(A \cup B)} = \dfrac{0.1 + 0.3}{0.1 + 0.1 + 0.3 + 0.3} = \dfrac{0.4}{0.8} = 0.5$

9 a All of the people who have the disease test positive, which means that there are no people in A who are not in $A \cap B$. There are also 10 people who test positive but do not have the disease. These people lie in B but do not lie in A, i.e. they lie in $B \cap A'$. There are 100 – 10 – 5 = 85 people who do not have the disease and do not test positive, so they lie in $A' \cap B'$. Therefore the Venn diagram should show:



b $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.05}{0.15} = \dfrac{1}{3} = 0.333$ (3 s.f.)

c The test would allow the doctor to find all of the people who have the disease, but only one third of those who tested positive would actually have the disease. This means that two thirds of the people who were told they had the disease would actually not have it.

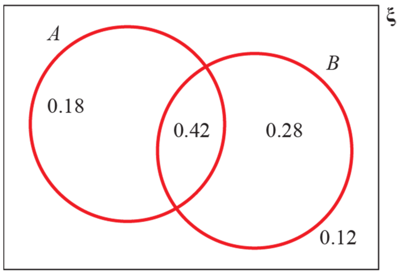
10 a Since $P(A' \cap B') = 0.12$, this means that $P(A \cup B) = 1 - 0.12 = 0.88$

Now find $P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.6 + 0.7 - 0.88 = 0.42$

This allows a Venn diagram of the probabilties of the two events to be produced, which can be used to answer each part of the question.



$$P(B|A') = \frac{P(B \cap A')}{P(A')} = \frac{0.28}{0.28 + 0.12} = \frac{0.28}{0.4} = 0.7$$

b $$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.42}{0.6} = 0.7$$

c Since $P(B|A) = P(B|A') = P(B)$, the events A and B are independent.

11 $P(A|B) = P(B')$

$$\Rightarrow \frac{P(A \cap B)}{P(B)} = P(B') = 0.2 + 0.1 = 0.3$$

$\Rightarrow P(A \cap B) = 0.3P(B)$

$\Rightarrow x = 0.3(x + y)$

The probabilities must sum to 1, so $0.2 + x + y + 0.1 = 1 \Rightarrow x + y = 0.7$

Substituting for $x + y$ gives

$x = 0.3(0.7) = 0.21$

$y = 0.7 - x = 0.7 - 0.21 = 0.49$

12 $\mathrm{P}(A|B) = \mathrm{P}(A')$

$$\Rightarrow \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \mathrm{P}(A')$$

$$\Rightarrow \frac{c}{c+d} = d + 0.2 \qquad \mathbf{(1)}$$

The probabilities must sum to 1, so

$$0.3 + c + d + 0.2 = 1 \Rightarrow c + d = 0.5 \qquad \mathbf{(2)}$$

Substituting for $c + d$ in the equation (**1**) gives

$$\frac{c}{c+d} = d + 0.2 \Rightarrow c = 0.5d + 0.1 \qquad \mathbf{(3)}$$

Substituting this equation for c in equation (**2**) gives

$$0.5d + 0.1 + d = 0.5 \Rightarrow 1.5d = 0.4 \Rightarrow d = \frac{4}{15}$$

Finally, using equation (**3**) gives

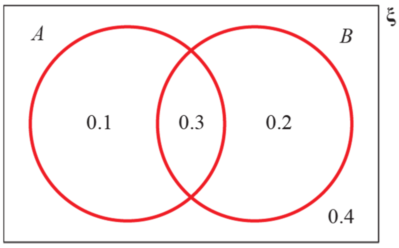
$$c = 0.5 \times \frac{4}{15} + 0.1 = \frac{4}{30} + \frac{1}{10} = \frac{4}{30} + \frac{3}{30} = \frac{7}{30}$$

Conditional probability 2D

1 a Rewrite the addition formula to obtain

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.5 - 0.6 = 0.3$$

Use this result to complete a Venn diagram to help answer the remaining parts of the question.



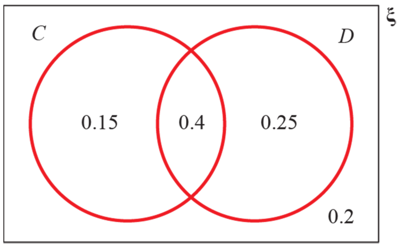
b $P(A') = 0.2 + 0.4 = 0.6$

c $P(A \cup B') = 0.4 + 0.4 = 0.8$

d $P(A' \cup B) = 0.5 + 0.4 = 0.9$

2 a $P(C \cup D) = P(C) + P(D) - P(C \cap D) = 0.55 + 0.65 - 0.4 = 0.8$

b



i The required region is the part 'outside' of C and D, which can be found since all of the probabilities must sum to 1.

$$P(C' \cap D') = 1 - P(C \cup D) = 1 - 0.8 = 0.2$$

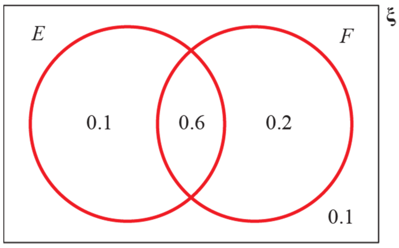
ii $P(C|D) = \dfrac{P(C \cap D)}{P(D)} = \dfrac{0.4}{0.65} = 0.615$ (3 s.f.)

iii $P(C|D') = \dfrac{P(C \cap D')}{P(D')} = \dfrac{0.15}{0.35} = \dfrac{3}{7} = 0.429$ (3 s.f.)

c From part **bii**, it is known that $P(C|D) \neq P(C)$ so the two events are not independent. Alternatively, show that $P(C) \times P(D) \neq P(C \cap D)$.

3 **a** $\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F) = 0.7 + 0.8 - 0.6 = 0.9$

b



i The required region is within E as well as everything outside F. It includes three of the four regions in the Venn diagram.
$\mathrm{P}(E \cup F') = 0.1 + 0.6 + 0.1 = 0.8$

ii The required region is that part of F that does not intersect E.
$\mathrm{P}(E \cap F') = 0.2$

iii $\mathrm{P}(E|F') = \dfrac{\mathrm{P}(E \cap F')}{\mathrm{P}(F')} = \dfrac{0.1}{0.1 + 0.1} = \dfrac{1}{2} = 0.5$

4 **a**

$$\begin{aligned} \mathrm{P}(T \cup Q) &= \mathrm{P}(T) + \mathrm{P}(Q) - \mathrm{P}(T \cap Q) \\ 0.75 &= 3\mathrm{P}(T \cap Q) + 3\mathrm{P}(T \cap Q) - \mathrm{P}(T \cap Q) \\ 5\mathrm{P}(T \cap Q) &= 0.75 \\ \mathrm{P}(T \cap Q) &= 0.15 \end{aligned}$$

b As $\mathrm{P}(T) = \mathrm{P}(Q)$, using $\mathrm{P}(T \cup Q) = \mathrm{P}(T) + \mathrm{P}(Q) - \mathrm{P}(T \cap Q)$ gives

$0.75 = 2\mathrm{P}(T) - 0.15$

$\Rightarrow 2\mathrm{P}(T) = 0.9$

$\Rightarrow \mathrm{P}(T) = 0.45$

c $\mathrm{P}(Q') = 1 - \mathrm{P}(Q) = 1 - \mathrm{P}(T) = 1 - 0.45 = 0.55$

d $\mathrm{P}(T' \cap Q') = 1 - \mathrm{P}(T \cup Q) = 1 - 0.75 = 0.25$

e $\mathrm{P}(T \cap Q') = \mathrm{P}(T) - \mathrm{P}(T \cap Q) = 0.45 - 0.15 = 0.3$

5 Let F be the event has a freezer and D be the event has a dishwasher. The question requires finding $\mathrm{P}(F \cap D)$. Use the addition formula
$\mathrm{P}(F \cap D) = \mathrm{P}(F) + \mathrm{P}(D) - \mathrm{P}(F \cup D) = 0.7 + 0.2 - 0.8 = 0.1$

6 a Use the multiplication formula for conditional probability to find $\mathrm{P}(A \cap B)$

$\mathrm{P}(A \cap B) = \mathrm{P}(A \mid B) \times \mathrm{P}(B) = 0.4 \times 0.5 = 0.2$

Now use the multiplication formula again to find $\mathrm{P}(B|A)$

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{0.2}{0.4} = \frac{1}{2} = 0.5$$

b Use the addition formula to find $\mathrm{P}(A \cup B)$

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.4 + 0.5 - 0.2 = 0.7$

Now $\mathrm{P}(A' \cap B')$ can be found as it is the region outside $\mathrm{P}(A \cup B)$

$\mathrm{P}(A' \cap B') = 1 - \mathrm{P}(A \cup B) = 1 - 0.7 = 0.3$

c $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.5 - 0.2 = 0.3$

7 a First use the addition formula to find $\mathrm{P}(A \cap B)$

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = \frac{1}{4} + \frac{1}{2} - \frac{3}{5} = \frac{3}{20}$$

Now use the multiplication formula to

$$\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\frac{3}{20}}{\frac{1}{2}} = \frac{3}{10} = 0.3$$

b $\mathrm{P}(A' \cap B) = \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{1}{2} - \frac{3}{20} = \frac{7}{20} = 0.35$

c $\mathrm{P}(A' \cap B') = 1 - P(A \cup B) = 1 - \frac{3}{5} = \frac{2}{5} = 0.4$

8 a $\mathrm{P}(C \cap D) = \mathrm{P}(C|\mathrm{D}) \times \mathrm{P}(D) = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12} = 0.0833$ (3 s.f.)

b $\mathrm{P}(C \cap D') = \mathrm{P}(C|\mathrm{D}') \times \mathrm{P}(D') = \frac{1}{5} \times \left(1 - \frac{1}{4}\right) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20} = 0.15$

c $\mathrm{P}(C) = \mathrm{P}(C \cap D') + \mathrm{P}(C \cap D) = \frac{3}{20} + \frac{1}{12} = \frac{9}{60} + \frac{5}{60} = \frac{14}{60} = \frac{7}{30} = 0.233$ (3 s.f.)

d $\mathrm{P}(D|C) = \dfrac{\mathrm{P}(D \cap C)}{\mathrm{P}(C)} = \dfrac{\frac{1}{12}}{\frac{7}{30}} = \dfrac{30}{84} = \dfrac{5}{14} = 0.357$ (3 s.f.)

e $\mathrm{P}(D'|C) = 1 - \mathrm{P}(D|C) = 1 - \frac{5}{14} = \frac{9}{14} = 0.643$ (3 s.f.)

8 f $\mathrm{P}(D'|C') = \dfrac{\mathrm{P}(C' \cap D')}{\mathrm{P}(C')} = \dfrac{1-\mathrm{P}(C \cup D)}{\mathrm{P}(C')}$

However, $\mathrm{P}(C') = 1 - \mathrm{P}(C) = \dfrac{7}{30} = \dfrac{23}{30}$

And $\mathrm{P}(C \cup D) = \mathrm{P}(C) + \mathrm{P}(D) - \mathrm{P}(C \cap D) = \dfrac{7}{30} + \dfrac{1}{4} - \dfrac{1}{12} = \dfrac{24}{60} = \dfrac{2}{5}$

So $\mathrm{P}(D'|C') = \dfrac{\mathrm{P}(C' \cap D')}{\mathrm{P}(C')} = \dfrac{1-\mathrm{P}(C \cup D)}{\mathrm{P}(C')} = \dfrac{1-\frac{2}{5}}{\frac{23}{30}} = \dfrac{3}{5} \times \dfrac{30}{23} = \dfrac{18}{23} = 0.783$ (3 s.f.)

9 a $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.42 + 0.37 - 0.12 = 0.67$

b $\mathrm{P}(A|B') = \dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \dfrac{\mathrm{P}(A) - \mathrm{P}(A \cap B)}{1 - \mathrm{P}(B)} = \dfrac{0.42 - 0.12}{1 - 0.37} = \dfrac{0.3}{0.63} = 0.476$ (3 s.f.)

c Since the events A and C are independent, $\mathrm{P}(A \cap C) = \mathrm{P}(A) \times \mathrm{P}(C) = 0.42 \times 0.3 = 0.126$

d Since B and C are mutually exclusive, there is no need to have an intersection between B and C on the diagram. Work out the probabilities associated with each region as follows:

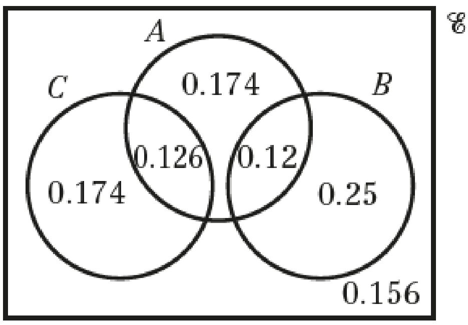
$\mathrm{P}(C \cap A') = \mathrm{P}(C) - \mathrm{P}(A \cap C) = 0.3 - 0.126 = 0.174$

$\mathrm{P}(B \cap A') = \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.37 - 0.12 = 0.25$

$\mathrm{P}(A \cap B' \cap C') = \mathrm{P}(A) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) = 0.42 - 0.12 - 0.126 = 0.174$

$\mathrm{P}(A \cup B \cup C) = 0.174 + 0.126 + 0.174 + 0.12 + 0.25 = 0.844$

$\mathrm{P}(A' \cap B' \cap C') = 1 - \mathrm{P}(A \cup B \cup C) = 1 - 0.844 = 0.156$



e $\mathrm{P}((A' \cup C)') = 1 - \mathrm{P}(A' \cup C)$

Use the Venn diagram to find $\mathrm{P}(A' \cup C) = 0.174 + 0.126 + 0.25 + 0.156 = 0.706$

So $\mathrm{P}((A' \cup C)') = 1 - 0.706 = 0.294$

10 a B and C are independent: $\mathrm{P}(B \cap C) = \mathrm{P}(B) \times \mathrm{P}(C) = 0.7 \times 0.4 = 0.28$

b Using part **a**, $\mathrm{P}(B|C) = \dfrac{\mathrm{P}(B \cap C)}{\mathrm{P}(C)} = \dfrac{0.28}{0.4} = \dfrac{7}{10} = 0.7$

c $\mathrm{P}(A|B') = \dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')} = \dfrac{\mathrm{P}(A) - \mathrm{P}(A \cap B)}{1 - \mathrm{P}(B)} = \dfrac{0.4 - 0.3}{1 - 0.7} = \dfrac{0.1}{0.3} = 0.333$ (3 s.f.)

d $\mathrm{P}((B \cap C)|A') = \dfrac{\mathrm{P}((B \cap C) \cap A')}{\mathrm{P}(A')} = \dfrac{\mathrm{P}(B \cap C) - P(A \cap B \cap C)}{1 - \mathrm{P}(A)}$

As A and C are mutually exclusive, $\mathrm{P}(A \cap B \cap C) = 0$

So $\mathrm{P}((B \cap C)|A') = \dfrac{\mathrm{P}(B \cap C)}{1 - \mathrm{P}(A)} = \dfrac{0.28}{1 - 0.4} = \dfrac{0.28}{0.6} = 0.467$ (3 s.f.)

11 a This requires finding $\mathrm{P}(A \cap B)$
First find $\mathrm{P}(A \cup B)$
$\mathrm{P}(A \cup B) = 0.9$ as $\mathrm{P}(A \cup B) + \mathrm{P}(A' \cap B') = 1$
Using the addition rule gives
$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.3 + 0.7 - 0.9 = 0.1$

b This requires finding $\mathrm{P}(A|B)$
$\mathrm{P}(A|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \dfrac{0.1}{0.7} = 0.143$ (3 s.f.)

c Test whether the events are independent
$\mathrm{P}(A) \times \mathrm{P}(B) = 0.3 \times 0.7 = 0.21$, $\mathrm{P}(A \cap B) = 0.1$
So the events are not independent. If Anna is late, Bella is *less* likely to be late and vice versa.

12 a The probability that both John and Kayleigh win their matches is $\mathrm{P}(J \cap K)$
$\mathrm{P}(J \cap K) = \mathrm{P}(J) + \mathrm{P}(K) - \mathrm{P}(J \cup K) = 0.6 + 0.7 - 0.8 = 0.5$

b $\mathrm{P}(J|K') = \dfrac{\mathrm{P}(J \cap K')}{\mathrm{P}(K')} = \dfrac{\mathrm{P}(J) - \mathrm{P}(J \cap K)}{1 - \mathrm{P}(K)} = \dfrac{0.6 - 0.5}{1 - 0.7} = \dfrac{0.1}{0.3} = 0.333$ (3 s.f.)

c $\mathrm{P}(K|J) = \dfrac{\mathrm{P}(J \cap K)}{\mathrm{P}(J)} = \dfrac{0.5}{0.6} = 0.833$ (3 s.f.)

d $\mathrm{P}(K|J) = 0.833$ (3 s.f.), $\mathrm{P}(K) = 0.7$, so $\mathrm{P}(K|J) \neq \mathrm{P}(K)\mathrm{P}(K|J) = 0.833\ldots \neq \mathrm{P}(K) = 0.7$. So J and K are not independent.

Challenge

a The probability function must sum to 1. Therefore $k+2k+3k+4k+5k=1 \Rightarrow 15k=1 \Rightarrow k=\frac{1}{15}$

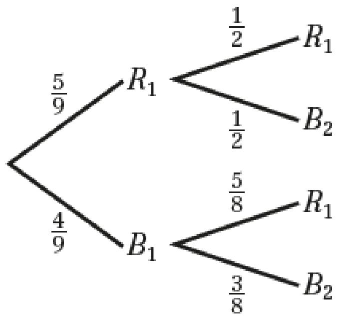
b $\mathrm{P}(X=5|X>2)=\dfrac{\mathrm{P}(X=5)}{\mathrm{P}(X>2)}=\dfrac{\frac{5}{15}}{1-\mathrm{P}(X=1\cup X=2)}=\dfrac{\frac{5}{15}}{1-\frac{3}{15}}=\dfrac{\frac{5}{15}}{\frac{12}{15}}=\dfrac{5}{12}$

c

$$\mathrm{P}(X\text{ is odd}|X\text{ is prime})=\frac{\mathrm{P}(X\text{ is odd and prime})}{\mathrm{P}(X\text{ is prime})}=\frac{\mathrm{P}(X=3\cup X=5)}{\mathrm{P}(X=2\cup X=3\cup X=5)}=\frac{\frac{3+5}{15}}{\frac{2+3+5}{15}}=\frac{8}{10}=\frac{4}{5}$$

Conditional probability 2E

1 a When the first token removed is red, there are 8 tokens remaining in the bag, 4 red and 4 blue. When the first token removed is blue, there are 8 tokens remaining in the bag, 5 red and 3 blue.



b The answer can be read off from the tree diagram, following the lower branch (first blue) and then the red branch.

So $\mathrm{P(second\ red|first\ blue)} = \frac{5}{8}$

c $$\mathrm{P(first\ red|second\ blue)} = \frac{\mathrm{P(first\ red\ and\ second\ blue)}}{\mathrm{P(second\ blue)}} = \frac{\frac{5}{9}\times\frac{1}{2}}{\left(\frac{5}{9}\times\frac{1}{2}\right)+\left(\frac{4}{9}\times\frac{3}{8}\right)} = \frac{\frac{5}{18}}{\frac{32}{72}} = \frac{20}{32} = \frac{5}{8}$$

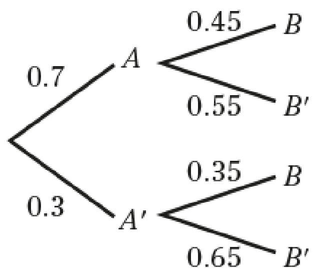
d $$\mathrm{P(first\ blue|tokens\ different\ colours)} = \frac{\mathrm{P(first\ blue\ and\ second\ red)}}{\mathrm{P(first\ blue\ and\ second\ red)} + \mathrm{P(first\ red\ and\ second\ blue)}}$$

$$= \frac{\frac{4}{9}\times\frac{5}{8}}{\left(\frac{4}{9}\times\frac{5}{8}\right)+\left(\frac{5}{9}\times\frac{1}{2}\right)} = \frac{\frac{20}{72}}{\frac{40}{72}} = \frac{20}{40} = \frac{1}{2}$$

e $$\mathrm{P(tokens\ same\ colour|second\ token\ red)} = \frac{\mathrm{P(first\ red\ and\ second\ red)}}{\mathrm{P(first\ red\ and\ second\ red)} + \mathrm{P(first\ blue\ and\ second\ red)}}$$

$$= \frac{\frac{5}{9}\times\frac{1}{2}}{\left(\frac{5}{9}\times\frac{1}{2}\right)+\left(\frac{4}{9}\times\frac{5}{8}\right)} = \frac{\frac{5}{18}}{\frac{40}{72}} = \frac{\frac{20}{72}}{\frac{40}{72}} = \frac{20}{40} = \frac{1}{2}$$

2 a $P(A) = 0.7 \Rightarrow P(A') = 1 - 0.7 = 0.3$
$P(B|A) = 0.45 \Rightarrow P(B'|A) = 1 - 0.45 = 0.55$
$P(B|A') = 0.35 \Rightarrow P(B'|A') = 1 - 0.35 = 0.65$
Therefore the completed tree diagram should be:



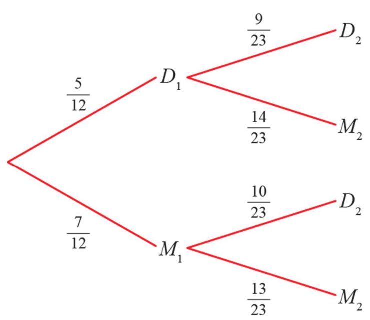
b i $P(A \cap B) = P(A) \times P(B|A) = 0.7 \times 0.45 = 0.315$

ii $P(A' \cap B') = P(A') \times P(B'|A') = 0.3 \times 0.65 = 0.195$

iii $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.315}{P(A \cap B) + P(A' \cap B)} = \dfrac{0.315}{0.315 + 0.3 \times 0.35} = \dfrac{0.315}{0.42} = 0.75$

3 a There are 10 dark chocolates in a box of 24, meaning the probability of choosing a dark chocolate is $\frac{10}{24} = \frac{5}{12}$. Similarly there are 14 milk chocolates out of the 24, and so the probability of choosing a dark chocolate is $\frac{14}{24} = \frac{7}{12}$.

Once Linda has eaten one chocolate, there are 23 chocolates left in the box. If the first chocolate she ate was a dark one, the probability of choosing another dark chocolate is $\frac{9}{23}$, and the probability of choosing a milk chocolate is $\frac{14}{23}$. If the first chocolate she ate was a milk one, the probability of a dark chocolate is $\frac{10}{23}$, and the probability of choosing another milk chocolate is $\frac{13}{23}$.



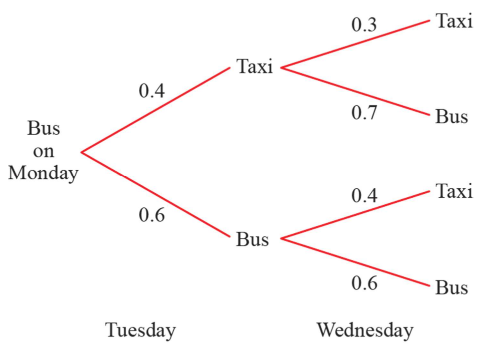
b $P(\text{dark and dark}) = \dfrac{5}{12} \times \dfrac{9}{23} = \dfrac{15}{92} = 0.163$ (3 s.f.)

3 c $P(\text{one dark and one milk}) = P(D_1 \cap M_2) + P(M_1 \cap D_2)$

$$= \frac{5}{12} \times \frac{14}{23} + \frac{7}{12} \times \frac{10}{23}$$
$$= \frac{70}{276} + \frac{70}{276} = \frac{140}{276} = \frac{35}{69} = 0.507 \text{ (3 s.f.)}$$

d $$\text{P(dark and dark}|\text{at least one dark)} = \frac{\text{P(dark and dark)}}{\text{P(at least one dark)}}$$
$$= \frac{\mathrm{P}(D_1 \cap D_2)}{\mathrm{P}(D_1 \cap D_2) + \mathrm{P}(D_1 \cap M_2) + \mathrm{P}(M_1 \cap D_2)}$$
$$= \frac{\frac{5}{12} \times \frac{9}{23}}{\frac{5}{12} \times \frac{9}{23} + \frac{5}{12} \times \frac{14}{23} + \frac{7}{12} \times \frac{10}{23}} = \frac{\frac{45}{276}}{\frac{185}{276}} = \frac{45}{185} = \frac{9}{37} = 0.243 \text{ (3 s.f.)}$$

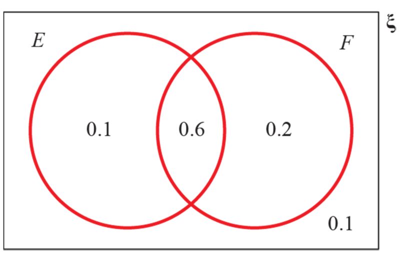
4 Use the information in the question to produce a tree diagram covering Jean's possible travel arrangements on Tuesday and Wednesday as follows:



Now sum the probabilities of Jean taking a taxi to work on Wednesday

$$\text{P(taxi on Wednesday)} = 0.4 \times 0.3 + 0.6 \times 0.4$$
$$= 0.12 + 0.24$$
$$= 0.36$$

5 Represent the information as a tree diagram. The coins are chosen at random, so there is a probability of $\frac{1}{2}$ of choosing each coin.

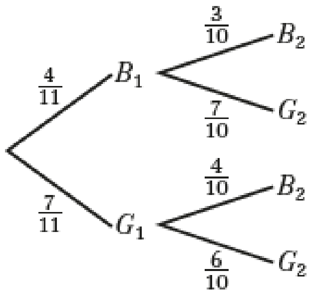


a $\text{P(head)} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = 0.25$

5 b $$\text{P(Fair}|\text{tail)} = \frac{\text{P(Fair and Tail)}}{\text{P(Tail)}}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1} = \frac{1}{3} = 0.333 \text{ (3 s.f.)}$$

6 a Since the first ball selected is not replaced, there are 10 balls in the bag when the second ball is selected.



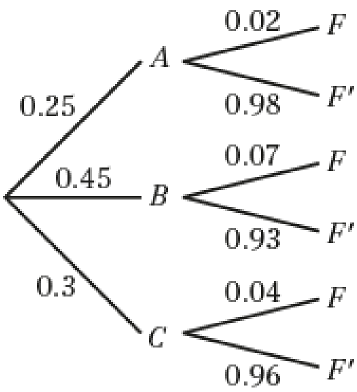
b $\text{P(second ball is green)} = \text{P}(B_1 \cap G_2) + \text{P}(G_1 \cap G_2)$

$$= \left(\frac{4}{11} \times \frac{7}{10}\right) + \left(\frac{7}{11} \times \frac{6}{10}\right) = \frac{28 + 42}{110} = \frac{7}{11} = 0.636 \text{ (3 s.f.)}$$

c $$\text{P(both balls are green} \mid \text{second ball is green)} = \frac{\text{P(both balls are green)}}{\text{P(second ball is green)}}$$

$$= \frac{\frac{7}{11} \times \frac{6}{10}}{\frac{70}{110}} = \frac{\frac{42}{110}}{\frac{70}{110}} = \frac{42}{70} = \frac{3}{5} = 0.6$$

7 a The probability of the sheet coming from *A, B* or *C* is given in the question. In each case, the probability that a sheet is flawed immediately provides the probability that it is not flawed since the two probabilities must sum to 1. Therefore the completed tree diagram is:



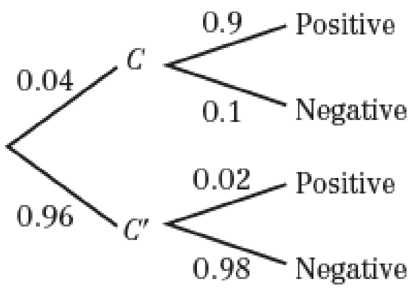
b i $\text{P(produced by } B \cap \text{flawed)} = 0.45 \times 0.07 = 0.0315$

7 b ii P(flawed) can be found by summing $\text{P(produced by } A \cap \text{flawed)}$, $\text{P(produced by } B \cap \text{flawed)}$ and $\text{P(produced by } C \cap \text{flawed)}$. Therefore

$$\text{P(flawed)} = 0.25 \times 0.02 + 0.0315 + 0.3 \times 0.04 = 0.0485$$

c $\text{P(produced by } A|\text{flawed}) = \dfrac{\text{P(produced by } A \cap \text{flawed)}}{\text{P(flawed)}} = \dfrac{0.25 \times 0.02}{0.0485} = 0.103 \text{ (3 s.f.)}$

8 a The reliability of the test depends on whether the person has the condition (C) or not (C').



b $\text{P(tests negative)} = \text{P}(C \cap \text{tests negative}) + \text{P}(C' \cap \text{tests negative})$
$= 0.04 \times 0.1 + 0.96 \times 0.98 = 0.9448 = 0.945 \text{ (3 s.f.)}$

c $\text{P(has condition|tests negative)} = \dfrac{\text{P}(C \cap \text{tests negative})}{\text{P(tests negative)}}$
$= \dfrac{0.04 \times 0.1}{0.9448} = 0.00423 \text{ (3 s.f.)}$

d From the data in the question, the test fails to find 10% of the people with the condition (since it has a 0.1 chance of producing a negative result when a person has the condition).

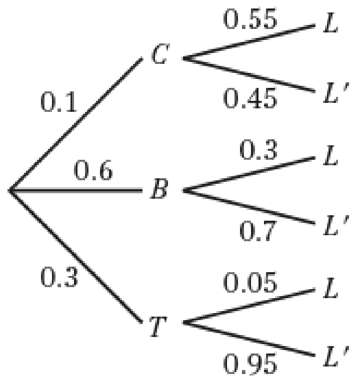
Consider also false positives, the case of a person who does not have the condition returning a positive result.

$\text{P(does not have the condition|tests positive)} = \dfrac{\text{P}(C' \cap \text{tests positive})}{\text{P(tests positive)}} = \dfrac{0.96 \times 0.02}{1 - 0.9448} = 0.348 \text{ (3 s.f.)}$

So over one third of the positive tests are false positives.

This means that if the test was used on the entire population, 10% of the people with the condition would not be identified and over one third of the people with a positive result would actually not have the condition.

9 a Since the probabilities of being late are given, the probabilities for being on time (i.e. not late) for each type of transport are known, sine the probabilities must sum to 1. Therefore the completed tree diagram should be as follows:



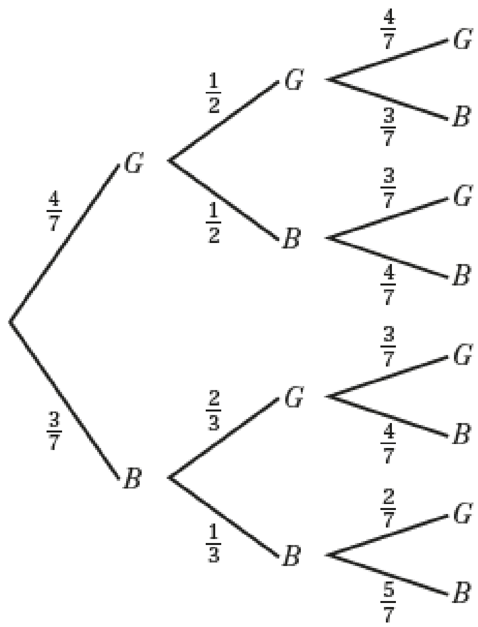
b i $\text{P(Bill travels by train and is late)} = 0.3 \times 0.05 = 0.015$

ii To find P(Bill is late), sum P(Bill travels by car and is late), P(Bill travels by bus and is late) and P(Bill travels by train and is late).

$\text{P(Bill is late)} = 0.1 \times 0.55 + 0.6 \times 0.3 + 0.3 \times 0.05 = 0.25$

c $\text{P(Bill travels by bus or train|Bill is late)} = \dfrac{0.6 \times 0.3 + 0.3 \times 0.05}{0.25} = 0.78$

10 a The two counters being drawn from box A can be modelled using a tree diagram. In each case, the number of counters of each colour in box B is then known, and so the third set of branches can be labelled to represent the drawing of the counter from box B. Therefore the completed tree diagram should be:



b $\text{P}(C) = \text{P}(GG) + \text{P}(BB) = \left(\dfrac{4}{7} \times \dfrac{1}{2}\right) + \left(\dfrac{3}{7} \times \dfrac{1}{3}\right) = \dfrac{2}{7} + \dfrac{1}{7} = \dfrac{3}{7}$

10 c $\text{P}(D) = \text{P}(GGB) + \text{P}(GBB) + \text{P}(BGB) + \text{P}(BBB)$

$$= \left(\frac{4}{7} \times \frac{1}{2} \times \frac{3}{7}\right) + \left(\frac{4}{7} \times \frac{1}{2} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{2}{3} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{1}{3} \times \frac{5}{7}\right)$$

$$= \frac{6}{49} + \frac{8}{49} + \frac{8}{49} + \frac{5}{49} = \frac{27}{49}$$

d The calculation will be similar to that for P(D), but with the first and second counters being the same colour.

$$\text{P}(C \cap D) = \text{P}(GGB) + \text{P}(BBB) = \left(\frac{4}{7} \times \frac{1}{2} \times \frac{3}{7}\right) + \left(\frac{3}{7} \times \frac{1}{3} \times \frac{5}{7}\right) = \frac{6}{49} + \frac{5}{49} = \frac{11}{49}$$

e Use the addition formula

$$\text{P}(C \cup D) = \text{P}(C) + \text{P}(D) - \text{P}(C \cap D) = \frac{3}{7} + \frac{27}{49} - \frac{11}{49} = \frac{21 + 27 - 11}{49} = \frac{37}{49}$$

f The required probability is:

$$\frac{\text{P}(GGG)}{\text{P}(GGG) + \text{P}(BBB)} = \frac{\frac{4}{7} \times \frac{1}{2} \times \frac{4}{7}}{\left(\frac{4}{7} \times \frac{1}{2} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{1}{3} \times \frac{5}{7}\right)} = \frac{\frac{8}{49}}{\frac{8}{49} + \frac{5}{49}} = \frac{8}{13} = 0.615 \text{ (3 s.f.)}$$

11 She has not taken into account the fact that after the first jelly bean is selected, there are only 9 jelly beans left in the box. So if the first jelly bean selected is sweet, the probability that the second bean is sweet is $\frac{6}{9}$ not $\frac{7}{10}$.

This is the correct solution.

$$\text{P(both jelly beans are sweet)} = \frac{7}{10} \times \frac{6}{9} = \frac{7}{15}$$

$$\text{P(at least one jelly bean is sweet)} = 1 - \text{P(neither jelly bean is sweet)} = 1 - \left(\frac{3}{10} \times \frac{2}{9}\right) = \frac{14}{15}$$

$$\text{P(both are sweet given at least one is sweet)} = \frac{\frac{7}{15}}{\frac{14}{15}} = \frac{7}{14} = 0.5$$

The correct answer is therefore 0.5.

Conditional probability Mixed exercise 2

1 a $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4 + 0.35 - 0.2 = 0.55$

b $P(A' \cap B') = 1 - P(A \cup B) = 1 - 0.55 = 0.45$

c $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.2}{0.4} = 0.5$

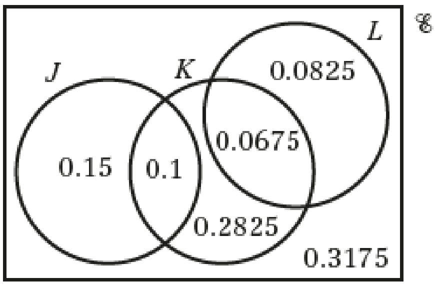
d $P(A'|B) = \dfrac{P(A' \cap B)}{P(B)} = \dfrac{P(B) - P(A \cap B)}{P(B)} = \dfrac{0.15}{0.35} = 0.429$ (3 s.f.)

2 a Work out each region of the Venn diagram from the information provided in the question.

First, as J and L are mutually exclusive, $P(J \cap L) = \varnothing$
So $P(J \cap K' \cap L') = P(J) - P(J \cap K) = 0.25 - 0.1 = 0.15$

As K and L are independent $P(K \cap L) = P(K) \times P(L) = 0.45 \times 0.15 = 0.0675$
So $P(L \cap K') = P(L) - P(L \cap K) = 0.15 - 0.0625 = 0.0825$
And $P(K \cap J' \cap L') = P(K) - P(J \cap K) - P(K \cap L) = 0.45 - 0.1 - 0.0675 = 0.2825$

Find the outer region by subtracting the sum of all the other regions from 1
$P(J' \cap K' \cap L') = 1 - 0.15 - 0.1 - 0.2825 - 0.0675 - 0.0825 = 0.3175$



b i $P(J \cup K) = 0.15 + 0.1 + 0.2825 + 0.0675 = 0.6$

ii $P(J' \cap L') = 0.2825 + 0.3175 = 0.6$

iii $P(J|K) = \dfrac{P(J \cap K)}{P(K)} = \dfrac{0.1}{0.45} = 0.222$ (3 s.f.)

iv $P(K|J' \cap L') = \dfrac{P(K \cap (J' \cap L'))}{P(J' \cap L')} = \dfrac{0.2825}{0.6} = 0.471$ (3 s.f.)

3 a $P(F \cap S') + P(S \cap F') = P(F) - P(F \cap S) + P(F) - P(F \cap S)$

$$= \frac{35 - 27 + 45 - 27}{60} = \frac{26}{60} = 0.433 \text{ (3 s.f.)}$$

b $P(F|S) = \dfrac{P(F \cap S)}{P(S)} = \dfrac{27}{45} = 0.6$

c $P(S|F') = \dfrac{P(S \cap F')}{P(F')} = \dfrac{45 - 27}{60 - 35} = \dfrac{18}{25} = 0.72$

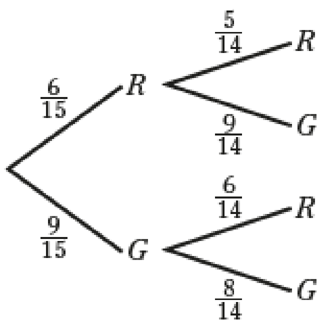
d There are 6 students that study just French and wear glasses ($8 \times 0.75 = 6$) and 9 students that study just Spanish and wear glasses ($18 \times 0.5 = 9$), so the required probability is

$$\text{P(studies one language and wears glasses)} = \frac{6+9}{60} = \frac{15}{60} = 0.25$$

e There are 26 students studying one language (from part **a**). Of these, 15 wear glasses (from part **d**).

$$\text{P(wears glasses|studies one language)} = \frac{15}{26} = 0.577 \text{ (3 s.f.)}$$

4 a



b i $P(GG) = \dfrac{9}{15} \times \dfrac{8}{14} = \dfrac{3}{5} \times \dfrac{4}{7} = \dfrac{12}{35} = 0.343 \text{ (3 s.f.)}$

ii There are two different ways to obtain balls that are different colours:

$$P(RG) + P(GR) = \left(\frac{6}{15} \times \frac{9}{14}\right) + \left(\frac{9}{15} \times \frac{6}{14}\right) = \frac{2 \times 9}{5 \times 7} = \frac{18}{35} = 0.514 \text{ (3 s.f.)}$$

c There are 4 different outcomes:

$P(RRR) + P(RGR) + P(GRR) + P(GGR)$

$$= \left(\frac{6}{15} \times \frac{5}{14} \times \frac{4}{13}\right) + \left(\frac{6}{15} \times \frac{9}{14} \times \frac{5}{13}\right) + \left(\frac{9}{15} \times \frac{6}{14} \times \frac{5}{13}\right) + \left(\frac{9}{15} \times \frac{8}{14} \times \frac{6}{13}\right)$$

$$= \frac{120 + 270 + 270 + 432}{2730} = \frac{1092}{2730} = 0.4$$

4 d The only way for this to occur is to draw a green ball each time. The corresponding probability is:

$$\text{P}(GGGG) = \frac{9}{15} \times \frac{8}{14} \times \frac{7}{13} \times \frac{6}{12} = \frac{3\times 2}{5\times 13} = \frac{6}{65} = 0.0923 \text{ (3 s.f.)}$$

5 a Either Colin or Anne must win both sets. Therefore the required probability is:

$$\text{P(match over in two sets)} = (0.7\times 0.8) + (0.3\times 0.6) = 0.56 + 0.18 = 0.74$$

b $\text{P(Anne wins|match over in two sets)} = \dfrac{0.7\times 0.8}{0.74} = \dfrac{0.56}{0.74} = 0.757 \text{ (3 s.f.)}$

c The three ways that Anne can win the match are: wins first set, wins second set; wins first set, loses second set, wins tiebreaker; loses first set, wins second set, wins tiebreaker.

$$\begin{aligned}\text{P(Anne wins match)} &= (0.7\times 0.8) + (0.7\times 0.2\times 0.55) + (0.3\times 0.4\times 0.55)\\ &= 0.56 + 0.077 + 0.066 = 0.703\end{aligned}$$

6 a There are 20 kittens with neither black nor white paws (75 – 26 – 14 – 15 = 20).

$$\text{P(neither white or black paws)} = \frac{20}{75} = \frac{4}{15} = 0.267 \text{ (3 s.f.)}$$

b There are 41 kittens with some black paws (26 + 15 = 41).

$$\text{P(black and white paws|some black paws)} = \frac{15}{41} = 0.366 \text{ (3 s.f.)}$$

c This is selection without replacement (since the first kitten chosen is not put back).

$$\text{P(both kittens have all black paws)} = \frac{26}{75} \times \frac{25}{74} = \frac{13}{3\times 37} = \frac{13}{111} = 0.117 \text{ (3 s.f.)}$$

d There are 29 kittens with some white paws (14 + 15 = 29).

$$\text{P(both kittens have some white paws)} = \frac{29}{75} \times \frac{28}{74} = \frac{812}{5550} = 0.146 \text{ (3 s.f.)}$$

7 a Using the fact that A and B are independent: $\text{P}(A)\times\text{P}(B) = \text{P}(A\cap B) \Rightarrow \text{P}(B) = \dfrac{0.12}{0.4} = 0.3$

b Use the addition formula to find $\text{P}(A\cup B)$

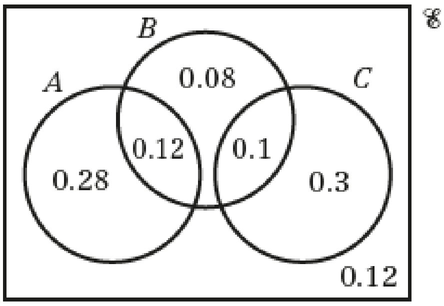
$\text{P}(A\cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A\cap B) = 0.4 + 0.3 - 0.12 = 0.58$

$\text{P}(A'\cap B') = 1 - \text{P}(A\cup B) = 1 - 0.58 = 0.42$

7 c As A and C are mutually exclusive

$P(A \cap B' \cap C') = P(A) - P(A \cap B) = 0.4 - 0.12 = 0.28$
$P(C \cap A' \cap B') = P(C) - P(B \cap C) = 0.4 - 0.1 = 0.3$
$P(B \cap A' \cap C') = P(B) - P(A \cap B) - P(B \cap C) = 0.3 - 0.12 - 0.1 = 0.08$

Find the outer region by subtracting the sum of all the other regions from 1
$P(A' \cap B' \cap C') = 1 - 0.28 - 0.12 - 0.08 - 0.1 - 0.3 = 0.12$



d i $P(B|C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{0.1}{0.4} = 0.25$

ii The required region must be contained within A, and not include B (the condition on C is irrelevant since A and C are mutually exclusive). Therefore, $P(A \cap (B' \cup C)) = 0.28$

8 a It may be that neither team scores in the match, and it is a 0–0 draw.

b $P(\text{team } A \text{ scores first}) = P(\text{team } A \text{ scores first and wins}) + P(\text{team } A \text{ scores first and does not win})$
So $P(\text{team } A \text{ scores first and does not win}) = 0.6 - 0.48 = 0.12$

c From the question $P(A \text{ wins}|B \text{ scores first}) = 0.3$. Using the multiplication formula gives

$$P(A \text{ wins}|B \text{ scores first}) = \frac{P(A \text{ wins} \cap B \text{ scores first})}{P(B \text{ scores first})} = 0.3$$

$\Rightarrow P(A \text{ wins} \cap B \text{ scores first}) = 0.3 \times 0.35 = 0.105$

Now find the required probability

$$P(B \text{ scores first}|A \text{ wins}) = \frac{P(A \text{ wins} \cap B \text{ scores first})}{P(A \text{ wins})} = \frac{0.105}{0.48 + 0.105} = \frac{0.105}{0.585} = 0.179 \text{ (3 s.f.)}$$

Challenge

a Let $P(A \cap B) = k$
As $P(A \cap B) \leqslant P(B) \Rightarrow k \leqslant 0.2$
A and B could be mutually exclusive, meaning $P(A \cap B) = 0$, so $0 \leqslant k \leqslant 2$

Now, $P(A \cap B') = P(A) - P(A \cap B)$, so $p = 0.6 - k \Rightarrow 0.4 \leqslant p \leqslant 0.6$

Challenge

b Use the fact that $\mathrm{P}(A\cap C)=\mathrm{P}(A\cap B\cap C)+\mathrm{P}(A\cap B'\cap C)$
So $\mathrm{P}(A\cap B'\cap C)=\mathrm{P}(A\cap C)-\mathrm{P}(A\cap B\cap C)=\mathrm{P}(A\cap C)-0.1$

Consider the range of $\mathrm{P}(A\cap C)$
$\mathrm{P}(A\cap C)\leqslant \mathrm{P}(A)\Rightarrow \mathrm{P}(A\cap C)\leqslant 0.6$

By the multiplication formula $\mathrm{P}(A\cup C)=\mathrm{P}(A)+\mathrm{P}(C)-\mathrm{P}(A\cap C)$
So $\mathrm{P}(A\cap C)=\mathrm{P}(A)+\mathrm{P}(C)-\mathrm{P}(A\cup C)=1.3-\mathrm{P}(A\cup C)$
As $\mathrm{P}(A\cup C)\leqslant 1\Rightarrow \mathrm{P}(A\cap C)\geqslant 0.3$

So $0.3\leqslant \mathrm{P}(A\cap C)\leqslant 0.6$ and as $\mathrm{P}(A\cap B'\cap C)=\mathrm{P}(A\cap C)-0.1$ this gives the result that $0.3-0.1\leqslant \mathrm{P}(A\cap B'\cap C)\leqslant 0.6-0.1$, so $0.2\leqslant q\leqslant 0.5$

The normal distribution 3A

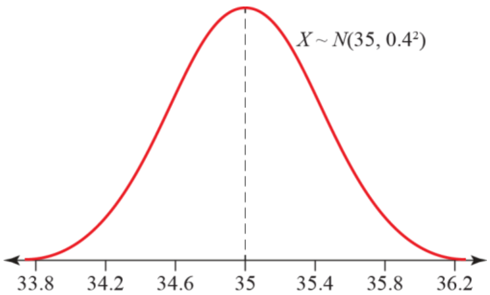
1 **a** Continuous – lengths can take any value.

b Discrete – scores can only take certain values.

c Continuous – masses can take any value.

d Discrete – shoe sizes can only take certain values.

2 Since the mean is 35 mm, the distribution should be symmetrical about this value. Since the standard deviation is 0.4 mm, 68% of the data should lie in the range 34.6 mm to 35.4 mm and 95% of the data should lie in the range 34.2 mm to 35.8 mm.



3 One of the key features of normal distributions is that they are symmetrical about the mean (which is equal to the mode and the median). This curve shows that the bank employees' incomes are not equally distributed either side of its peak (the modal income), so the normal distribution is not a suitable model.

4 **a** Since $\sigma^2 = 16$, the standard deviation is $\sqrt{16} = 4$ cm and the mean is 120 cm.
68% of the pupils are expected to have an armspan within one standard deviation of the mean, i.e. within the interval 116 cm to 124 cm.

b The given interval, 112 cm to 128 cm, includes all of the pupils whose armspans are up to two standard deviations either side of the mean. Therefore 95% of the pupils can be expected to have an armspan within this range.

5 If 68% of the adders have a length between 93 cm (100 cm − 7 cm) and 107 cm (100 cm + 7 cm), then the standard deviation is 7 cm. Therefore the variance, σ^2, is $7^2 = 49$.

6 Since 95% of the data should lie within two standard deviations of the mean, 2.5% of the data should be two standard deviations or more below the mean and 2.5% of the data should be two standard deviations or more above the mean. Since 2.5% of the dormice weigh more than 70 grams, this means that 70 grams is two standard deviations above the mean. The standard deviation is 5 grams (the square root of the variance) so the mean is 60 grams.

7 In a normal distribution, 68% of the data lies within one standard deviation, σ, of the mean, μ, and 32% lies outside of this range. Therefore 16% of the data lies below $\sigma - \mu$, and 16% lies above $\sigma + \mu$. Also, 95% of the data lies between $\mu - 2\sigma$ and $\mu + 2\sigma$. Therefore 2.5% lies below $\mu - 2\sigma$ and 2.5% lies above $\mu + 2\sigma$.

The question states that 84% of the pigs weigh more than 52 kg, so 16% weigh less than 52 kg. Hence $52 = \mu - \sigma$. Also, 97.5% of the pigs weigh more than 47.5 kg, so 2.5% weigh less than 47.5 kg and $47.5 = \mu - 2\sigma$. From these two equations, $\sigma = 52 - 47.5 = 4.5$ kg and so $\sigma^2 = (4.5)^2 = 20.25$.
Hence $\mu = 52 + \sigma = 52 + 4.5 = 56.5$ kg.

8 a Since the normal distribution is symmetrical and the mean is equal to the median and the mode, half the scores should be above 45 and half should be below. Therefore $\mathrm{P}(S > 45) = 0.5$.

b Since the data within one standard deviation of the mean should be 68% of the sample, $\mathrm{P}(30 < S < 60) = 0.68$.

c Since the data within two standard deviations of the mean should be 95% of the sample, $\mathrm{P}(15 < S < 75) = 0.95$.

d Alexia is incorrect: although $\mathrm{P}(X > 100) > 0$, the value is very small as 100 as more than three standard deviations from the mean, so the model as a whole is still reasonable.

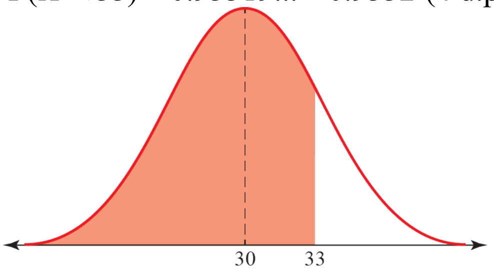
9 a The mean of the normal distribution is where the highest point on the curve appears. From the sketch, this is around 36 cm.

b The points of inflection of the normal occur at $\mu + \sigma$ and $\mu - \sigma$. From the sketch, these points lie somewhere in the intervals [33, 34] and [38, 39]. Since the mean is around 36 cm, this means that the standard deviation should be somewhere between 2 cm and 3 cm.

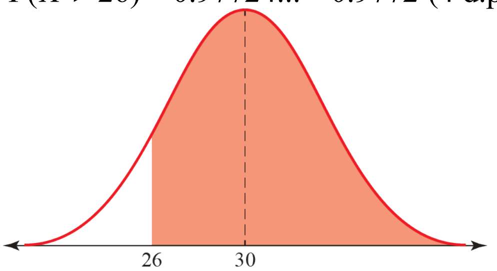
The normal distribution 3B

1 Use the Normal CD function on your calculator, with $\mu = 30$ and $\sigma = 2$.

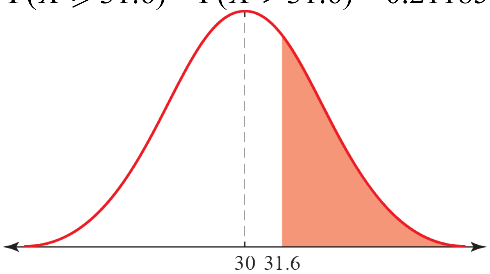
a Set a small value for the lower limit, e.g. 0.
$\text{P}(X < 33) = 0.93319... = 0.9332$ (4 d.p.)



b Set a large value for the upper limit, e.g. 1000.
$\text{P}(X > 26) = 0.97724... = 0.9772$ (4 d.p.)

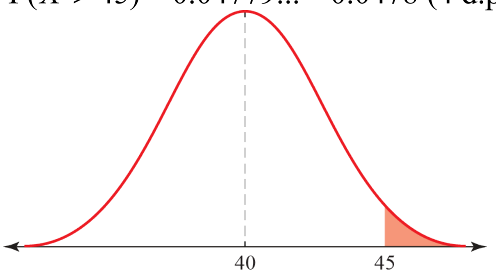


c Set a large value for the upper limit, e.g. 1000.
$\text{P}(X \geqslant 31.6) = \text{P}(X > 31.6) = 0.21185... = 0.2119$ (4 d.p.)

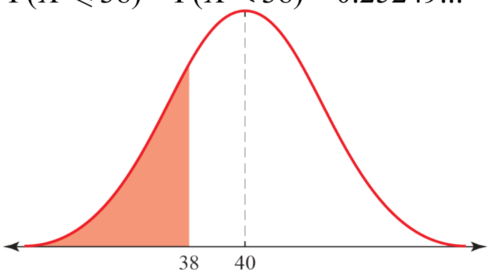


2 Use the Normal CD function on your calculator, with $\mu = 40$ and $\sigma = \sqrt{9} = 3$.

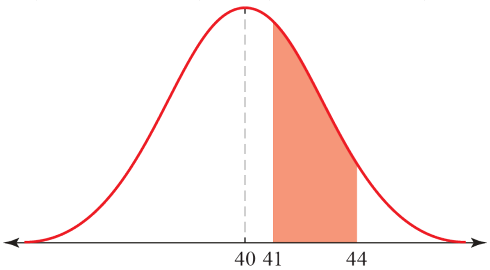
a Set a large value for the upper limit, e.g. 1000.
$\text{P}(X > 45) = 0.04779... = 0.0478$ (4 d.p.)



2 **b** Set a small value for the lower limit, e.g. 0.
$P(X \leqslant 38) = P(X < 38) = 0.25249... = 0.2525$ (4 d.p.)

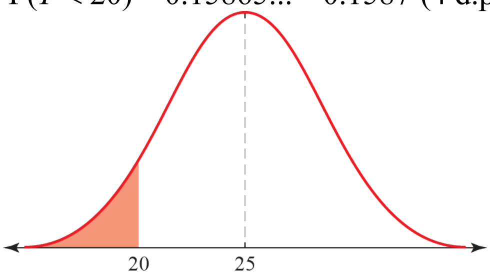


c $P(41 \leqslant X \leqslant 44) = P(41 < X < 44) = 0.27823... = 0.2782$ (4 d.p.)

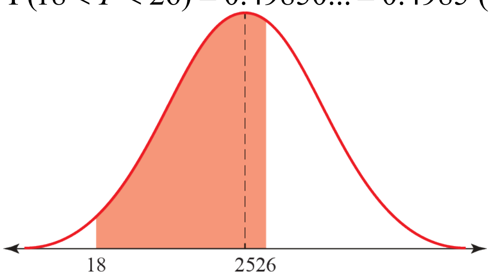


3 Use the Normal CD function on your calculator, with $\mu = 25$ and $\sigma = \sqrt{25} = 5$.

a Set a small value for the lower limit, e.g. 0.
$P(Y < 20) = 0.15865... = 0.1587$ (4 d.p.)



b $P(18 < Y < 26) = 0.49850... = 0.4985$ (4 d.p.)



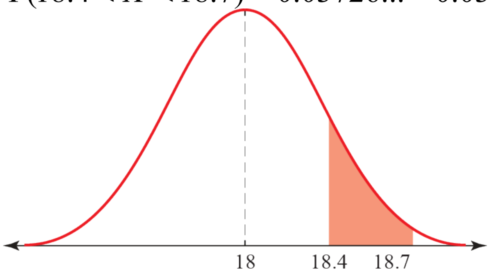
c Set a large value for the upper limit, e.g. 1000.
$P(Y > 23.8) = 0.59483... = 0.5948$ (4 d.p.)

4 Use the Normal CD function on your calculator, with $\mu = 18$ and $\sigma = \sqrt{10}$.

a Set a large value for the upper limit, e.g. 1000.
$P(X \geqslant 20) = P(X > 20) = 0.26354... = 0.2635$ (4 d.p.)

b Set a small value for the lower limit, e.g. 0.
$P(X < 15) = 0.17139... = 0.1714$ (4 d.p.)

4 c $P(18.4 < X < 18.7) = 0.03726... = 0.0373$ (4 d.p.)



5 Use the Normal CD function on your calculator, with $\mu = 15$ and $\sigma = 1.5$.

a i Set a large value for the upper limit, e.g. 1000.
$P(M > 14) = 0.74750... = 0.7474$ (4 d.p.)

ii Set a small value for the lower limit, e.g. 0.
$P(M < 14) = 0.25249... = 0.2525$ (4 d.p.)

b $P(M > 14) + P(M < 14) = 0.7475 + 0.2525 = 1$
The sum is 1, as the combined probabilities include every possible value.

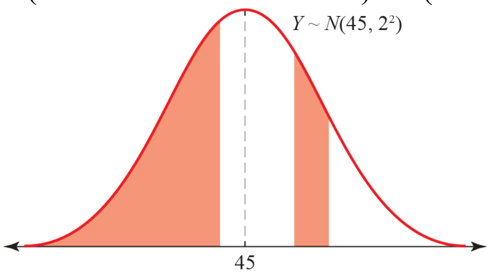
6 a Use the Normal CD function on your calculator, with $\mu = 4.5, \sigma = \sqrt{0.4}$ and a small value for the lower limit.
$P(T < 4.2) = 0.31762... = 0.3176$ (4 d.p.)

b $P(T > 4.2) = 1 - P(T < 4.2) = 1 - 0.3176 = 0.6824$ (4 d.p.)

7 Use the Normal CD function on your calculator, with $\mu = 45$ and $\sigma = 2$.

a $P(Y < 41 \text{ or } Y > 47) = 1 - P(41 < Y < 47)$
Using your calculator, $P(41 < Y < 47) = 0.81859...$
So $P(Y < 41 \text{ or } Y > 47) = 1 - 0.81859... = 0.1814$ (4 d.p.)

b $P(Y < 44 \text{ or } 46.5 < Y < 47.5) = P(Y < 44) + P(46.5 < Y < 47.5)$



Using your calculator, $P(Y < 44) = 0.30853...$ and $P(46.5 < Y < 47.5) = 0.12097...$
so $P(Y < 44 \text{ or } 46.5 < Y < 47.5) = 0.30853... + 0.12097... = 0.4295$ (4 d.p.)

8 Use the Normal CD function on your calculator, with $\mu = 6$ and $\sigma = 0.8$.

a i A suitable upper limit is 10, giving $P(X < 7) = 0.10564... = 0.1056$ (4 d.p.)

ii A suitable lower limit is 2, giving $P(X < 5) = 0.10564... = 0.1056$ (4 d.p.)

b Since these are independent events, the probability is $P(X < 5)^3$, i.e.
$(0.10564...)^3 = 0.00117... = 0.0012$ (4 d.p.)

9 Use the Normal CD function on your calculator, with $\mu = 500$ and $\sigma = 14$.

a **i** A suitable upper limit is 570, giving $P(W > 505) = 0.36049... = 0.3605$ (4 d.p.)

ii A suitable lower limit is 430, giving $P(W < 490) = 0.23752... = 0.2375$ (4 d.p.)

b Since these are independent events, the probability is $P(W > 490)^4$.
$P(W > 490) = 1 - P(W < 490) = 1 - 0.23752... = 0.76248...$
So the probability is $(0.76248\ldots)^4 = 0.33799\ldots = 0.3380$ (4 d.p.).

10 Use the Normal CD function on your calculator, with $\mu = 165$ and $\sigma = 3.5$.

a A suitable lower limit is 10, giving $P(h < 160) = 0.07656... = 0.0766$ (4 d.p.)

b $P(168 < h < 174) = 0.19061... = 0.1906$ (4 d.p.)

c Use the binomial distribution $X \sim B(20, 0.1906)$.
Using the binomial CD function on your calculator:
$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.67035... = 0.3296$ (4 d.p.)

11 Use the Normal CD function on your calculator, with $\mu = 13$ and $\sigma = 0.1$.

a A suitable lower limit is 12.5, giving $P(D < 12.8) = 0.02274... = 0.0227$ (4 d.p.)

b P('perfect') = $P(12.9 < D < 13.1) = 0.68268... = 0.6827$ (4 d.p.)
Use the binomial distribution $X \sim B(40, 0.6827)$.
Using the binomial CD function on your calculator:
$P(X > 25) = 1 - P(X \leqslant 25) = 1 - 0.26549... = 0.7345$ (4 d.p.)

12 Use the Normal CD function on your calculator, with $\mu = 480$ and $\sigma = 40$.

a A suitable upper limit is 680, giving $P(X > 490) = 0.40129... = 0.4013$ (4 d.p.)

b $P(470 < X < 490) = 0.19741... = 0.1974$ (4 d.p.)
Use the binomial distribution $Y \sim B(30, 0.1974)$.
Using the binomial CD function on your calculator:
$P(X \geqslant 15) = 1 - P(X \leqslant 14) = 1 - 0.99980141... = 0.0001986$ (4 s.f.)

The normal distribution 3C

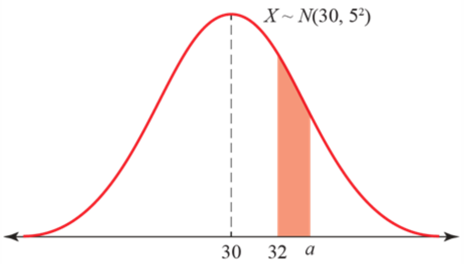
1 Use the inverse normal distribution function on your calculator, with $\mu = 30$ and $\sigma = 5$.

a $\text{P}(X < a) = 0.3 \Rightarrow a = 27.377... = 27.38$ (2 d.p.).

b $\text{P}(X < a) = 0.75 \Rightarrow a = 33.372... = 33.37$ (2 d.p.).

c $\text{P}(X > a) = 0.4 \Rightarrow \text{P}(X < a) = 0.6 \Rightarrow a = 31.266... = 31.27$ (2 d.p.).

d Since $P(32 < X < a) = 0.2$, it must be that $a > 32$.



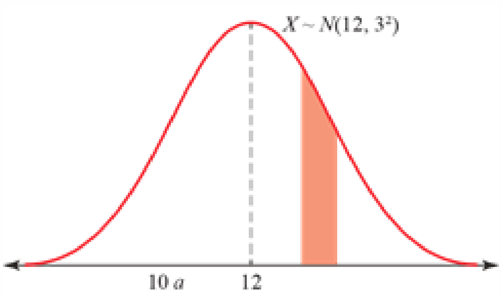
$\text{P}(32 < X < a) = 0.2$
$\Rightarrow \text{P}(X < a) - \text{P}(X < 32) = 0.2$
$\Rightarrow \text{P}(X < a) = 0.2 + \text{P}(X > 32) = 0.2 + 0.6554 = 0.8554$
$\Rightarrow a = 35.299... = 35.30$ (2 d.p.)

2 Use the inverse normal distribution function on your calculator, with $\mu = 12$ and $\sigma = 3$.

a $\text{P}(X < a) = 0.1 \Rightarrow a = 8.155... = 8.16$ (2 d.p.).

b $\text{P}(X > a) = 0.65 \Rightarrow \text{P}(X < a) = 0.35 \Rightarrow a = 10.844... = 10.84$ (2 d.p.).

c



$\text{P}(10 \leqslant X \leqslant a) = \text{P}(X \leqslant a) - \text{P}(X \leqslant 10) = 0.25$
$\Rightarrow \text{P}(X < a) = 0.25 + \text{P}(X < 10) = 0.25 + 0.2525 = 0.5025$
$\Rightarrow a = 12.018... = 12.02$ (2 d.p.)

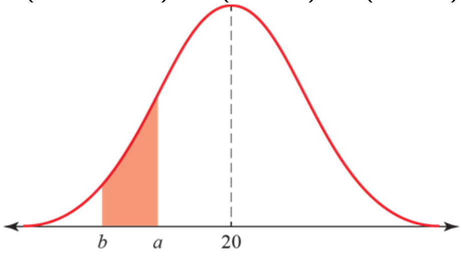
d $\text{P}(a < X < 14) = \text{P}(X < 14) - \text{P}(X < a) = 0.32$
$\Rightarrow \text{P}(X < a) = \text{P}(X < 14) - 0.32 = 0.7475 - 0.32 = 0.4275$
$\Rightarrow a = 11.451... = 11.45$ (2 d.p.)

3 Use the inverse normal distribution function on your calculator, with $\mu = 20$ and $\sigma = \sqrt{12}$.

a i $\text{P}(X < a) = 0.40 \Rightarrow a = 19.12... = 19.1$ (1 d.p.)

ii $\text{P}(X > b) = 0.6915 \Rightarrow \text{P}(X < b) = 0.3085 \Rightarrow b = 18.26... = 18.3$ (1 d.p.)

3 **b** $P(b < X < a) = P(X < a) - P(X < b) = 0.40 - (1 - 0.6915) = 0.0915$

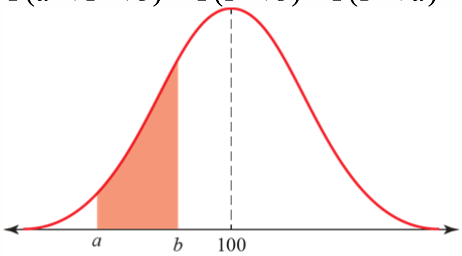


4 Use the inverse normal distribution function on your calculator, with $\mu = 100$ and $\sigma = 15$.

a **i** $P(Y > a) = 0.975 \Rightarrow P(Y < a) = 0.025 \Rightarrow a = 70.60... = 70.6$ (1 d.p.)

ii $P(Y < b) = 0.10 \Rightarrow b = 80.77... = 80.8$ (1 d.p.)

b $P(a < Y < b) = P(Y < b) - P(Y < a) = 0.10 - (1 - 0.975) = 0.10 - 0.025 = 0.075$

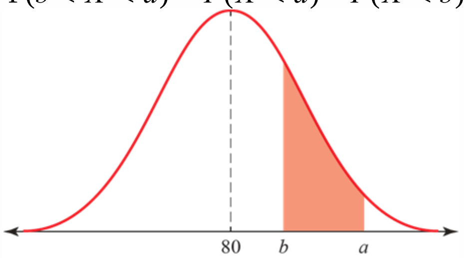


5 Use the inverse normal distribution function on your calculator, with $\mu = 80$ and $\sigma = \sqrt{16} = 4$.

a **i** $P(X > a) = 0.40 \Rightarrow P(X < a) = 0.60 \Rightarrow a = 81.01... = 81.0$ (1 d.p.)

ii $P(X < b) = 0.5636 \Rightarrow b = 80.64... = 80.6$ (1 d.p.)

b $P(b < X < a) = P(X < a) - P(X < b) = (1 - 0.40) - 0.5636 = 0.60 - 0.5636 = 0.0364$ (4 d.p.)



6 Use the inverse normal distribution function on your calculator, with $\mu = 4.5$ and $\sigma = 0.6$.

a By definition, the lower quartile is the point Q_1 such that $P(M < Q_1) = 0.25$.
$P(M < Q_1) = 0.25 \Rightarrow Q_1 = 4.0953... = 4.095$ kg (3 d.p.).

b Let a be the 80th percentile, so that $P(M < a) = 0.8$.
$P(M < a) = 0.8 \Rightarrow a = 5.0049... = 5.005$ kg (3 d.p.)

c The mean is 4.5, and since the data is normally distributed, this means that 50% of the badgers will have a mass less than 4.5 kg, i.e. $Q_2 = 4.5$ kg.

7 Use the inverse normal distribution function on your calculator, with $\mu = 72$ and $\sigma = 6$.

a $P(X < a) = 0.6 \Rightarrow a = 73.520... = 73.52$ (2 d.p.).

7 b $P(X < Q_1) = 0.25 \Rightarrow Q_1 = 67.953...$ and $P(X < Q_3) = 0.75 \Rightarrow Q_3 = 76.046...$
So the interquartile range = $Q_3 - Q_1 = 76.046 - 67.953 = 8.093 = 8.09$ (2 d.p.)

8 Use the inverse normal distribution function on your calculator, with $\mu = 60$ and $\sigma = 2$.

a $\mathrm{P}(Y > y) = 0.2 \Rightarrow \mathrm{P}(Y < y) = 0.8 \Rightarrow y = 61.683... = 61.68$ (2 d.p.)

b $\mathrm{P}(X < a) = 0.1 \Rightarrow a = 57.436...$ and $\mathrm{P}(X < b) = 0.9 \Rightarrow b = 62.563...$
So the 10% to 90% interpercentile range of masses is $b - a = 5.127 = 5.13$ grams (2 d.p.).

c Tom is correct: the data is assumed to be normally distributed, so the median is equal to the mean.

9 a The short coat should be suitable for the shortest 30% of the men.
Since $\mathrm{P}(H < a) = 0.3 \Rightarrow a = 165$, this means that the short coat should be suitable for the men who are up to 165 cm tall.
The long coat should be suitable for the tallest 20% of the men.
Since $\mathrm{P}(H > b) = 0.2 \Rightarrow \mathrm{P}(H < b) = 0.8 \Rightarrow b = 178$, this means that the long coat should be suitable for the men who are more than 178 cm tall.
The regular coat is then suitable for those in between, i.e. those men who are between 165 cm and 178 cm tall.

b There are many assumptions made by the model. It is assumed, for example, that the men likely to shop at the stores selling these frock coats have the same distribution of heights as those in the 'large group'; that the men's measurements from their necks to near the floor are normally distributed; that arm lengths are also normally distributed; that people are 'in proportion' (that, generally, taller men have longer arm lengths and higher shoulders than shorter men); and that the population follows the normal distribution over the whole range of values, i.e. that there are no extreme outliers.

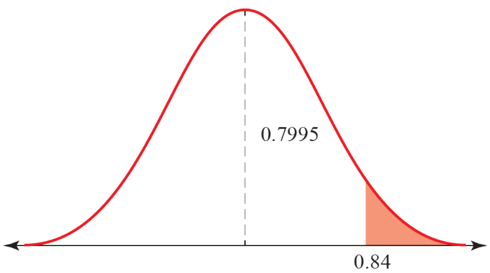
The normal distribution 3D

1 Use the Normal CD function on your calculator, with $\mu = 0, \sigma = 1$ and a small value for the lower limit, e.g. −10.

a $P(Z < 2.12) = 0.98299... = 0.9830$ (4 d.p.)

b $P(Z < 1.36) = 0.91308... = 0.9131$ (4 d.p.)

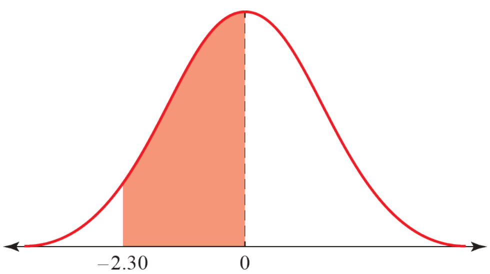
c



$P(Z > 0.84)$
$= 1 - P(Z < 0.84)$
$= 1 - 0.79954...$
$= 0.20045... = 0.2005$ (4 d.p.)

d $P(Z < -0.38) = 0.35197... = 0.3520$ (4 d.p.)

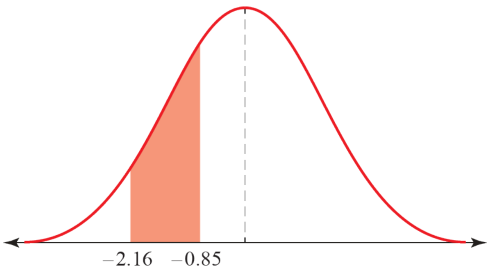
e



$P(-2.30 < Z < 0)$
$= 0.5 - P(Z < -2.30)$
$= 0.5 - 0.1072...$
$= 0.48929... = 0.4893$ (4 d.p.)

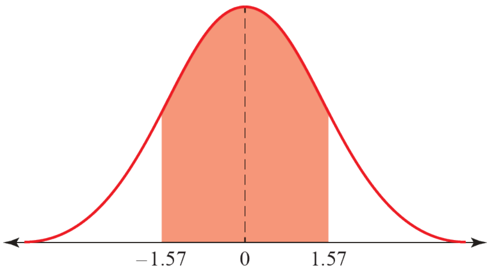
f $P(Z < -1.63) = 0.05155... = 0.0516$ (4 d.p.)

1 g



$P(-2.16 < Z < -0.85)$
$= P(Z < -0.85) - P(Z < -2.16)$
$= 0.19766... - 0.01538...$
$= 0.18227... = 0.1823$ (4 d.p.)

h



$P(-1.57 < Z < 1.57)$
$= 2 \times (0.5 - P(Z < -1.57)$
$= 2 \times (0.5 - 0.05820...)$
$= 2 \times 0.44179...$
$= 0.88358... = 0.8836$ (4 d.p.)

2 Use the inverse normal distribution function on your calculator, with $\mu = 0$ and $\sigma = 1$.

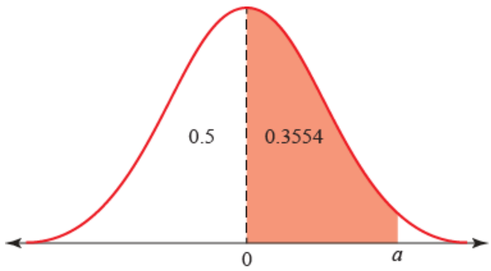
a $P(Z < a) = 0.9082 \Rightarrow a = 1.32975... = 1.3298$ (4 d.p.)

b $P(Z > a) = 0.0314$
$\Rightarrow P(Z < a) = 0.9686$
$\Rightarrow a = 1.86060... = 1.8606$ (4 d.p.)

c $P(Z > a) = 0.15$
$\Rightarrow P(Z < a) = 0.85$
$\Rightarrow a = 1.03643... = 1.0364$ (4 d.p.)
(Alternatively, use the table of percentage points with $p = 0.15 \Rightarrow a = 1.0364$)

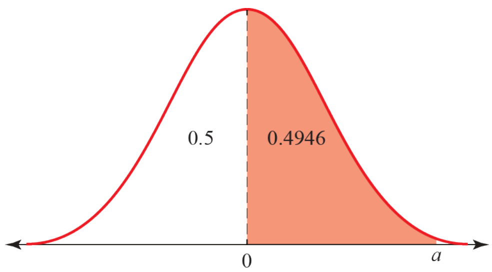
d $P(Z > a) = 0.95$
$\Rightarrow P(Z < a) = 0.05$
$\Rightarrow a = -1.64485... = -1.6449$ (4 d.p.)
(Alternatively, use the table of percentage points with $p = 0.05 \Rightarrow -a = 1.6449 \Rightarrow a = -1.6449$)

2 **e**



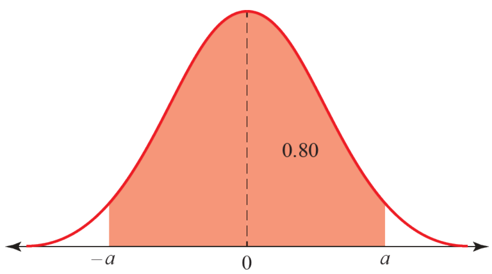
$P(0 < Z < a) = 0.3554$
$\Rightarrow P(Z < a) = 0.8554$
$\Rightarrow a = 1.05987... = 1.0599$ (4 d.p.)

f



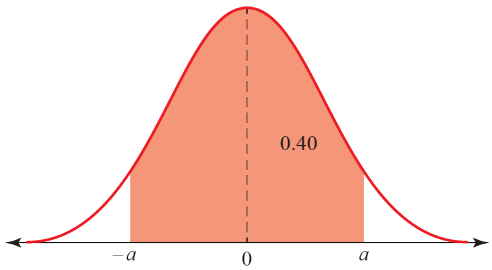
$P(0 < Z < a) = 0.4946$
$\Rightarrow P(Z < a) = 0.9946$
$\Rightarrow a = 2.54910... = 2.5491$ (4 d.p.)

g



$P(-a < Z < a) = 0.80$
$\Rightarrow P(-a < Z < 0) = 0.40$
$\Rightarrow P(-a < Z) = 0.10$
$\Rightarrow -a = -1.28155...$
$\Rightarrow a = 1.2816$ (4 d.p.)
(Alternatively, use the table of percentage points with $p = 0.10 \Rightarrow a = 1.2816$)

2 h



$\mathrm{P}(-a < Z < a) = 0.40$
$\Rightarrow \mathrm{P}(-a < Z < 0) = 0.20$
$\Rightarrow \mathrm{P}(-a < Z) = 0.30$
$\Rightarrow -a = -0.52440...$
$\Rightarrow a = 0.5244$ (4 d.p.)
(Alternatively, use the table of percentage points with $p = 0.30 \Rightarrow a = 0.5244$)

3 a $x = 0.8 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{0.8 - 0.8}{0.05} = 0$

b $x = 0.792 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{0.792 - 0.8}{0.05} = -0.16$

c $x = 0.81 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{0.81 - 0.8}{0.05} = 0.2$

d $x = 0.837 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{0.837 - 0.8}{0.05} = 0.74$

4 a $x = 154 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{154 - 154}{12} = 0 \Rightarrow \mathrm{P}(X < 154) = \Phi(0)$

b $x = 160 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{160 - 154}{12} = 0.5 \Rightarrow \mathrm{P}(X < 160) = \Phi(0.5)$

c $x = 151 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{151 - 154}{12} = -0.25 \Rightarrow \mathrm{P}(X > 151) = 1 - \mathrm{P}(X < 151) = 1 - \Phi(-0.25)$

d $x = 140 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{140 - 154}{12} = -\dfrac{7}{6}$

$x = 155 \Rightarrow z = \dfrac{x-\mu}{\sigma} = \dfrac{155 - 154}{12} = \dfrac{1}{12}$

$\Rightarrow \mathrm{P}(140 < X < 155) = \mathrm{P}(X < 155) - \mathrm{P}(X < 140) = \Phi\left(\dfrac{1}{12}\right) - \Phi\left(-\dfrac{7}{6}\right)$

5 a $\mathrm{P}(Z > z) = 0.025 \Rightarrow p = 0.025$
Using the percentage points table, $p = 0.025 \Rightarrow z = 1.96$

5 b Using the formula $z = \dfrac{x-\mu}{\sigma}$:

$$1.96 = \frac{x-80}{4}$$
$$x - 80 = 4 \times 1.96$$
$$x = 80 + 7.84$$
$$= 87.84$$

A score of 87.8 (3 s.f.) is needed to get on the programme.

6 a From the percentage points table, $p = 0.15 \Rightarrow z = 1.0364$
Therefore $\mathrm{P}(Z > 1.0364) = 0.15,$ hence $\mathrm{P}(Z < -1.0364) = 0.15,$ so $z = -1.0364$

b Using the formula $z = \dfrac{x-\mu}{\sigma}$:

$$-1.0364 = \frac{x-57}{2}$$
$$x - 57 = 2 \times (-1.0364)$$
$$x = 57 - 2.0728$$
$$= 54.9272$$

The size of a 'petite' hat is 54.9 cm (3 s.f.).

7 a The 90th percentile corresponds to $p = 0.1$.
From the percentage points table, $p = 0.10 \Rightarrow z = 1.2816$
By the symmetry of the normal distribution, the 10th percentile is at $z = -1.2816$
So the 10% to 90% interpercentile range corresponds to $-1.2816 < z < 1.2816$

b A 'standard' light bulb should have a range of life within the above range, but for $\mathrm{N}(1175,\ 56)$.

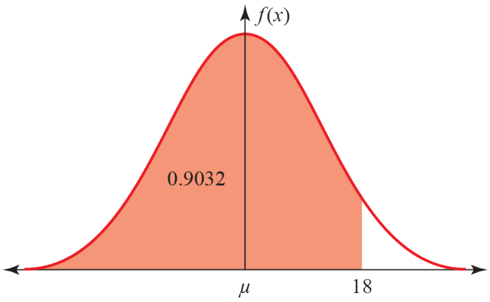
Using the formula $z = \dfrac{x-\mu}{\sigma}$ with $z = -1.2816$:

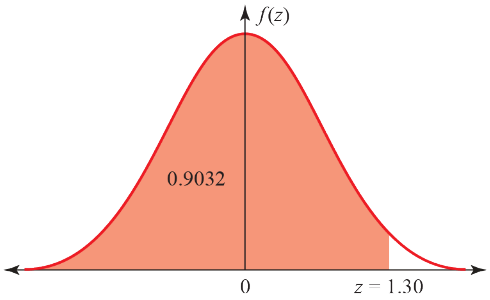
$$-1.2816 = \frac{x-1175}{56}$$
$$x - 1175 = 56 \times (-1.2816)$$
$$x = 1175 - 71.7696$$
$$= 1103.2304$$

Similarly, for $z = 1.2816$, $x = 1175 + 71.7696 = 1246.7696$.
So the range of life for a 'standard' bulb is 1103 to 1247 hours.

The normal distribution 3E

1 $P(X < 18) = 0.9032 \Rightarrow P\left(Z < \frac{18-\mu}{5}\right) = 0.9032$



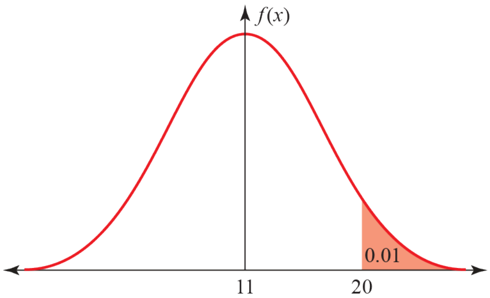


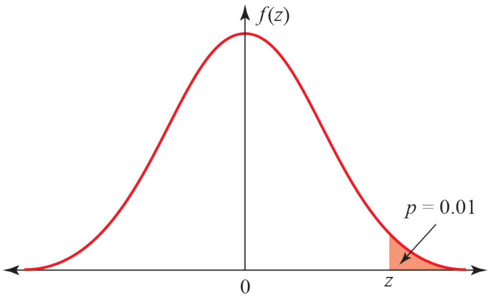
Using the inverse normal function, $z = 1.30$

so $1.30 = \frac{18-\mu}{5}$

$\mu = 18 - 5 \times 1.30 = 11.5$

2 $\mathrm{P}(X > 20) = 0.01 \Rightarrow \mathrm{P}\left(Z < \frac{20-11}{\sigma}\right) = 0.01$





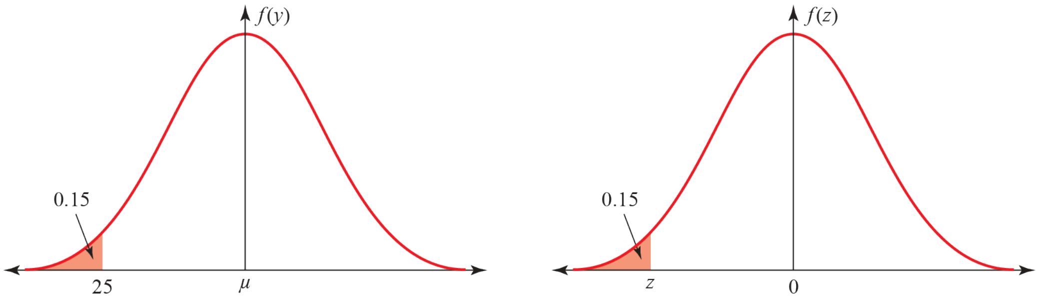
Using the inverse normal function, $z = 2.3263...$

so $2.3263... = \frac{20-11}{\sigma}$

$\sigma = \frac{9}{2.3263...}$

$= 3.8687... = 3.87$ (3 s.f.)

3 $P(Y < 25) = 0.15 \Rightarrow P\left(Z < \frac{25-\mu}{\sqrt{40}}\right) = 0.15$

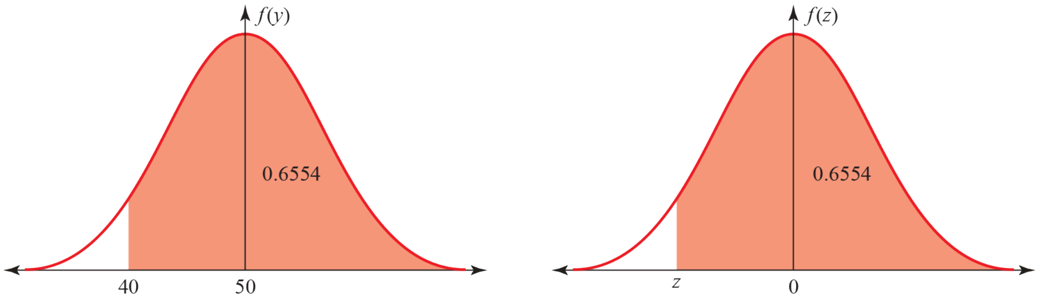


Using the inverse normal function, $z = -1.0364...$

so $-1.0364... = \frac{25-\mu}{\sqrt{40}}$

$$\mu = \sqrt{40} \times (-1.0364...)$$
$$= 31.554... = 31.6 \text{ (3 s.f.)}$$

4 $P(Y > 40) = 0.6554 \Rightarrow P\left(Z > \frac{40-50}{\sigma}\right) = 0.6554$



Using the inverse normal function, $z = -0.3999...$

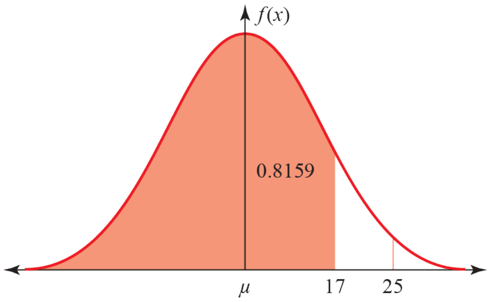
so $-0.3999... = \frac{40-50}{\sigma}$

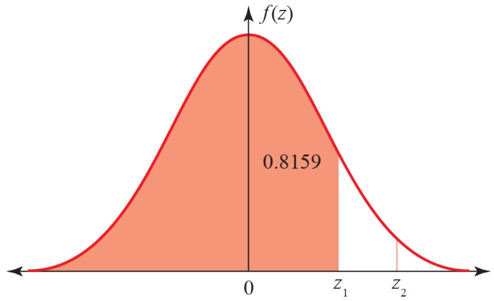
$$\sigma = \frac{10}{0.3999...} = 25.0 \text{ (3 s.f.)}$$

5 Using the inverse normal function,

$$P(X < 17) = 0.8159 \Rightarrow P\left(Z < \frac{17-\mu}{\sigma}\right) = 0.8159 \Rightarrow z_1 = 0.8998...$$

$$P(X < 25) = 0.9970 \Rightarrow P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.9970 \Rightarrow z_2 = 2.7477...$$





So $0.8998\sigma = 17 - \mu$ **(1)**

and $2.7477\sigma = 25 - \mu$ **(2)**

(2) − **(1)**: $1.8479\sigma = 8$

$\sigma = 4.329...$

Substituting into **(2)**:

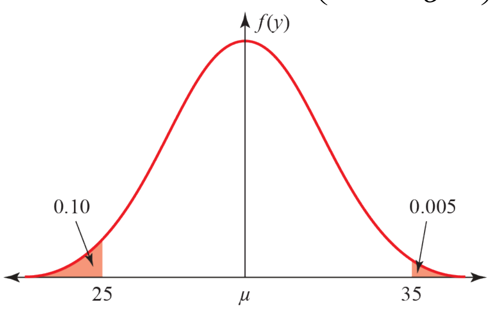
$\mu = 17 - 0.8998 \times 4.329... = 13.104...$

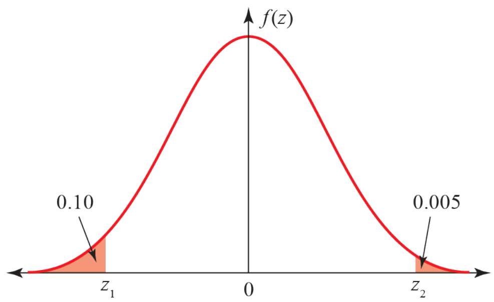
So $\mu = 13.1$ and $\sigma = 4.33$ (3 s.f.)

6 Using the inverse normal function (or the percentage points table),

$$P(Y < 25) = 0.10 \Rightarrow P\left(Z < \frac{25-\mu}{\sigma}\right) = 0.10 \Rightarrow z_1 = -1.28155...$$

$$P(Y > 35) = 0.005 \Rightarrow P\left(Z > \frac{35-\mu}{\sigma}\right) = 0.005 \Rightarrow z_2 = 2.57582...$$





So $-1.2816\sigma = 25 - \mu$ **(1)**

and $2.5758\sigma = 35 - \mu$ **(2)**

(2) − **(1)**: $3.8574\sigma = 10$

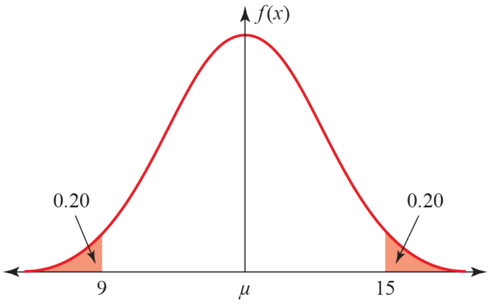
$\sigma = 2.5924...$

Substituting into **(2)**:

$\mu = 35 - 2.5758 \times 2.5924... = 28.322...$

So $\mu = 28.3$ and $\sigma = 2.59$ (3 s.f.)

7



By symmetry, $\mu = \frac{1}{2}(9+15) = 12$

$$\mathrm{P}(X > 15) = 0.20 \Rightarrow \mathrm{P}\left(Z > \frac{15-12}{\sigma}\right) = 0.20$$

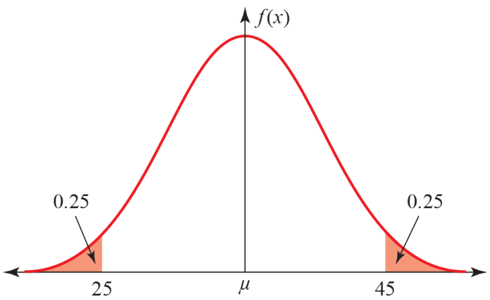
Using the inverse normal function (or the percentage points table), $z = 0.8416...$

so $0.8416 = \dfrac{3}{\sigma}$

$$\sigma = \frac{3}{0.8416} = 3.564...$$

So $\mu = 12$ and $\sigma = 3.56$ (3 s.f.)

8



By symmetry, $\mu = \frac{1}{2}(25+45) = 35$

$$\mathrm{P}(X > 45) = 0.25 \Rightarrow \mathrm{P}\left(Z > \frac{45-35}{\sigma}\right) = 0.25$$

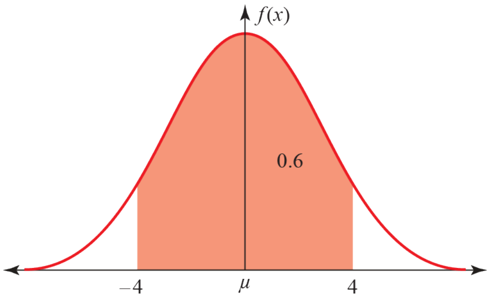
Using the inverse normal function, $z = 0.6744...$

so $0.6744 = \dfrac{10}{\sigma}$

$$\sigma = \frac{10}{0.6744} = 14.82...$$

So $\mu = 35$ and $\sigma = 14.8$ (3 s.f.)

9 $\mu = 0$ (given) so $\mathrm{P}(X > 4) = 0.2$ and $\mathrm{P}\left(Z > \frac{4-0}{\sigma}\right) = 0.2$



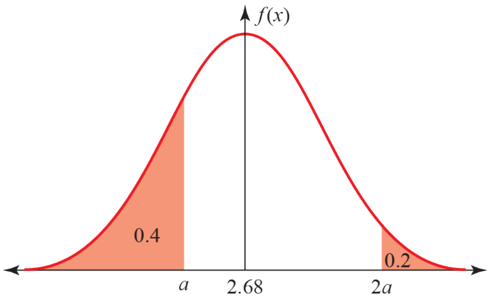
Using the inverse normal function (or the percentage points table), $z = 0.8416...$

so $0.8416 = \frac{4}{\sigma}$

$$\sigma = \frac{4}{0.8416} = 4.752...$$

So $\sigma = 4.75$ (3 s.f.)

10



Using the inverse normal function (or the percentage points table),

$\mathrm{P}(X > 2a) = 0.2 \Rightarrow \mathrm{P}\left(Z < \frac{2a-2.68}{\sigma}\right) = 0.2 \Rightarrow z_1 = 0.8416...$

$\mathrm{P}(X < a) = 0.4 \Rightarrow \mathrm{P}\left(Z < \frac{a-2.68}{\sigma}\right) = 0.4 \Rightarrow z_2 = -0.2533...$

So $\quad 0.8416\sigma = 2a - 2.68 \qquad$ **(1)**

and $\quad -0.2533\sigma = a - 2.68 \qquad$ **(2)**

(2) $\times 2$: $\quad -0.5066\sigma = 2a - 5.36 \qquad$ **(3)**

(1) − **(3)**: $\quad 1.3482\sigma = 2.68$

$$\sigma = 1.9878...$$

Substituting into **(2)**:

$a = 2.68 - 0.2533 \times 1.9878... = 2.176...$

So $\sigma = 1.99$ and $a = 2.18$ (3 s.f.)

11 a The distribution is $D \sim \text{N}(\mu, 5^2)$.

$$\text{P}(D > 200) = 0.75 \Rightarrow \text{P}(D < 200) = 0.25 \Rightarrow \text{P}\left(Z < \frac{200 - \mu}{5}\right) = 0.25$$

Using the inverse normal function, $z = -0.6744...$

$$\text{so } -0.6744... = \frac{200 - \mu}{5}$$

$$\mu = 200 + 5 \times 0.6744...$$

$$= 203.37... = 203\text{ mm (3 s.f.)}$$

b $\text{P}(204 < D < 206) = \text{P}(D < 206) - \text{P}(D < 204)$

$$= \text{P}\left(Z < \frac{206 - 203.37...}{5}\right) - \text{P}\left(Z < \frac{204 - 203.37...}{5}\right)$$

$$= \text{P}(Z < 0.5256) - \text{P}(Z < 0.1256)$$

$$= 0.70041... - 0.54997...$$

$$= 0.15045 = 0.1504 \text{ (4 d.p.)}$$

c $\text{P}(D > 205) = \text{P}(Z > 0.3256) = 1 - 0.62763... = 0.37237...$
So the probability that all three bowls are greater than 205 mm in diameter is $\text{P}(D > 205)^3$,
i.e. $(0.37237...)^3 = 0.05163... = 0.0516$ (3 s.f.).

12 a The distribution is $T \sim \text{N}(2.5, \sigma)$.

$$\text{P}(T < 2.55) = 0.65 \Rightarrow \text{P}\left(Z < \frac{2.55 - 2.5}{\sigma}\right) = 0.65$$

Using the inverse normal function, $z = 0.38532...$

$$\text{so } \frac{2.55 - 2.5}{\sigma} = 0.38532...$$

$$0.05 = \sigma \times 0.38532...$$

$$\sigma = 0.12976... = 0.1298 \text{ (4 d.p.)}$$

b $\text{P}(2.4 < T < 2.6) = \text{P}(T < 2.6) - \text{P}(T < 2.4)$ (4 s.f.).

$$= \text{P}\left(Z < \frac{2.6 - 2.5}{0.12976...}\right) - \text{P}\left(Z < \frac{2.4 - 2.5}{0.12976...}\right)$$

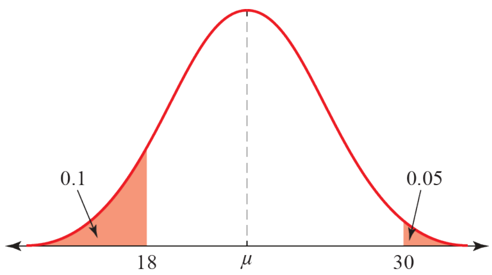
$$= \text{P}(Z < 0.77065...) - \text{P}(Z < -0.77065...)$$

$$= 0.77954... - 0.22045...$$

$$= 0.55908... = 0.5591 \text{ (4 d.p.)}$$

c Use the binomial distribution $X \sim \text{B}(20, 0.5591)$.
Using the binomial CD function,
$\text{P}(X \geqslant 15) = 1 - \text{P}(X \leqslant 14) = 1 - 93501... = 0.06497...$
So the probability that at least 15 table cloths can be sold is 0.0650 (4 d.p.)

13 a



b $P(M < 18) = 0.10 \Rightarrow P\left(Z < \frac{18-\mu}{\sigma}\right) = 0.10 \Rightarrow z_1 = -1.28155...$

$P(M > 30) = 0.05 \Rightarrow P\left(Z > \frac{30-\mu}{\sigma}\right) = 0.05 \Rightarrow z_2 = 1.64485...$

So $\quad -1.28155... \times \sigma = 18 - \mu \quad$ **(1)**

and $\quad 1.64485... \times \sigma = 30 - \mu \quad$ **(2)**

(2) − **(1)**: $\quad 2.92640... \times \sigma = 12$

$\sigma = 4.10059...$

Substituting into **(2)**:

$\mu = 30 - 1.64485... \times 4.10059... = 23.25512...$

So $\mu = 23.26$ and $\sigma = 4.101$ (4 s.f.)

c $P(M > 25) = 1 - P\left(Z > \frac{25 - 23.25512...}{4.10059...}\right) = 0.33522...$

Use the binomial distribution $X \sim B(10, 0.33522...)$

Using the binomial CD function,

$P(X \geqslant 4) = 1 - P(X \leqslant 3) = 1 - 0.55408... = 0.44591...$

So the probability that at least 4 have a mass greater than 25 kg is 0.4459 (4 d.p.)

14 a $P(L < 16) = 0.20 \Rightarrow P\left(Z < \frac{16-\mu}{\sigma}\right) = 0.20 \Rightarrow z_1 = -0.84162...$

$P(L > 18) = 0.10 \Rightarrow P\left(Z > \frac{18-\mu}{\sigma}\right) = 0.10 \Rightarrow z_2 = 1.28155...$

So $\quad -0.84162... \times \sigma = 16 - \mu \quad$ **(1)**

and $\quad 1.28155... \times \sigma = 18 - \mu \quad$ **(2)**

(2) − **(1)**: $\quad 2.12317... \times \sigma = 2$

$\sigma = 0.94198...$

Substituting into **(2)**:

$\mu = 18 - 1.28155... \times 0.94198... = 16.79279...$

So $\mu = 16.79$ and $\sigma = 0.9420$ (4 s.f.)

b $P(L < Q_1) = 0.25 \Rightarrow z_1 = -0.67448... \Rightarrow \frac{Q_1 - 16.79279...}{0.94198...} = -0.67448... \Rightarrow Q_1 = 16.15743...$

$P(L < Q_3) = 0.75 \Rightarrow z_2 = 0.67448... \Rightarrow \frac{Q_3 - 16.79279...}{0.94198...} = 0.67448... \Rightarrow Q_3 = 17.42814...$

The interquartile range is $Q_3 - Q_1 = 17.42814... - 16.15743... = 1.2701... = 1.27$ (2 d.p.)

Challenge

a Let the quartiles be $Q_3 = \mu + z\sigma$ and $Q_1 = \mu + z\sigma$.
Then the interquartile range is $Q_3 - Q_1 = q = (\mu + z\sigma) - (\mu - z\sigma) = 2z\sigma$
z is such that $\Phi(z) = 0.75$ so, using the inverse normal function, $z = 0.67448...$
So $q = 2 \times 0.67448... \times \sigma = 1.34987... \times \sigma$
and hence $\sigma = 0.74130... \times p = 0.741p$ (3 s.f.)

b Since $q = (\mu + z\sigma) - (\mu - z\sigma) = 2z\sigma$ (i.e. the μ s cancel), q is not dependent on μ.
So it is not possible to write an equation for q in terms of μ, and vice versa.

The normal distribution 3F

1 a i Yes, since $n = 120$ is large (> 50) and $p = 0.6$ is close to 0.5.

ii $\mu = np = 120 \times 0.6 = 72$ and $\sigma = \sqrt{np(1-p)} = \sqrt{72 \times 0.4} = \sqrt{28.8} = 5.37$ (3 s.f.)
$X \sim \text{B}(72, 5.37^2)$

b i No, $n = 6$ is not large enough (< 50).

c i Yes, since $n = 250$ is large (> 50) and $p = 0.52$ is close to 0.5.

ii $\mu = np = 250 \times 0.52 = 130$ and $\sigma = \sqrt{np(1-p)} = \sqrt{130 \times 0.48} = \sqrt{62.4} = 7.90$ (3 s.f.)
$X \sim \text{B}(130, 7.90^2)$

d i No, $p = 0.98$ is too far from 0.5.

e i Yes, since $n = 400$ is large (> 50) and $p = 0.48$ is close to 0.5.

ii $\mu = np = 400 \times 0.48 = 192$ and $\sigma = \sqrt{np(1-p)} = \sqrt{192 \times 0.52} = \sqrt{99.84} = 9.99$ (3 s.f.)
$X \sim \text{B}(192, 9.99^2)$

f i Yes, since $n = 1000$ is large (> 50) and $p = 0.58$ is close to 0.5.

ii $\mu = np = 1000 \times 0.58 = 580$ and $\sigma = \sqrt{np(1-p)} = \sqrt{580 \times 0.42} = \sqrt{243.6} = 15.6$ (3 s.f.)
$X \sim \text{B}(580, 15.6^2)$

2 A normal approximation is valid since $n = 150$ is large (> 50) and $p = 0.45$ is close to 0.5.
$\mu = np = 150 \times 0.45 = 67.5$ and $\sigma = \sqrt{np(1-p)} = \sqrt{67.5 \times 0.55} = \sqrt{37.125} = 6.093$ (4 s.f.)

a $\text{P}(X \leqslant 60) \approx \text{P}(Y < 60.5) = 0.1253$ (4 s.f.)

b $\text{P}(X > 75) \approx \text{P}(Y > 75.5) = 0.0946$ (4 s.f.)

c $\text{P}(65 \leqslant X \leqslant 80) \approx \text{P}(64.5 < Y < 80.5) = 0.6723$ (4 s.f.)

3 A normal approximation is valid since $n = 200$ is large (> 50) and $p = 0.53$ is close to 0.5.
$\mu = np = 200 \times 0.53 = 106$ and $\sigma = \sqrt{np(1-p)} = \sqrt{106 \times 0.47} = \sqrt{49.82} = 7.058$ (4 s.f.)

a $\text{P}(X < 90) \approx \text{P}(Y < 89.5) = 0.0097$ (4 s.f.)

b $\text{P}(100 \leqslant X < 110) \approx \text{P}(99.5 < Y < 109.5) = 0.5115$ (4 s.f.)

c $\text{P}(X = 105) \approx \text{P}(104.5 < Y < 105.5) = 0.0559$ (4 s.f.)

4 A normal approximation is valid since $n = 100$ is large (> 50) and $p = 0.6$ is close to 0.5.
$\mu = np = 100 \times 0.6 = 60$ and $\sigma = \sqrt{np(1-p)} = \sqrt{60 \times 0.4} = \sqrt{24} = 4.899$ (4 s.f.)

a $\text{P}(X > 58) \approx \text{P}(Y > 58.5) = 0.6203$ (4 s.f.)

b $\text{P}(60 < X \leqslant 72) \approx \text{P}(60.5 < Y < 72.5) = 0.4540$ (4 s.f.)

c $\text{P}(X = 70) \approx \text{P}(69.5 < Y < 70.5) = 0.0102$ (4 s.f.)

5 Let X = number of heads in 70 tosses of a fair coin, so $X \sim \text{B}(70, 0.5)$.
Since $p = 0.5$ and 70 is large, X can be approximated by the normal distribution $Y \sim \text{N}(\mu, \sigma^2)$,
where $\mu = 70 \times 0.5 = 35$ and $\sigma = \sqrt{70 \times 0.5 \times 0.5} = \sqrt{17.5}$
So $Y \sim \text{N}(35, 17.5)$
$\text{P}(X > 45) \approx \text{P}(Y \geqslant 45.5) = 0.0060$

6 A normal approximation is valid since $n = 1200$ is large and p is close to 0.5.
$\mu = np = 1200 \times \frac{50}{101} = 594.059$
and $\sigma = \sqrt{np(1-p)} = \sqrt{594.059 \times \frac{51}{101}} = \sqrt{299.97059\ldots} = 17.32$ (4 s.f.)
So $Y \sim \text{N}(594.059, 299.97\ldots)$
$\text{P}(X \geqslant 600) \approx \text{P}(Y > 599.5) = 0.3767$ (4 s.f.)

7 a The number of trials, n, must be large (> 50), and the success probability, p, must be close to 0.5.

b Using the binomial distribution, $\text{P}(X = 10) = \binom{20}{10} \times 0.45^{10} \times 0.55^{10} = 0.1593$ (4 s.f.)

c A normal approximation is valid since $n = 240$ is large and $p = 0.45$ is close to 0.5.
$\mu = np = 240 \times 0.45 = 108$ and $\sigma = \sqrt{np(1-p)} = \sqrt{108 \times 0.55} = \sqrt{59.4} = 7.707$ (4 s.f.)
So $Y \sim \text{N}(108, 59.4)$
$\text{P}(X < 110) \approx \text{P}(Y < 109.5) = 0.5772$ (4 s.f.)

d $\text{P}(X \geqslant q) = 0.2 \Rightarrow \text{P}(Y > (q - 0.5)) = 0.2$
Using the inverse normal function,
$\text{P}(Y > (q - 0.5)) = 0.2 \Rightarrow q - 0.5 = 114.485 \Rightarrow q = 114.985$
So $q = 115$

8 a Using the cumulative binomial function with $N = 30$ and $p = 0.52$,
$\text{P}(X < 17) = \text{P}(X \leqslant 16) = 0.6277$ (4 s.f.)

b A normal approximation is valid since $n = 600$ is large and $p = 0.52$ is close to 0.5.
$\mu = np = 600 \times 0.52 = 312$ and $\sigma = \sqrt{np(1-p)} = \sqrt{312 \times 0.48} = \sqrt{149.76} = 12.24$ (4 s.f.)
So $Y \sim \text{N}(312, 149.76)$
$\text{P}(300 \leqslant X \leqslant 350) \approx \text{P}(299.5 < Y < 350.5) = 0.8456$ (4 s.f.)

9 **a** Using the binomial distribution, $P(X = 55) = \binom{100}{55} \times 0.56^{55} \times 0.44^{45} = 0.07838$ (4 s.f.)

b A normal approximation is valid since $n = 100$ is large and $p = 0.56$ is close to 0.5.

$\mu = np = 100 \times 0.56 = 56$ and $\sigma = \sqrt{np(1-p)} = \sqrt{56 \times 0.44} = \sqrt{24.64} = 4.964$ (4 s.f.)

So $Y \sim B(56, 24.64)$

$P(X = 55) \approx P(54.5 < Y < 55.5) = 0.07863$ (4 s.f.)

Percentage error $= \dfrac{0.07838... - 0.07863...}{0.07838...} \times 100 = -0.31\%$ (2 d.p.)

The normal distribution 3G

1 a $H_0 : \mu = 21$, $H_1 : \mu \neq 21$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X \sim N(21, 1.5^2)$ and $\overline{X} \sim \left(21, \frac{1.5^2}{20}\right)$ or $\overline{X} \sim (21, 0.0.3354^2)$

Using the cumulative normal function, $P(\overline{X} > 21.2) = 0.2755$ (4 d.p.)
$0.2755 > 0.025$ so not significant. Do not reject H_0.

b $H_0 : \mu = 100$, $H_1 : \mu < 100$, one-tailed test at the 5% level.

Assume H_0, so that $X \sim N(100, 5^2)$ and $\overline{X} \sim \left(100, \frac{5^2}{36}\right)$ or $\overline{X} \sim (100, 0.8333^2)$

$P(\overline{X} < 98.5) = 0.0359$ (4 d.p.)
$0.0359 < 0.05$ so significant. Do not reject H_0.

c $H_0 : \mu = 5$, $H_1 : \mu \neq 5$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X \sim N(5, 3^2)$ and $\overline{X} \sim \left(5, \frac{3^2}{25}\right)$ or $\overline{X} \sim (5, 0.6^2)$

$P(\overline{X} > 6.1) = 0.0334$ (4 d.p.)
$0.0334 > 0.025$ so not significant. Do not reject H_0.

d $H_0 : \mu = 15$, $H_1 : \mu > 15$, one-tailed test at the 1% level.

Assume H_0, so that $X \sim N(15, 3.5^2)$ and $\overline{X} \sim \left(15, \frac{3.5^2}{40}\right)$ or $\overline{X} \sim (5, 0.5534^2)$

$P(\overline{X} > 16.5) = 0.0034$ (4 d.p.)
$0.0034 < 0.01$ so significant. Reject H_0.

e $H_0 : \mu = 50$, $H_1 : \mu \neq 50$, two-tailed test with 0.5% in each tail.

Assume H_0, so that $X \sim N(50, 4^2)$ and $\overline{X} \sim \left(50, \frac{4^2}{60}\right)$ or $\overline{X} \sim (50, 0.5163^2)$

$P(\overline{X} < 48.9) = 0.0166$ (4 d.p.)
$0.0166 > 0.005$ so not significant. Do not reject H_0.

2 a $H_0 : \mu = 120$, $H_1 : \mu < 120$, one-tailed test at the 5% level.

Assume H_0, so that $X \sim N(120, 2^2)$ and $\overline{X} \sim \left(120, \frac{2^2}{30}\right)$

Let $Z = \dfrac{\overline{X} - 120}{\frac{2}{\sqrt{30}}}$

Using the inverse normal function, $P(Z < z) = 0.05 \Rightarrow z = -1.6449$

$-1.6449 = \dfrac{\overline{X} - 120}{\frac{2}{\sqrt{30}}} \Rightarrow \overline{X} = 120 - 1.6449 \times \dfrac{2}{\sqrt{30}} = 119.39...$

So the critical region is $\overline{X} < 119.39...$ or 119 (3 s.f.).

2 **b** $\mathrm{H}_0 : \mu = 12.5$, $\mathrm{H}_1 : \mu > 12.5$, one-tailed test at the 1% level.

Assume H_0, so that $X \sim \mathrm{N}(12.5, 1.5^2)$ and $\overline{X} \sim \left(12.5, \frac{1.5^2}{25}\right)$

Let $Z = \frac{\overline{X} - 12.5}{\frac{1.5}{\sqrt{25}}} = \frac{\overline{X} - 12.5}{\frac{1.5}{5}}$

$\mathrm{P}(Z > z) = 0.01 \Rightarrow z = 2.3263$

$2.3263 = \frac{\overline{X} - 12.5}{\frac{1.5}{5}} \Rightarrow \overline{X} = 12.5 + 2.3263 \times \frac{3}{10} = 13.197...$

So the critical region is $\overline{X} > 13.197...$ or 13.2 (3 s.f.) (3 s.f.).

c $\mathrm{H}_0 : \mu = 85$, $\mathrm{H}_1 : \mu > 85$, one-tailed test at the 10% level.

Assume H_0, so that $X \sim \mathrm{N}(85, 4^2)$ and $\overline{X} \sim \left(85, \frac{4^2}{50}\right)$

Let $Z = \frac{\overline{X} - 85}{\frac{4}{\sqrt{50}}}$

$\mathrm{P}(Z < z) = 0.10 \Rightarrow z = -1.2816$

$-1.2816 = \frac{\overline{X} - 85}{\frac{4}{\sqrt{50}}} \Rightarrow \overline{X} = 85 - 1.2816 \times \frac{4}{\sqrt{50}} = 84.275...$

So the critical region is $\overline{X} < 84.275...$ or 84.3 (3 s.f.).

d $\mathrm{H}_0 : \mu = 0$, $\mathrm{H}_1 : \mu \neq 0$, two-tailed test with 2.5% in each tail.

Assume H_0, so that $X \sim \mathrm{N}(0, 3^2)$ and $\overline{X} \sim \left(0, \frac{3^2}{45}\right)$

Let $Z = \frac{\overline{X} - 0}{\frac{3}{\sqrt{45}}}$

$\mathrm{P}(Z < z) = 0.025 \Rightarrow z = -1.9600 \Rightarrow \overline{X} = -1.96 \times \frac{3}{\sqrt{45}} = -0.8765...$

$\mathrm{P}(Z > z) = 0.025 \Rightarrow z = 1.9600 \Rightarrow \overline{X} = 1.96 \times \frac{3}{\sqrt{45}} = 0.8765...$

So the critical region is $\overline{X} < -0.877$ or $\overline{X} > 0.877$ (3 s.f.).

e $\mathrm{H}_0 : \mu = -8$, $\mathrm{H}_1 : \mu \neq -8$, two-tailed test with 0.5% in each tail.

Assume H_0, so that $X \sim \mathrm{N}(-8, 1.2^2)$ and $\overline{X} \sim \left(-8, \frac{1.2^2}{20}\right)$

Let $Z = \frac{\overline{X} - (-8)}{\frac{1.2}{\sqrt{20}}}$

$\mathrm{P}(Z < z) = 0.005 \Rightarrow z = -2.5758 \Rightarrow \overline{X} = -8 - 2.5758 \times \frac{1.2}{\sqrt{20}} = -8.6911...$

$\mathrm{P}(Z > z) = 0.005 \Rightarrow z = 2.5758 \Rightarrow \overline{X} = -8 + 2.5758 \times \frac{1.2}{\sqrt{20}} = -7.3088...$

So the critical region is $\overline{X} < -8.69$ or $\overline{X} > -7.31$ (3 s.f.).

3 $\sigma = 15,\ n = 25,\ \overline{x} = 179$

$\mathrm{H_0} : \mu = 185$ (no improvement), $\mathrm{H_1} : \mu < 185$ (shorter time), one-tailed test at the 5% level.

Assume $\mathrm{H_0}$, so that $X \sim \mathrm{N}(185, 15^2)$ and $\overline{X} \sim \left(15, \dfrac{15^2}{25}\right)$ or $\overline{X} \sim \left(15, 3^2\right)$

Using the cumulative normal function, $\mathrm{P}(\overline{X} < 179) = 0.0227$ (4 d.p.)

$0.0227 < 0.05$ so significant, so reject $\mathrm{H_0}$.

There is evidence that the new formula is an improvement.

4 a The psychologist wishes to test whether the score has increased (not just changed). Therefore $\mathrm{H_0} : \mu = 100,\ \ \mathrm{H_1} : \mu > 100$, one-tailed test at the 2.5% level.

Assume $\mathrm{H_0}$, so that $X \sim \mathrm{N}(100, 15^2)$ and $\overline{X} \sim \left(100, \dfrac{15^2}{80}\right)$ or $\overline{X} \sim (100, 1.6771^2)$

Using the inverse normal function, $\mathrm{P}(\overline{X} > \overline{x}) = 0.025 \Rightarrow \overline{x} = 103.287\ldots$

So the critical region is $\overline{X} > 103.287...$ or 103.29 (3 s.f.).

b $102.5 < 103.29$ so there is not sufficient evidence to reject $\mathrm{H_0}$, i.e. there is not sufficient evidence to say, at the 2.5% level, that eating chocolate before taking an IQ test improves the result.

5 a $\sigma = 0.15,\ n = 30,\ \overline{x} = 8.95$

$\mathrm{H_0} : \mu = 9$ (no change), $\mathrm{H_1} : \mu \neq 9$ (change in mean diameter)

Two-tailed test with 2.5% in each tail

Assume $\mathrm{H_0}$, so that $X \sim \mathrm{N}(9.0, 0.15^2)$ and $\overline{X} \sim \left(9.0, \dfrac{0.15^2}{30}\right)$ or $\overline{X} \sim \left(9.0, 0.0274^2\right)$

Using the cumulative normal function, $\mathrm{P}(\overline{X} < 8.95) = 0.033944... = 0.0340$ (4 d.p.)

$0.0340 > 0.025$ so not significant, so do not reject $\mathrm{H_0}$.

There is not enough evidence to conclude that there has been a change in the mean diameter.

b Two-tailed test so double to probability to find the p-value

p-value = $0.033944\ldots \times 2 = 0.0678$ (4 d.p.)

6 a First find the mean of the distribution.

$\mathrm{P}(D > 5.62) = 0.05$

Using the inverse normal function (or the percentage points table), $p = 0.05 \Rightarrow z = 1.6449$

Using the formula $z = \dfrac{x - \mu}{\sigma}$, $\dfrac{5.62 - \mu}{0.1} = 1.6449 \Rightarrow 5.62 - \mu = 0.16449 \Rightarrow \mu = 5.4555$

The probability that a randomly chosen bolt can be sold is $\mathrm{P}(5.1 \leqslant D \leqslant 5.6)$

Using the cumulative normal function, $\mathrm{P}(5.1 \leqslant D \leqslant 5.6) = 0.92558...$

So the probability that a randomly chosen bolt can be sold is 0.9256 (4 d.p.).

b Use the binomial distribution $N \sim \mathrm{B}(12, 1 - 0.9256)$ or $N \sim \mathrm{B}(12, 1 - 0.0744)$.

Using the cumulative binomial function, $\mathrm{P}(N < 3) = \mathrm{P}(N \leqslant 2) = 0.94549...$

So the probability that fewer than three cannot be sold is 0.9455 (4 d.p.).

6 c Test to determine whether the mean diameter is less than 5.7 mm. Therefore
$H_0 : \mu = 5.7$, $H_1 : \mu < 5.7$, one-tailed test at the 2.5% level.

Assume H_0, so that $Y \sim N(5.7, 0.08^2)$ and $\overline{Y} \sim \left(5.7, \frac{0.08^2}{10}\right)$ or $\overline{Y} \sim (5.7, 0.025298^2)$

Using the cumulative normal function, $P(\overline{Y} < 5.65) = 0.02405... < 0.025$ (one-tailed)
so reject H_0.

There is sufficient evidence to suggest that the mean diameter is less than 5.7 mm.

7 a $P(M > 160) = 0.025$
Using the inverse normal function (or the percentage points table), $p = 0.025 \Rightarrow z = 1.9599$

Using the formula $z = \frac{x - \mu}{\sigma}$, $\frac{160 - \mu}{12} = 1.9599 \Rightarrow 160 - \mu = 23.52 \Rightarrow \mu = 136.48$

So the mean mass of a European water vole is 136.48 g (2 d.p.).

b Using the cumulative normal function, $P(M > 150) = 0.1299$
Use the binomial distribution $N \sim B(8, 0.1299)$.
Using the cumulative binomial function, $P(N \geqslant 4) = 1 - P(N \leqslant 3) = 1 - 0.98708... = 0.01291...$
The probability that at least 4 voles have a mass greater than 150 g is 0.0129 (4 d.p.).

7 c Test to determine whether the mean mass is different from 860 grams. Therefore
$H_0 : \mu = 860$, $H_1 : \mu \neq 860$, two-tailed test with 5% in each tail.

Assume H_0, so that $N \sim N(860, 85^2)$ and $\overline{N} \sim \left(860, \frac{85^2}{15}\right)$ or $\overline{N} \sim (860, 21.946^2)$

Using the cumulative normal function, $P(\overline{N} > 875) = 0.24715...$
$0.24715 > 0.05$ so not significant, do not reject H_0.
There is insufficient evidence to suggest that the mean mass of all water rats is different from 860 g.

8 Test to determine whether the daily mean windspeed is greater than 9.5 knots. Therefore
$H_0 : \mu = 9.5$, $H_1 : \mu > 9.5$, one-tailed test at the 2.5% level.

Assume H_0, so that $X \sim N(9.5, 3.1^2)$ and $\overline{X} \sim \left(9.5, \frac{3.1^2}{25}\right)$ or $\overline{X} \sim (9.5, 0.62^2)$

Using the inverse normal function, $P(\overline{X} > \overline{x}) = 0.025 \Rightarrow \overline{x} = 10.715...$

So the critical region is $\overline{X} \geqslant 10.715...$
$12.2 > 10.715$ so there is sufficient evidence to reject H_0, i.e. there is sufficient evidence to say, at the 2.5% level, that the daily mean windspeed is greater than 9.5 knots.

The normal distribution Mixed exercise 3

1 $H \sim \mathrm{N}(178, 4^2)$

a Using the normal CD function, $\mathrm{P}(H > 185) = 0.04059... = 0.0401$ (4 d.p.)

b Using the normal CD function, $\mathrm{P}(H < 180) = 0.69146...$
The probability that three men, selected at random, all satisfy this criterion is
$\mathrm{P}(H < 180)^3 = 0.33060... = 0.3306$ (4 d.p.) .

c Using the inverse normal function, $\mathrm{P}(H > h) = 0.005 \Rightarrow h = 188.03...$
To the nearest centimetre, the height of a door frame needs to be at least 188 cm.

2 $W \sim \mathrm{N}(32.5,\ 2.2^2)$

a Using the normal CD function, $\mathrm{P}(W < 30) = 0.12790...$
The percentage of sheets weighing less than 30kg is 12.8% (3 s.f.).

b Using the normal CD function, $\mathrm{P}(31.6 < W < 34.8) = 0.51085...$
So 51.1% of sheets satisfy Bob's requirements.

3 $T \sim \mathrm{N}(48,\ 8^2)$

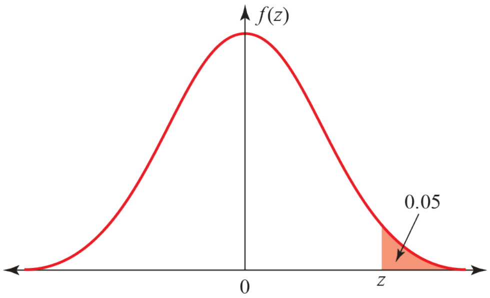
a Using the normal CD function, $\mathrm{P}(T > 60) = 0.06680...$
The probability that a battery will last for more than 60 hours is 0.0668 (4 d.p.).

b Using the normal CD function, $\mathrm{P}(T < 35) = 0.05208...$
The probability that a battery will last for less than 35 hours is 0.0521 (4 d.p.).

c Use the binomial distribution $X \sim \mathrm{B}(30, 0.05208\ldots)$
Using the binomial CD function, $\mathrm{P}(X \leqslant 3) = 0.93145...$
The probability that three or fewer last less than 35 hours is 0.9315 (4 d.p.).

4 $X \sim N(24, \sigma^2)$

a $P(X > 30) = 0.05 \Rightarrow P\left(Z > \frac{30-\mu}{\sigma}\right) = 0.05$



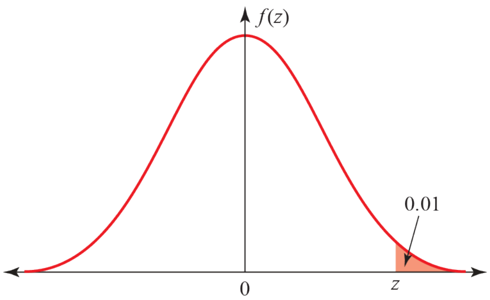
Using the inverse normal function, $z = -1.64485...$

so $1.64485... = \frac{30-24}{\sigma}$

$$\sigma = \frac{6}{1.64485...} = 3.647... = 3.65 \text{ (3 s.f.)}$$

b Using the normal CD function, $P(X < 20) = 0.13636... = 0.136$ (3 d.p.)

c $P(X > d) = 0.01 \Rightarrow P\left(Z > \frac{d-\mu}{\sigma}\right) = 0.01$



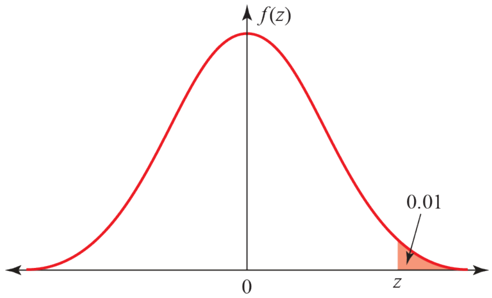
Using the inverse normal function, $z = 2.32634...$

so $2.32634... = \frac{d-24}{3.647...}$

$$d = 32.485... = 32.5 \text{ (3 s.f.)}$$

5 $L \sim \mathrm{N}(120,\ \sigma^2)$

a $\mathrm{P}(L > 140) = 0.01 \Rightarrow \mathrm{P}\left(Z > \dfrac{140-\mu}{\sigma}\right) = 0.01$



Using the inverse normal function, $z = 2.32634...$

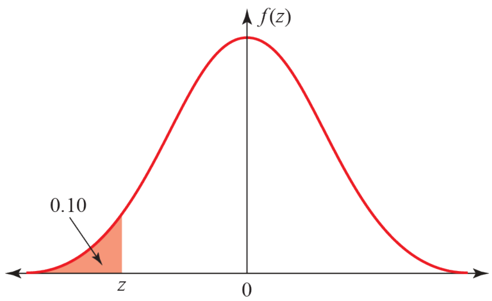
$$\text{so } 2.32634... = \frac{140-120}{\sigma}$$

$$\sigma = \frac{20}{2.32634...} = 8.59716...$$

So the standard deviation of the volume dispensed is 8.60 ml (3 s.f.).

b Using the normal CD function, $\mathrm{P}(L < 110) = 0.12237...$
The probability that the machine dispenses less than 110ml is 0.122 (3 s.f.).

c $\mathrm{P}(L < c) = 0.10 \Rightarrow \mathrm{P}\left(Z < \dfrac{c-\mu}{\sigma}\right) = 0.10$



Using the inverse normal function, $z = -1.28155...$

$$\text{so } -1.2816 = \frac{c-120}{8.59716...}$$

$$c = 108.982...$$

To the nearest millilitre, the largest volume leading to a refund is 109 ml.

6 a $P(X < 20) = 0.25$ and $P(X < 40) = 0.75$

Using the inverse normal function (or the percentage points table),

$$P(X < 20) = 0.25 \Rightarrow P\left(Z < \frac{20-\mu}{\sigma}\right) = 0.25 \Rightarrow z_1 = -0.67448...$$

$$P(X < 40) = 0.75 \Rightarrow P\left(Z < \frac{40-\mu}{\sigma}\right) = 0.75 \Rightarrow z_2 = 0.67448...$$

So $-0.6745\sigma = 20 - \mu$ **(1)**

and $0.6745\sigma = 40 - \mu$ **(2)**

(2) − **(1)**: $1.3489\sigma = 20$

$\sigma = 14.826...$

Substituting into **(2)**:

$\mu = 40 - 0.6745 \times 14.826... = 29.99...$

So $\mu = 30$ and $\sigma = 14.8$ (3 s.f.)

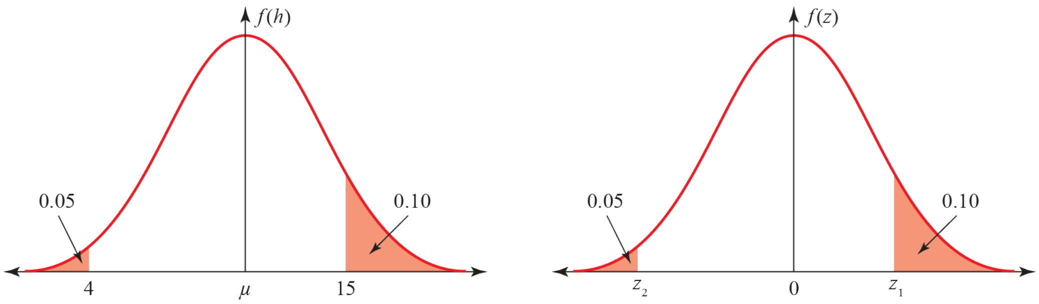
b Using the inverse normal CD function with $\mu = 30$ and $\sigma = 14.826...$,

$P(X < a) = 0.1 \Rightarrow a = 10.999...$ and $P(X < b) = 0.9 \Rightarrow b = 49.000...$

So the 10% to 90% interpercentile range is $49.0 - 11.0 = 38.0$

7 $$P(H > 15) = 0.10 \Rightarrow P\left(Z > \frac{15-\mu}{\sigma}\right) = 0.10 \Rightarrow z_1 = 1.28155...$$

$$P(H < 4) = 0.05 \Rightarrow P\left(Z > \frac{4-\mu}{\sigma}\right) = 0.05 \Rightarrow z_2 = -1.64485...$$



So $-1.6449\sigma = 4 - \mu$

$1.2816\sigma = 15 - \mu$

Subtract $2.9265\sigma = 11$

$\sigma = 3.7587... = 3.76\,\text{cm}$ (3 s.f.)

$\mu = 15 - 1.2816\sigma = 10.2\,\text{cm}$

8 a $T \sim N(80, 10^2)$

Using the normal CD function, $P(T > 85) = 0.30853... = 0.3085$ (4 d.p.)

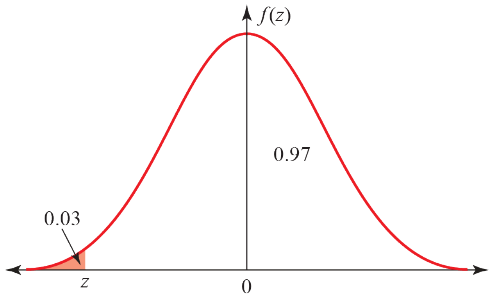
b $S \sim N(100, 15^2)$

Using the normal CD function, $P(S > 105) = 0.36944... = 0.3694$ (4 d.p.)

c The student's score on the first test was better, since fewer of the students got this score or higher.

9 $J \sim \mathrm{N}(108,\ \sigma^2)$

a $\mathrm{P}(J < 100) = 0.03 \Rightarrow \mathrm{P}\left(Z < \frac{100-\mu}{\sigma}\right) = 0.03$



Using the inverse normal function, $z = -1.88079...$

so $-1.88079... = \frac{100-108}{\sigma}$

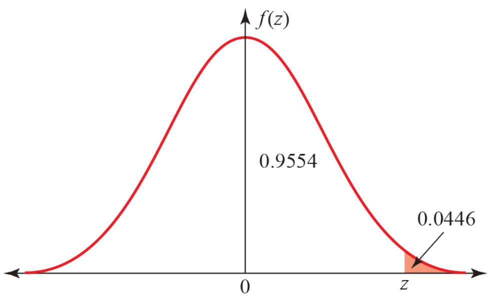
$$\sigma = 4.2535... = 4.25\text{ g } (3\text{ s.f.}).$$

The standard deviation is 4.25 g (3 s.f.).

b Using the normal CD function, $\mathrm{P}(J > 115) = 0.0499... = 0.050$ (3 d.p.)

c Use the binomial distribution $X \sim \mathrm{B}(25, 0.05)$
Using the binomial CD function,
$\mathrm{P}(X \leqslant 2) = 0.87289... = 0.8729$ (4 d.p.)

10 $T \sim \mathrm{N}(\mu, 3.8^2)$ and $\mathrm{P}(T > 15) = 0.0446$



a $\mathrm{P}(T > 15) = 0.0446 \Rightarrow \mathrm{P}\left(Z > \frac{X-\mu}{\sigma}\right) = 0.0446 \Rightarrow z = 1.70$

so $1.70 = \frac{15-\mu}{3.8}$

$$\mu = 15 - 3.8 \times 1.70$$
$$= 8.54 \text{ minutes } (3\text{ s.f.})$$

b $\mathrm{P}(T < 5) = \mathrm{P}\left(Z < \frac{5-8.54}{3.8}\right)$

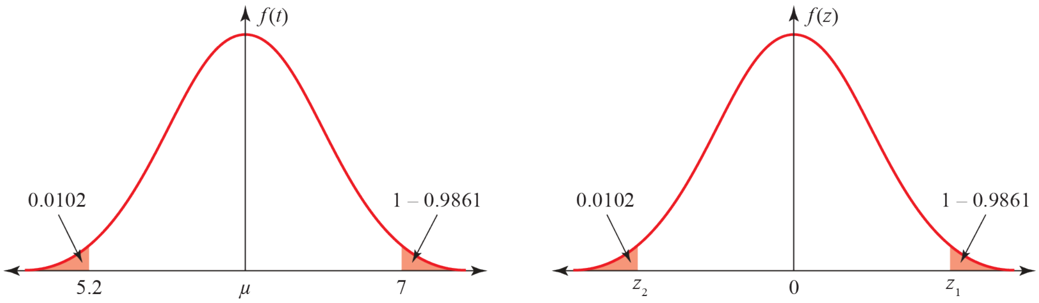
$$= \mathrm{P}(Z < -0.93...)$$
$$= 0.17577... = 0.1758 \text{ (4 d.p.)}$$

11 $T \sim \mathrm{N}(\mu, \sigma^2)$

Using the inverse normal function,

$\mathrm{P}(T < 7) = 0.9861 \Rightarrow \mathrm{P}\left(Z < \dfrac{7-\mu}{\sigma}\right) = 0.9861 \Rightarrow z_1 = 2.20009...$

$\mathrm{P}(T < 5.2) = 0.0102 \Rightarrow \mathrm{P}\left(Z < \dfrac{5.2-\mu}{\sigma}\right) = 0.0102 \Rightarrow z_2 = -2.31890...$



So $\quad 2.2001\sigma = 7 - \mu \quad$ **(1)**

and $\quad -2.3189\sigma = 5.2 - \mu \quad$ **(2)**

(1) − **(2)**: $\quad 4.5190\sigma = 1.8$

$\sigma = 0.3983...$

Substituting into **(1)**:

$\mu = 7 - 2.2001 \times 0.3983... = 6.123...$

So the mean thickness of the shelving is 6.12 mm and the standard deviation is 0.398 mm (3 s.f.).

12 Let X = number of heads in 60 tosses of a fair coin, so $X \sim \mathrm{B}(60, 0.5)$.

Since $p = 0.5$ and 60 is large, X can be approximated by the normal distribution $Y \sim \mathrm{N}(\mu,\sigma^2)$,

where $\mu = 60 \times 0.5 = 30$ and $\sigma = \sqrt{60 \times 0.5 \times 0.5} = \sqrt{15}$

So $Y \sim \mathrm{N}(30,15)$

$\mathrm{P}(X < 25) \approx \mathrm{P}(Y < 24.5) = 0.07779... = 0.0778$ (3 s.f.)

13 a The distribution is binomial, $\mathrm{B}(100, 0.40)$.

The binomial distribution can be approximated by the normal distribution when n is large (> 50) and p is close to 0.5. Here $n = 100$ and $p = 0.4$ so both of these conditions are satisfied.

b $\mu = np = 100 \times 0.4 = 40$ and $\sigma = \sqrt{np(1-p)} = \sqrt{40 \times 0.6} = \sqrt{24} = 4.899$ (4 s.f.)

c $\mathrm{P}(X \geqslant 50) \approx \mathrm{P}(Y \geqslant 49.5) = 0.02623... = 0.0262$ (3 s.f.)

14 a $\mathrm{P}(X = 65) = \dbinom{120}{65} \times 0.46^{65} \times 0.54^{55} = 0.01467$ (4 s.f.) or 0.0147 (4 d.p.)

b The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 120$ is large (> 50) and $p = 0.46$ is close to 0.5.

$\mu = np = 120 \times 0.46 = 55.2$ and $\sigma = \sqrt{np(1-p)} = \sqrt{55.2 \times 0.54} = \sqrt{29.808} = 5.460$ (4 s.f.)

14 c $Y \sim N(55.2, 5.460^2)$

Using the normal CD function, $P(X = 65) \approx P(64.5 < Y < 65.5) = 0.01463...$

Percentage error $= \dfrac{0.01467 - 0.01463}{0.01467} \times 100 = \dfrac{0.00004}{0.01467} \times 100 = 0.27\%$

15 a The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 300$ is large (> 50) and $p = 0.6$ is close to 0.5.

b $\mu = np = 300 \times 0.6 = 180$ and $\sigma = \sqrt{np(1-p)} = \sqrt{180 \times 0.4} = \sqrt{72} = 8.485$ (4 s.f.)

So $Y \sim N(180, 8.485^2)$

$P(150 < Y \leqslant 180) \approx P(150.5 < N < 180.5) = 0.52324... = 0.5232$ (4 s.f.)

c Using the inverse normal distribution, $P(N < a) = 0.05 \Rightarrow a = 166.04$

So $P(N < 166.5) > 0.05$ and $P(N < 165.5) < 0.05$

So $165.5 < y < 166.5$, i.e. the smallest value of y such that $P(Y < y) < 0.05$ is $y = 166$.

16 The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 80$ is large (> 50) and $p = 0.4$ is close to 0.5.

$\mu = np = 80 \times 0.4 = 32$ and $\sigma = \sqrt{np(1-p)} = \sqrt{32 \times 0.6} = \sqrt{19.2} = 4.382$ (4 s.f.)

So $Y \sim N(32, 4.382^2)$

$P(X > 30) \approx P(Y > 30.5) = 0.63394... = 0.6339$ (4 s.f.)

17 a Use the binomial distribution $X \sim B(20, 0.55)$

Using the binomial CD function, $P(X > 10) = 1 - P(X \leqslant 10) = 1 - 0.40863... = 0.5914$ (4 s.f.)

b A normal approximation is valid since $n = 200$ is large (> 50) and $p = 0.55$ is close to 0.5.

$\mu = np = 200 \times 0.55 = 110$ and $\sigma = \sqrt{np(1-p)} = \sqrt{110 \times 0.45} = \sqrt{49.5} = 7.036$ (4 s.f.)

So $Y \sim N(110, 7.036^2)$

$P(X \leqslant 95) \approx P(Y < 95.5) = 0.01965... = 0.0197$ (3 s.f.)

c It seems unlikely that the company's claim is correct: if it were true, the chance of only 95 (or fewer) seedlings producing apples from a sample of 200 seedlings would be less than 2%.

18 a Use the binomial distribution $X \sim B(25, 0.52)$

Using the binomial CD function, $P(X > 12) = 1 - P(X \leqslant 12) = 1 - 0.41992... = 0.5801$ (4 s.f.)

b A normal approximation is valid since $n = 300$ is large (> 50) and $p = 0.52$ is close to 0.5.

$\mu = np = 300 \times 0.52 = 156$ and $\sigma = \sqrt{np(1-p)} = \sqrt{156 \times 0.48} = \sqrt{74.88} = 8.653$ (4 s.f.)

So $Y \sim N(156, 8.653^2)$

$P(X \geqslant 170) \approx P(Y > 169.5) = 0.05936... = 0.0594$ (4 s.f.)

c There is a greater than 5% chance that 170 people out of 300 would be cured, therefore there is insufficient evidence for the herbalist's claim that the new remedy is more effective than the original remedy.

19 $X \sim \mathrm{N}(\mu, 2^2)$

$\mathrm{H}_0 : \mu = 7$, $\mathrm{H}_1 : \mu > 7$, one-tailed test at the 5% level.

Assume H_0, so that $X \sim \mathrm{N}(7, 2^2)$ and $\overline{X} \sim \left(7, \frac{2^2}{25}\right)$

Let $Z = \frac{\overline{X} - 7}{\frac{2}{\sqrt{25}}}$

Using the inverse normal function, $\mathrm{P}(Z > z) = 0.05 \Rightarrow z = 1.6449$

$1.6449 = \frac{\overline{X} - 7}{\frac{2}{5}} \Rightarrow \overline{X} = 7 + 1.6449 \times \frac{2}{5} = 7.6579...$

So the critical region is $\overline{X} > 7.6579...$ or 7.66 cm (3 s.f.).

20 Let B represent the amount of water in a bottle, so $B \sim \mathrm{N}(\mu, 2^2)$.

$\mathrm{H}_0 : \mu = 125$, $\mathrm{H}_1 : \mu < 125$, one-tailed test at the 5% level.

Assume H_0, so that $B \sim \mathrm{N}(125, 2^2)$ and $\overline{B} \sim \left(125, \frac{2^2}{15}\right)$

Using the normal CD function, $\mathrm{P}(\overline{B} < 124.2) = 0.06066... = 0.0607$ (3 s.f.)

$0.0607 > 0.05$ so not significant, so accept H_0.

There is insufficient evidence to conclude that the mean content of a bottle is lower than the manufacturer's claim.

21 Let B represent the breaking strength, so $B \sim \mathrm{N}(170.2, 10.5^2)$.

a Using the normal CD function, $\mathrm{P}(174.5 < B < 175.5) = 0.03421... = 0.0342$ (3 s.f.)

b $n = 50$ so $\overline{B} \sim \mathrm{N}\left(170.2, \frac{10.5^2}{50}\right)$

Using the normal CD function, $\mathrm{P}(\overline{B} > 172.4) = 0.06922... = 0.0692$ (3 s.f.)

c $\mathrm{H}_0 : \mu = 170.2$, $\mathrm{H}_1 : \mu > 170.2$ one-tailed test at the 5% level.

Assume H_0, so that $B \sim \mathrm{N}(170.2, 10.5^2)$ and $\overline{B} \sim \left(170.2, \frac{10.5^2}{50}\right)$ (as before).

Using the normal CD function, $\mathrm{P}(\overline{B} > 172.4) = 0.06922... = 0.0692$ (3 s.f.)

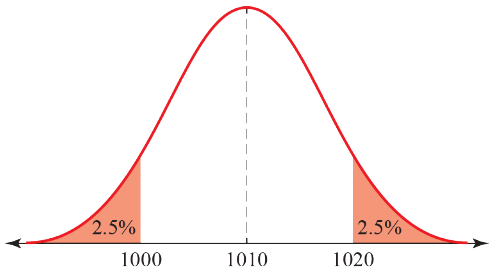
This is the p-value for the hypothesis test.

$0.0692 > 0.05$ so not significant, so accept H_0.

There is insufficient evidence to conclude that the mean breaking strength is increased.

22 Let W represent the weight of sugar in a packet, so $W \sim \mathrm{N}(1010, \sigma^2)$.

a $\mathrm{P}\left(1000 < W < 1020\right) = 0.95 \Rightarrow \mathrm{P}(W < 1000) = 0.025 \Rightarrow \mathrm{P}\left(Z < \dfrac{1000-1010}{\sigma}\right) = 0.025$



Using the inverse normal function, $z = -1.95996...$

so $\quad -1.95996... = \dfrac{1000-1010}{\sigma}$

$$\sigma = \frac{-10}{-1.95996..} = 5.1021...$$

$$\sigma^2 = 26.031... = 26.03 \text{ (2 d.p.)}$$

b $n = 8$ and $\sum x = 8109.1$, so $\bar{x} = 1013.6375$

$\mathrm{H}_0 : \mu = 1010$, $\mathrm{H}_1 : \mu \neq 1010$, two-tailed test with 1% in each tail.

Assume H_0, so that $W \sim \mathrm{N}(1010, 26.03)$ and $\overline{W} \sim \left(1010, \dfrac{26.03}{8}\right)$

Using the normal CD function, $\mathrm{P}(\overline{W} > 1013.6375) = 0.02187... = 0.0219$ (3 s.f.)

$0.0219 > 0.01$ so not significant, so accept H_0.

There is insufficient evidence of a deviation in the mean from 1010, so we can assume that condition **i** is being met.

23 Let D represent the diameter of a little-gull egg, so $D \sim \mathrm{N}(4.11, 0.19^2)$.

a Using the normal CD function, $\mathrm{P}(3.9 < D < 4.5) = 0.84542... = 0.8454$ (4 s.f.)

b $\sigma = 0.19$, $n = 8$, $\sum d = 34.5$, $\bar{d} = 4.3125$

$\mathrm{H}_0 : \mu = 4.11$, $\mathrm{H}_1 : \mu \neq 4.11$, two-tailed test with 0.5% in each tail.

Assume H_0, so that $D \sim \mathrm{N}(4.11, 0.19^2)$ and $\overline{D} \sim \left(4.11, \dfrac{0.19^2}{8}\right)$

Using the normal CD function, $\mathrm{P}(\overline{D} > 4.3125) = 0.00128... = 0.0013$ (2 s.f.)

p-value $= 2 \times 0.00128\ldots = 0.00258 < 0.01$ so significant, so reject H_0.

There is evidence that the mean diameter of eggs from this island is different from elsewhere.

24 **a** $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$$\overline{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$$

24 b $\mathrm{P}\left(\left|\overline{X}-\mu\right|<15\right)=\mathrm{P}\left(|Z|<\dfrac{15}{\left(\frac{\sigma}{\sqrt{40}}\right)}\right)$

Require $\mathrm{P}\left(|Z|<\dfrac{15\sqrt{n}}{40}\right)>0.95 \Rightarrow \mathrm{P}\left(Z<\dfrac{15\sqrt{n}}{40}\right)<0.025$ (by symmetry)

Using the inverse normal function, $z=-1.95996...$

so $\dfrac{15\sqrt{n}}{40}>1.95996...$

$$\sqrt{n}>\frac{40\times 1.95996...}{15}=5.2266...$$

$$n>27.317...$$

So a sample of at least 28 is needed.

Challenge

a Use the binomial distribution $X \sim \mathrm{B}(15, 0.48)$.
Using the binomial CD function, $\mathrm{P}(X>8)=1-\mathrm{P}(X\leqslant 8)=1-0.74903...=0.2510$ (4 s.f.)

b The distribution satisfies the two necessary conditions for an approximation by a normal distribution to be valid: $n = 250$ is large (> 50) and $p = 0.48$ is close to 0.5.
$\mu=np=250\times 0.48=120$ and $\sigma=\sqrt{np(1-p)}=\sqrt{120\times 0.52}=\sqrt{62.4}=7.90$ (3 s.f.)
Test to determine whether the mean is different from 120 (the expected number of supporters based on the manager's claim):
$\mathrm{H}_0: \mu=120$, $\mathrm{H}_1: \mu\neq 120$, two-tailed test with 2.5% in each tail.
Assume H_0, so that $X\sim \mathrm{N}(120, 7.9^2)$ and $\overline{X}\sim\left(120, \dfrac{7.9^2}{250}\right)$
Using the inverse normal function,
$\mathrm{P}(\overline{X}<\overline{x})=0.025 \Rightarrow \overline{x}=104.516$ and $\mathrm{P}(\overline{X}>\overline{x})=0.025\Rightarrow \overline{x}=135.484$
So the critical region approximated for the binomial distribution is $\overline{X}\leqslant 105$ or $\overline{X}\geqslant 135$.
(Note: 105 lies in the critical region because, as part of the normal distribution, it includes the region between 104.5 and 105.5 and 104.513 is where the critical region starts. Similarly 135 lies in the critical region because, as part of the normal distribution, it includes the region between 134.5 and 135.5 and 135.487 is where the critical region starts.)

c Since $102<105$, there is sufficient evidence to reject H_0, i.e. there is sufficient evidence to say, at the 5% level, that the level of support for the manager is different from 48%.

Review exercise 1

1 a Produce a table for the values of log s and log t:

log s	0.3010	0.6532	0.7924	0.8633	0.9590
log t	−0.4815	0.0607	0.2455	0.3324	0.4698

which produces $r = 0.9992$

b Since r is very close to 1, this indicates that log s by log t is very close to being linear, which means that s and t are related by an equation of the form $t = as^n$ (beginning of Section 1.1).

c Rearranging the equation:

$\log t = -0.9051 + \log s^{1.4437}$

$\Rightarrow t = 10^{-0.9051+\log s^{1.4437}} = 10^{-0.9051} \times 10^{\log s^{1.4437}}$

$\Rightarrow t = 10^{-0.9051} \times s^{1.4437}$

and so $a = 10^{-0.9051} = 0.1244$ (4 s.f.) and $n = 1.4437$

2 a Rearranging the equation:

$y = -0.2139 + 0.0172x$

$\Rightarrow \log t = -0.2139 + 0.0172P$

$\Rightarrow t = 10^{-0.2139+0.0172P} = 10^{-0.2139} \times 10^{0.0172P}$

$\Rightarrow t = 10^{-0.2139} \times \left(10^{0.0172}\right)^P$

Therefore $a = 10^{-0.2139} = 0.611$ (3 s.f.) and $b = 10^{0.0172} = 1.04$ (3 s.f.).

b Not in the range of data (extrapolation).

3 a $r = \dfrac{59.524}{\sqrt{152.444 \times 26.589}}$

$= 0.93494$ (the formulae for this is under S1 in the formula book).

b Make sure your hypotheses are clearly written using the parameter ρ:

$H_0 : \rho = 0, \quad H_1 : \rho > 0$

Test statistic: r = 0.935

Critical value at 1% = 0.7155

(Look up the value under 0.01 in the table for product moment coefficient; quote the figure in full.)

$0.935 > 0.7155$

Draw a conclusion in the context of the question:

So reject H_0 : levels of serum and disease are positively correlated.

4 $r = -0.4063$, critical value for $n = 6$ is -0.6084, so no evidence.

5 a $H_0 : \rho = 0$

$H_1 : \rho < 0$

From the data, $r = -0.9313$. Since the critical value for $n = 5$ is -0.8783, there is sufficient evidence to reject H_0, i.e. at the 2.5% level of significance, there is sufficient evidence to say that there is negative correlation between the number of miles done by a one-year-old car and its value.

b If a 1% level of significance was used, then the critical value for $n = 5$ is -0.9343 and so there would not be sufficient evidence to reject H_0.

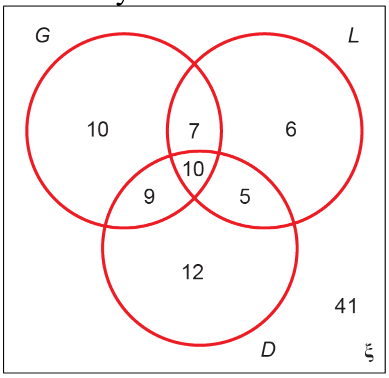
6 a
$$\begin{aligned}\mathrm{P}(\text{tourism}) &= \frac{50}{148}\\ &= \frac{25}{74}\\ &= 0.338 \text{ (3 s.f.)}\end{aligned}$$

b The words 'given that' in the question tell you to use conditional probability:
$$\begin{aligned}\mathrm{P}(\text{no glasses} \mid \text{tourism}) &= \frac{\mathrm{P}(G' \cap T)}{\mathrm{P}(T)}\\ &= \frac{\frac{23}{148}}{\frac{50}{148}}\\ &= \frac{23}{50}\\ &= 0.46\end{aligned}$$

c It often helps to write down which combinations you want:
$$\begin{aligned}\mathrm{P}(\text{right-handed}) &= \mathrm{P}(E \cap RH) + \mathrm{P}(T \cap RH) + \mathrm{P}(C \cap RH)\\ &= \frac{30}{148} \times 0.8 + \frac{50}{148} \times 0.7 + \frac{68}{148} \times 0.75\\ &= \frac{55}{74}\\ &= 0.743 \text{ (3 s.f.)}\end{aligned}$$

d The words 'given that' in the question tell you to use conditional probability:
$$\begin{aligned}\mathrm{P}(\text{engineering} \mid \text{right-handed}) &= \frac{\mathrm{P}(E \cap RH)}{\mathrm{P}(RH)}\\ &= \frac{\frac{30}{148} \times 0.8}{\frac{55}{74}}\\ &= \frac{12}{55}\\ &= 0.218 \text{ (3 s.f.)}\end{aligned}$$

7 a Start in the middle of the Venn diagram and work outwards. Remember the rectangle and those not in any of the circles. Your numbers should total 100.



b $\mathrm{P}(G, L', D') = \dfrac{10}{100}$

$= \dfrac{1}{10} = 0.1$

c $\mathrm{P}(G', L', D') = \dfrac{41}{100} = 0.41$

d $\mathrm{P}(\text{only two attributes}) = \dfrac{9+7+5}{100}$

$= \dfrac{21}{100} = 0.21$

e The word 'given' in the question tells you to use conditional probability:

$\mathrm{P}(G \mid L \cap D) = \dfrac{\mathrm{P}(G \mid L \cap D)}{\mathrm{P}(\mathrm{L} \mid D)}$

$= \dfrac{\frac{10}{100}}{\frac{15}{100}}$

$= \dfrac{10}{15}$

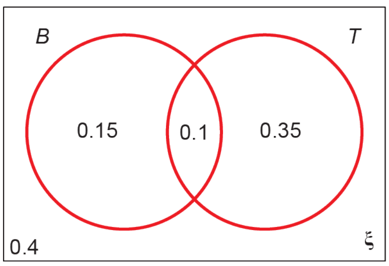
$= \dfrac{2}{3} = 0.667 \text{ (3 s.f.)}$

8 a $\mathrm{P}(B \cup T) = \mathrm{P}(B) + \mathrm{P}(T) - \mathrm{P}(B \cap T)$

$0.6 = 0.25 + 0.45 - \mathrm{P}(B \cap T)$

$\mathrm{P}(B \cap T) = 0.1$

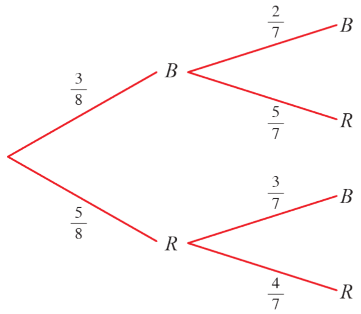
b When drawing the Venn diagram remember to draw a rectangle around the circles and add the probability 0.4.
Remember the total in circle $B = 0.25$ and the total in circle $T = 0.45$.



8 c The words 'given that' in the question tell you to use conditional probability:

$$\begin{aligned}P(B\cap T' \mid B\cup T) &= \frac{0.15}{0.6}\\ &= \frac{1}{4}\\ &= 0.25\end{aligned}$$

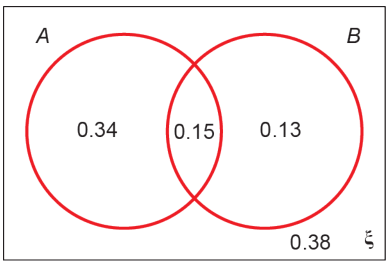
9 a



b i There are two different situations where the second counter drawn is blue. These are BB and RB. Therefore the probability is: $\left(\frac{3}{8}\times\frac{2}{7}\right)+\left(\frac{5}{8}\times\frac{3}{7}\right)=\frac{6+15}{56}=\frac{21}{56}=\frac{3}{8}=0.375$.

ii $\text{P(both blue} \mid \text{2nd blue)} = \dfrac{\text{P(both blue and 2nd blue)}}{\text{P(2nd blue)}} = \dfrac{\text{P(both blue)}}{\text{P(2nd blue)}} = \dfrac{\left(\frac{3}{8}\times\frac{2}{7}\right)}{\left(\frac{3}{8}\right)} = \dfrac{2}{7}$

10 a The first two probabilities allow two spaces in the Venn diagram to be filled in.
$P(A\cup B) = P(A\cap B') + P(A'\cap B) + P(A\cap B)$, and this can be rearranged to see that $P(A\cap B) = 0.15$
Finally, $P(A\cup B) = 0.62 \Rightarrow P((A\cup B)') = 0.38$. The completed Venn diagram is therefore:



b $P(A) = 0.34 + 0.15 = 0.49$ and $P(B) = 0.13 + 0.15 = 0.28$

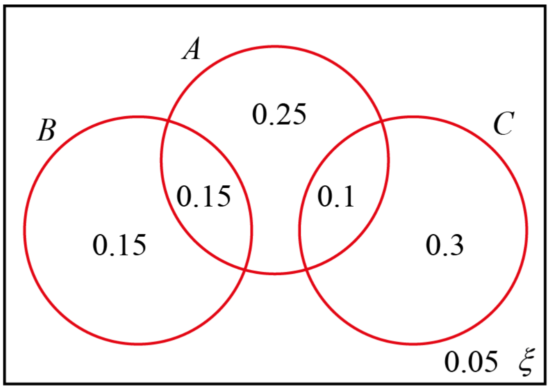
c $P(A \mid B') = \dfrac{P(A\cap B')}{P(B')} = \dfrac{0.34}{1-P(B)} = \dfrac{0.34}{0.72} = 0.472$ (3 d.p.).

d If A and B are independent, then $P(A) = P(A \mid B) = P(A \mid B')$. From parts **b** and **c**, this is not the case. Therefore they are not independent.

11 a $P(A\cap B) = P(A)\times P(B) \Rightarrow P(A) = P(A\cap B)\div P(B) = 0.15 \div 0.3 = 0.5$

11 b $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.5 + 0.3 - 0.15 = 0.65 \Rightarrow P(A' \cap B') = 1 - 0.65 = 0.35$

c Since B and C are mutually exclusive, they do not intersect.
The intersection of A and C should be 0.1 but $P(A) = 0.5$, allowing $P(A \cap B' \cap C')$ to be calculated. The filled-in probabilities sum to 0.95, and so $P(A' \cap B' \cap C') = 0.05$.
Therefore, the filled-in Venn diagram should look like:



d i $P(A \mid C) = \dfrac{P(A \cap C)}{P(C)} = \dfrac{0.1}{0.4} = 0.25$

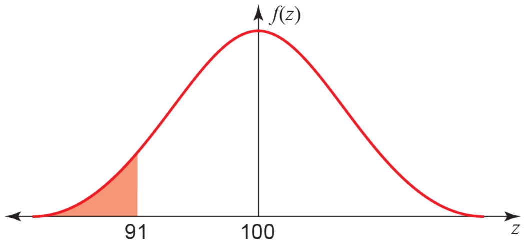
ii The set $A \cap (B \cup C')$ must be contained within A. First find the set $B \cup C'$: this is made up from four distinct regions on the above Venn diagram, with labels 0.15, 0.15, 0.25 and 0.05.
Restricting to those regions that are also contained within A leaves those labelled 0.15 and 0.25.
Therefore, $P(A \cap (B \cup C')) = 0.15 + 0.25 = 0.4$

iii From part **ii**, $P(B \cup C') = 0.15 + 0.15 + 0.25 + 0.05 = 0.6$. Therefore
$$P(A \mid (B \cup C')) = \frac{P(A \cap (B \cup C'))}{P(B \cup C')} = \frac{0.4}{0.6} = \frac{2}{3}$$

12 a There are two different events going on: ‘Joanna oversleeps’ (O) and ‘Joanna is late for college’ (L). From the context, we cannot assume that these are independent events.
Drawing a Venn diagram, none of the regions can immediately be filled in. We are told that $P(O) = 0.15$ and so $P(J \text{ does not oversleep}) = P(O') = 0.85$. The other two statements can be interpreted as $\dfrac{P(L \cap O)}{P(O)} = 0.75$ and $\dfrac{P(L \cap O')}{P(O')} = 0.1$
Filling in the first one:
$$\frac{P(L \cap O)}{P(O)} = 0.75 \Rightarrow \frac{P(L \cap O)}{0.15} = 0.75 \Rightarrow P(L \cap O) = 0.1125$$
Also, $\dfrac{P(L \cap O')}{0.85} = 0.1 \Rightarrow P(L \cap O') = 0.085$
Therefore, $P(L) = P(L \cap O) + P(L \cap O') = 0.1125 + 0.085 = 0.1975$

b $P(L \mid O) = \dfrac{P(L \cap O)}{P(O)} = \dfrac{0.1125}{0.1975} = \dfrac{45}{79} = 0.5696$ (4 s.f.).

13 a Drawing a diagram will help you to work out the correct area:

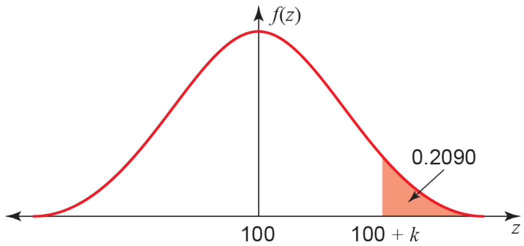


Using $z = \frac{x - \mu}{\sigma}$. As 91 is to the left of 100, your z value should be negative.

$$\begin{aligned} P(X < 91) &= P\left(Z < \frac{91 - 100}{15}\right) \\ &= P(Z < -0.6) \\ &= 1 - 0.7257 \\ &= 0.2743 \end{aligned}$$

(The tables give $P(Z < 0.6) = P(Z > -0.6)$, so you want $1 -$ this probability.)

b



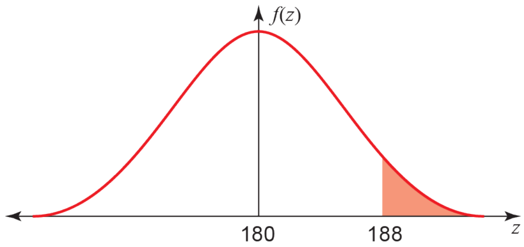
As 0.2090 is not in the table of percentage points you must work out the larger area:
$1 - 0.2090 = 0.7910$
Use the first table or calculator to find the z value. It is positive as $100 + k$ is to the right of 100.
$P(X > 100 + k) = 0.2090$ or $P(X < 100 + k) = 0.791$

$$\frac{100 + k - 100}{15} = 0.81$$

$$k = 12$$

14 a Let H be the random variable ~ height of athletes, so $H \sim \mathrm{N}\left(180,\ 5.2^2\right)$

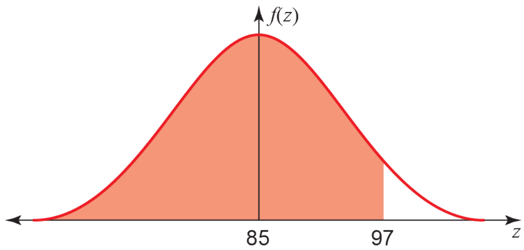
Drawing a diagram will help you to work out the correct area:



Using $z = \dfrac{x-\mu}{\sigma}$. As 188 is to the right of 180 your z value should be positive. The tables give $\mathrm{P}(Z < 1.54)$ so you want $1-$ this probability:

$$\begin{aligned}\mathrm{P}(H > 188) &= \mathrm{P}\left(Z > \frac{188-100}{5.2}\right)\\ &= \mathrm{P}(Z > 1.54)\\ &= 1 - 0.9382\\ &= 0.0618\end{aligned}$$

b Let W be the random variable ~ weight of athletes, so $W \sim \mathrm{N}\left(85,\ 7.1^2\right)$



Using $z = \dfrac{x-\mu}{\sigma}$. As 97 is to the right of 85, your z value should be positive.

$$\begin{aligned}\mathrm{P}(W < 97) &= \mathrm{P}\left(Z < \frac{97-85}{7.1}\right)\\ &= \mathrm{P}(Z < 1.69)\\ &= 0.9545\end{aligned}$$

c $\mathrm{P}(W > 97) = 1 - \mathrm{P}(W < 97)$, so

$$\begin{aligned}\mathrm{P}(H > 188\ \&\ W > 97) &= 0.618(1 - 0.9545)\\ &= 0.00281\end{aligned}$$

d Use the context of the question when you are commenting:
The evidence suggests that height and weight are positively correlated/linked, so assumption of independence is not sensible.

15 a Use the table of percentage points or calculator to find z. You must use at least the four decimal places given in the table.

$$\mathrm{P}(Z > a) = 0.2$$
$$a = 0.8416$$
$$\mathrm{P}(Z < b) = 0.3$$
$$b = -0.5244$$

0.5244 is negative since 1.65 is to the left of the centre. 0.8416 is positive as 1.78 is to the right of the centre.

Using $z = \dfrac{x-\mu}{\sigma}$:

$$\frac{1.78-\mu}{\sigma} = 0.8416 \Rightarrow 1.78-\mu = 0.8416\sigma \quad (\mathbf{1})$$
$$\frac{1.65-\mu}{\sigma} = -0.5244 \Rightarrow 1.65-\mu = 0.5244\sigma \quad (\mathbf{2})$$

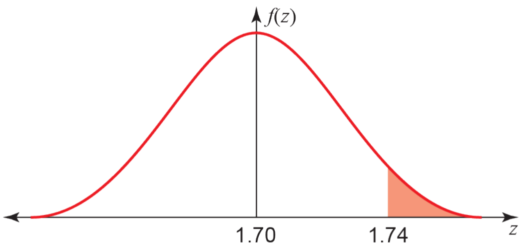
Solving simultaneously, $(\mathbf{1})-(\mathbf{2})$:

$$0.13 = 1.366\sigma$$
$$\sigma = 0.095 \text{ m}$$

Substitute in (**1**): $1.78-\mu = 0.8416 \times 0.095$

$$\mu = 1.70\,\text{m}$$

b



Using $z = \dfrac{x-\mu}{\sigma}$:

$$\mathrm{P}(\text{height} > 1.74) = \mathrm{P}\left(z > \frac{1.74-1.70}{0.095}\right)$$
$$= \mathrm{P}(z > 0.42) \text{ (the tables give P}(Z < 0.42)\text{ so you need } 1-\text{this probability)}$$
$$= 1-0.6628$$
$$= 0.3372 \text{ (calculator gives 0.3369)}$$

16 a $\mathrm{P}(D < 21.5) = 0.32$ and $\mathrm{P}(Z < a) = 0.32 \Rightarrow a = -0.467$. Therefore

$$\frac{21.5-\mu}{\sigma} = -0.467 \Rightarrow 21.5-22 = -0.467\sigma \Rightarrow \sigma = \frac{0.5}{0.467} = 1.071 \text{ (4 s.f.).}$$

b $\mathrm{P}(21 < D < 22.5) = \mathrm{P}(D < 22.5) - \mathrm{P}(D < 21) = 0.5045$ (4 s.f.).

c $\mathrm{P}(B \geq 10) = 1-\mathrm{P}(B \leq 9) = 1-0.01899 = 0.98101$ (using 4 s.f. for the value given by the binomial distribution) or 0.981 (4 s.f.).

17 a Let W be the random variable 'the number of white plants'. Then $W \sim \mathrm{B}(12,\ 0.45)$ ('batches of 12': $n = 12$; '45% have white flowers': $p = 0.45$).

$$\mathrm{P}(W = 5) = \binom{12}{5} 0.45^5\, 0.55^7 \text{ (you can also use tables: } P(W \leq 5) - P(W \leq 4))$$
$$= 0.2225$$

b Batches of 12, so: 7 white, 5 coloured; 8 white, 4 coloured; etc.

$$\mathrm{P}(W \geq 7) = 1 - \mathrm{P}(W \leq 6)$$
$$= 1 - 0.7393$$
$$= 0.2607$$

c Use your answer to part **b**: $p = 0.2607$, $n = 10$:

$$\mathrm{P}(\text{exactly } 3) = \binom{10}{3}(0.2607)^3\,(1 - 0.2607)^7$$
$$= 0.2567$$

d A normal approximation is valid, since n is large (> 50) and p is close to 0.5. Therefore $\mu = np = 150 \times 0.45 = 67.5$ and $\sigma = \sqrt{np(1-p)} = \sqrt{67.5 \times 0.55} = \sqrt{37.125} = 6.093$ (4 s.f.). Now $\mathrm{P}(X > 75) \approx \mathrm{P}(N > 75.5) = 0.0946$ (3 s.f.).

18 a Using the binomial distribution, $\mathrm{P}(B = 35) = \binom{80}{35} \times 0.48^{35} \times 0.52^{45} = 0.06703$.

b A normal approximation is valid, since n is large (> 50) and p is close to 0.5. Therefore $\mu = np = 80 \times 0.48 = 38.4$ and $\sigma = \sqrt{np(1-p)} = \sqrt{38.4 \times 0.52} = \sqrt{19.968} = 4.469$ (4 s.f.). Now $\mathrm{P}(B = 35) \approx \mathrm{P}(34.5 < N < 35.5) = 0.0668$ (3 s.f.).

Percentage error is $\dfrac{0.06703 - 0.0668}{0.06703} = 0.34\%$.

19 Remember to identify which is H_0 and which is H_1. This is a one-tail test since we are only interested in whether the time taken to solve the puzzle has reduced. You must use the correct parameter (μ):

$\mathrm{H}_0 : \mu = 18 \quad \mathrm{H}_1 : \mu < 18$

Using $z = \dfrac{x - \mu}{\frac{\sigma}{\sqrt{n}}}$, $z = \dfrac{(16.5 - 18)}{\left(\frac{3}{\sqrt{15}}\right)} = -1.9364...$

Using the percentage point table and quoting the figure in full:

5% one tail c.v. is $z = -1.6449$

$-1.9364 < -1.6449$, so

significant *or* reject H_0 *or* in critical region.

State your conclusion in the context of the question:

There is evidence that the (mean) time to complete the puzzles has reduced.

Or Robert is getting faster (at doing the puzzles).

20 a $\text{P}(Z < a) = 0.05 \Rightarrow -1.645$. Using that $\text{P}(L < 1.7) = 0.05$ means that
$\frac{1.7-\mu}{0.4} = -1.645 \Rightarrow 1.7 - \mu = -0.658 \Rightarrow \mu = 2.358$

b $\text{P}(L > 2.3) = 0.5576$ (4 s.f.) and so, using the binomial distribution,
$\text{P}(B \geq 6) = 1 - \text{P}(B \leq 5) = 1 - 0.4758 = 0.5242$ (4 s.f.).

c It is thought that the mean length of the female rattlesnakes is 1.9 m, and a hypothesis test is needed to conclude whether the mean length is not equal to 1.9 m. Therefore,
$H_0 : \mu = 1.9$
$H_1 : \mu \neq 1.9$
Sample size: 20. Therefore, the sample population is initially thought to have distribution
$\bar{M} \sim N\left(1.9, \frac{0.3^2}{20}\right)$. By using the inverse normal distribution, $\text{P}(\bar{M} < 1.768) = 0.025$ and
$\text{P}(\bar{M} > 2.032) = 0.025$, meaning that the critical region is below 1.768 and above 2.032

d There is sufficient evidence to reject H_0, since 2.09 > 2.032; i.e. there is sufficient evidence to say, at the 5% level, that the mean length of the female rattlesnakes is not equal to 1.9 metres.

21 It is thought that the daily mean temperature in Hurn is less than 12 °C, and so a hypothesis test is needed to conclude whether, at the 5% level of significance, the mean temperature is less than 12 °C. Therefore,
$H_0 : \mu = 12$
$H_1 : \mu < 12$
Sample size: 20. Therefore, the sample population is initially thought to have distribution
$\bar{T} \sim N\left(12, \frac{2.3^2}{20}\right)$. By using the inverse normal distribution, $\text{P}(\bar{T} < 11.154) = 0.05$, meaning that the critical region consists of all values below 11.154. Since 11.1 < 11.154, there is sufficient evidence to reject H_0; i.e. there is sufficient evidence to say, at the 5% level, that the mean daily temperature in Hurn is less than 12 °C.

Challenge

1 a Since A and B could be mutually exclusive, $\text{P}(A \cap B) \geq 0$. Since $\text{P}(A \cap B) \leq \text{P}(B) = 0.3$, we have that $0 \leq \text{P}(A \cap B) \leq 0.3$ and so $q = \text{P}(A \cap B') = \text{P}(A) - \text{P}(A \cap B)$. Therefore $0.4 \leq p \leq 0.7$

b First, $\text{P}(B \cap C) \leq \text{P}(B) = 0.3$ and so $q \leq \text{P}(B \cap C) - \text{P}(A \cap B \cap C) \leq 0.25$. Moreover, it is possible to draw a Venn diagram where $q = 0$, and so $0 \leq q \leq 0.25$

Challenge

2 a We wish to use a hypothesis test to determine (at the 10% significance level) whether the support for the politician is 53%. A normal distribution is suitable, and we use the model given by $\mu = np = 300 \times 0.53 = 159$ and $\sigma = \sqrt{np(1-p)} = \sqrt{159 \times 0.47} = \sqrt{74.73} = 8.645$ (4 s.f.).

Therefore,

$H_0 : \mu = 159$

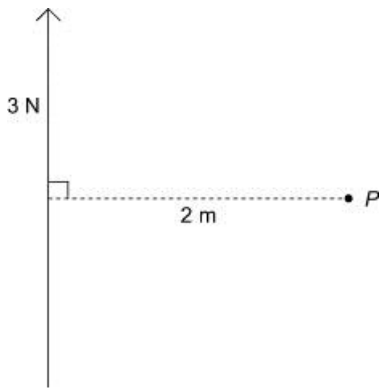
$H_1 : \mu \neq 159$

By using the inverse normal distribution, $\mathrm{P}(\bar{X} < 144.78) = 0.05$ and $\mathrm{P}(\bar{X} > 173.22) = 0.05$ (2 d.p.) and so the critical region consists of the values below 144.78 and above 173.22

b Since 173 is not within the critical region, there is not sufficient evidence to reject H_0 at the 10% significance level; i.e. there is not sufficient evidence to say, at the 10% level, that the politician's claim that they have support from 53% of the constituents is false.

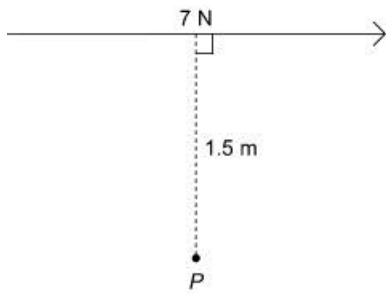
Moments 4A

1 a



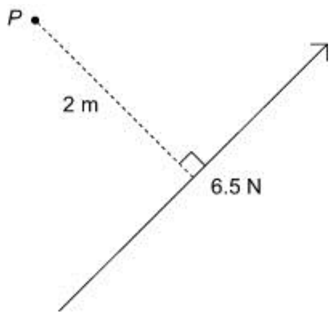
Moment $= 3 \times 2 = 6$ Nm clockwise

b



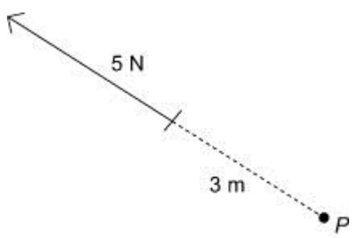
Moment $= 7 \times 1.5 = 10.5$ Nm clockwise

c



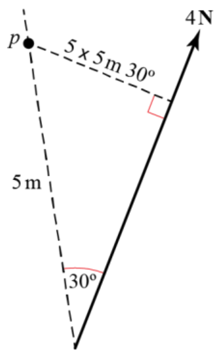
Moment $= 2 \times 6.5 = 13$ Nm anticlockwise

d



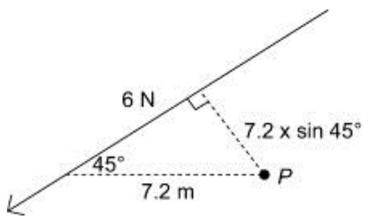
The line of action of the force acts through P, so moment $= 0$ Nm

2 a



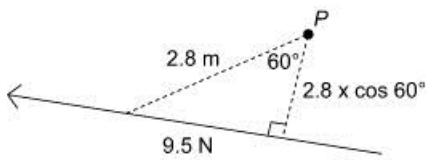
First, draw in the right-angled triangle.
Perpendicular distance $= 5 \times \sin 30°$
Moment $= 4 \times 5 \sin 30°$
$= 10$ Nm anticlockwise

b



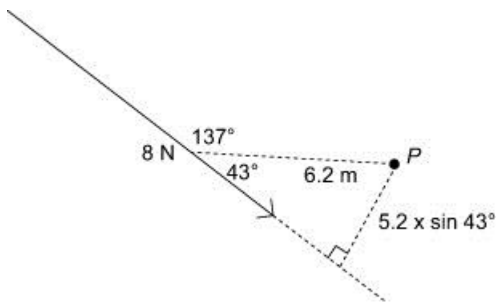
Distance $= 7.2 \times \sin 45°$
Moment $= 6 \times 7.2 \sin 45°$
$= 30.5$ Nm anticlockwise

c



Distance $= 2.8 \times \cos 60°$
Moment $= 9.5 \times 2.8 \cos 60°$
$= 13.3$ Nm clockwise

d



First, draw in the right-angled triangle.
Angle inside the triangle $= 180° - 137° = 43°$

2 d

$\text{Distance} = 6.2 \times \sin 43°$

$\text{Moment} = 8 \times 6.2 \sin 43°$

$= 33.8 \text{ Nm anticlockwise}$

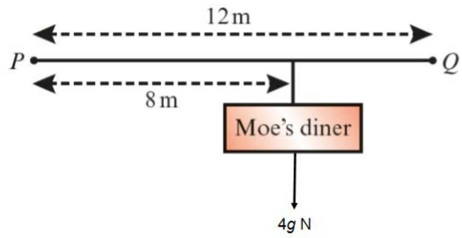
3 a i Moment = magnitude of force × perpendicular distance

$\text{Moment about } P = 4g \times 8$

$= 4 \times 9.8 \times 8$

$= 313.6$

The moment about P is 313.6 Nm clockwise.



ii Moment = magnitude of force × perpendicular distance

$\text{Moment about } Q = 4g \times (12 - 8)$

$= 4 \times 9.8 \times 4$

$= 156.8$

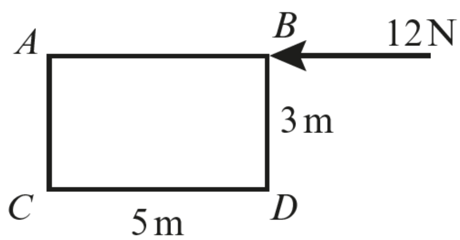
The moment about Q is 156.8 Nm anticlockwise.

b In these calculations, we have assumed that the sign is a particle – i.e. all the weight of the sign acts at its centre of mass.

4 a Moment = magnitude of force × perpendicular distance

$\text{Moment about } A = 12 \times 0$

$= 0 \text{ Nm}$



b Moment = magnitude of force × perpendicular distance

$\text{Moment about } B = 12 \times 0$

$= 0 \text{ Nm}$

c Moment = magnitude of force × perpendicular distance

$\text{Moment about } C = 12 \times 3$

$= 36 \text{ Nm anticlockwise}$

d Moment = magnitude of force × perpendicular distance

$\text{Moment about } D = 12 \times 3$

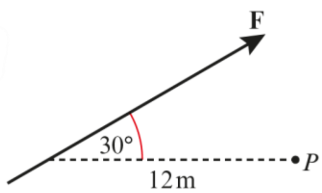
$= 36 \text{ Nm anticlockwise}$

5 Moment = magnitude of force × perpendicular distance

$15 = F \times 12 \sin 30°$

$F = \dfrac{15}{12 \sin 30°}$

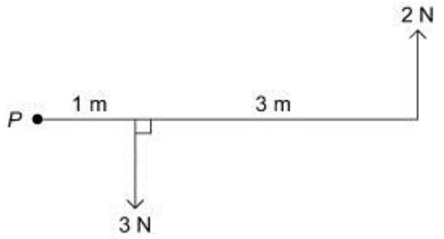
$= 2.5 \text{ Nm}$



Moments 4B

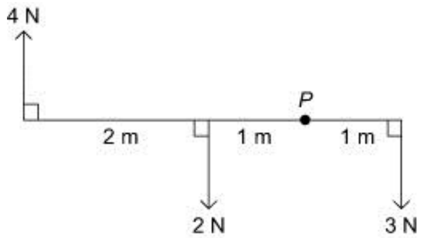
For each question in this exercise, clockwise is assumed to be the positive direction.

1 a



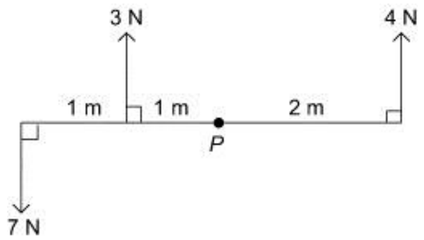
Moment of 3 N force
$= 3 \times 1 = 3$ Nm clockwise
Moment of 2 N force
$= (1+3) \times 2 = 8$ Nm anticlockwise
Resultant moment $= 8 - 3$
$= 5$ Nm anticlockwise

b



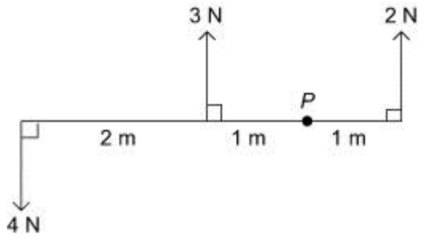
Moment of 4 N force
$= 4 \times (2+1) = 12$ Nm clockwise
Moment of 2 N force
$= 2 \times 1 = 2$ Nm anticlockwise
Moment of 3 N force
$= 3 \times 1 = 3$ Nm clockwise
Resultant moment $= 12 - 2 + 3 = 13$ Nm clockwise

c



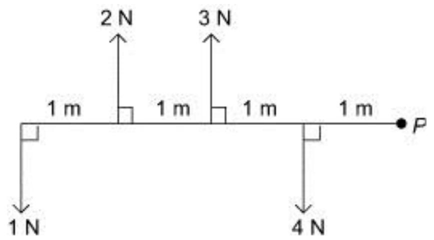
Moment of 7 N force
$= 7 \times (1+1) = 14$ Nm anticlockwise
Moment of 3 N force
$= 3 \times 1 = 3$ Nm clockwise
Moment of 4 N force
$= 4 \times 2 = 8$ Nm anticlockwise
Resultant moment $= -14 + 3 - 8 = -19$ Nm
The resultant moment is 19 Nm anticlockwise.

1 d



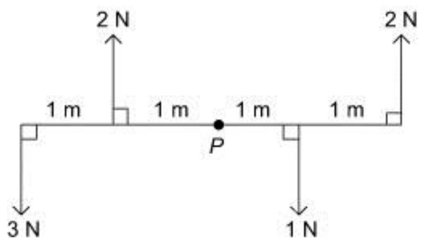
Moment of 4 N force
$= 4 \times (2+1) = 12$ Nm anticlockwise
Moment of 3 N force
$= 3 \times 1 = 3$ Nm clockwise
Moment of 2 N force
$= 2 \times 1 = 2$ Nm anticlockwise
Resultant moment $= -12 + 3 - 2 = -11$ Nm
The resultant moment is 11 Nm anticlockwise.

e



Moment of 1 N force
$= 1 \times (1+1+1+1) = 4$ Nm anticlockwise
Moment of 2 N force
$= 2 \times (1+1+1) = 6$ Nm clockwise
Moment of 3 N force
$= 3 \times (1+1) = 6$ Nm clockwise
Moment of 4 N force
$= 4 \times 1 = 4$ Nm anticlockwise
Resultant moment $= -4 + 6 + 6 - 4 = 4$ Nm clockwise

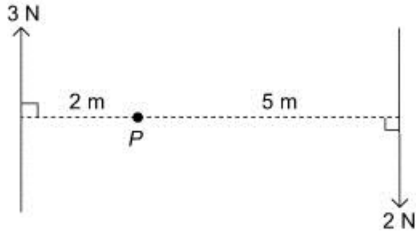
f



Moment of 3 N force
$= 3 \times (1+1) = 6$ Nm anticlockwise
Moment of 2 N force to the left of *P*
$= 2 \times 1 = 2$ Nm clockwise
Moment of 1 N force
$= 1 \times 1 = 1$ Nm clockwise

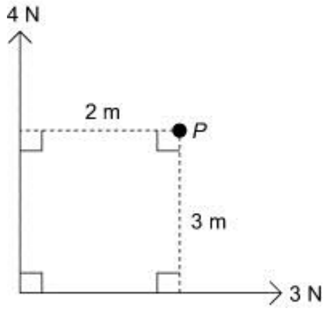
1 f Moment of 2 N force to the right of P
$= 2 \times (1+1) = 4$ Nm anticlockwise
Resultant moment $= -6+2+1-4 = -7$ Nm
The resultant moment is 7 Nm anticlockwise.

2 a



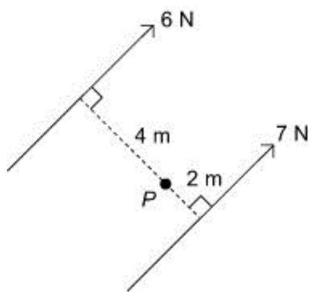
Moment of 3 N force
$= 3 \times 2 = 6$ Nm clockwise
Moment of 2 N force
$= 2 \times 5 = 10$ Nm clockwise
Resultant moment $= 6 + 10 = 16$ Nm clockwise

b



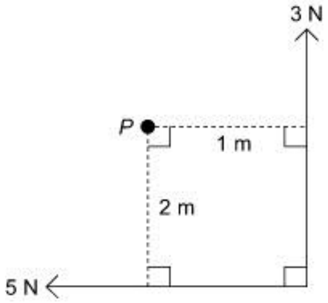
Moment of 4 N force
$= 4 \times 2 = 8$ Nm clockwise
Moment of 3 N force
$= 3 \times 3 = 9$ Nm anticlockwise
Resultant moment $= 8 - 9 = -1$ Nm
The resultant moment is 1 Nm anticlockwise.

c



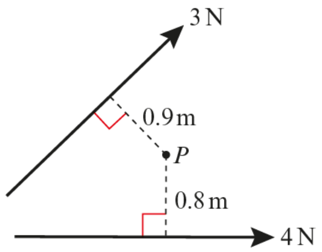
Moment of 6 N force
$= 6 \times 4 = 24$ Nm clockwise
Moment of 7 N force
$= 7 \times 2 = 14$ Nm anticlockwise
Resultant moment $= 24 - 14 = 10$ Nm clockwise

2 d



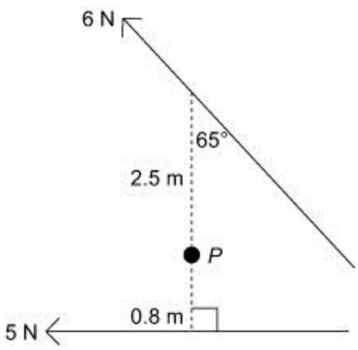
Moment of 3 N force
$= 3 \times 1 = 3$ Nm anticlockwise
Moment of 5 N force
$= 5 \times 2 = 10$ Nm clockwise
Resultant moment $= -3 + 10 = 7$ Nm clockwise

e



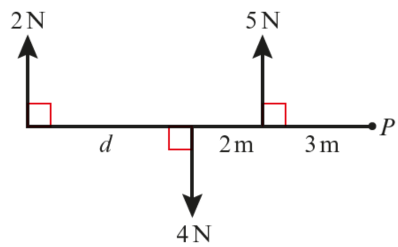
Moment of 3 N force
$= 3 \times 0.9 = 2.7$ Nm clockwise
Moment of 4 N force
$= 4 \times 0.8 = 3.2$ Nm anticlockwise
Resultant moment $= 2.7 - 3.2 = -0.5$ Nm
The resultant moment is 0.5 Nm anticlockwise.

f



Moment of 6 N force
$= 6 \times (2.5 \times \sin 65°) = 13.59$ Nm anticlockwise
Moment of 5 N force
$= 5 \times 0.8 = 4$ Nm clockwise
Resultant moment $= -13.59 + 4 = -9.59$ Nm
The resultant moment is 9.59 Nm anticlockwise.

3 Moment of 2 N force about P



= $2 \times (5 + d)$ Nm clockwise
Moment of 5 N force about P
= $5 \times 3 = 15$ Nm clockwise
Moment of 4 N force about P
= $4 \times (2 + 3) = 20$ Nm anticlockwise
Resultant moment = 17 Nm clockwise so:

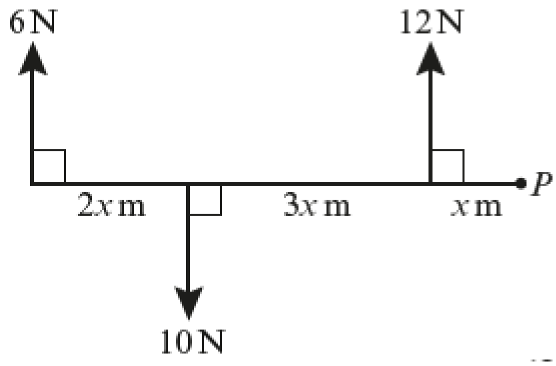
$$\begin{aligned} 2(5+d)+15-20 &= 17 \\ 5+2d &= 17 \\ 2d &= 12 \\ d &= 6 \end{aligned}$$

The distance d is 6 m.

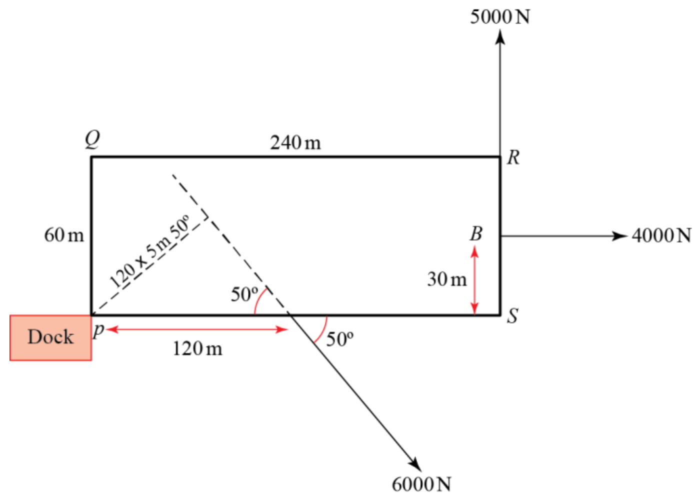
4 Moment of 6 N force about P
= $6 \times (2 + 3 + 1)x = 36x$ Nm clockwise.
Moment of 12 N force about P
= $12x$ Nm clockwise.
Moment of 10 N force about P
= $10 \times (3 + 1)x = 40x$ Nm anticlockwise.
Resultant moment = 12.8 Nm clockwise so:

$$\begin{aligned} 36x+12x-40x &= 12.8 \\ 8x &= 12.8 \\ x &= 1.6 \end{aligned}$$

The distance x is 1.6 m.

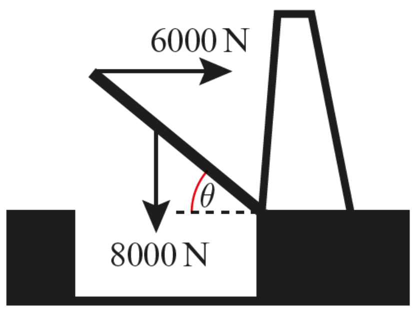


5 $AP = 240 \div 2 = 120$ m
$BS = 60 \div 2 = 30$ m
Moment of tug at A about P
= $6000 \times 120\sin 50 = 551\,552$ Nm clockwise.
Moment of tug at B about P
= $4000 \times 30 = 120\,000$ Nm clockwise.
Moment of tug at R about P
= $5000 \times 240 = 1\,200\,000$ Nm anticlockwise.
Resultant moment
= $551\,552 + 120\,000 - 1\,200\,000 = -528\,448$
The ship rotates anticlockwise and the resultant moment about P is 528 448 Nm.



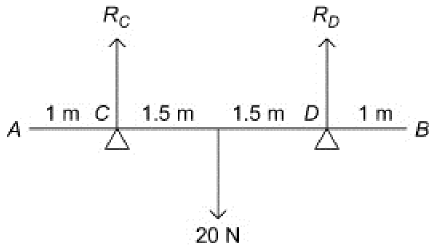
6 If drawbridge is rising, clockwise moment > anticlockwise moment
Let the length of the drawbridge be x

$$\begin{aligned} 6000 \times x\sin\theta &> 8000 \times \frac{1}{2}x\cos\theta \\ 6000\sin\theta &> 4000\cos\theta \\ \frac{\sin\theta}{\cos\theta} &> \frac{4000}{6000} \\ \tan\theta &> \frac{2}{3} \end{aligned}$$



Moments 4C

1 a



Resolving vertically:

$R_C + R_D = 20$

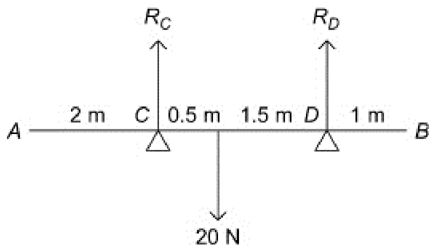
Taking moments about C:

$3 \times R_D = 1.5 \times 20$

$= 30$

$\Rightarrow R_D = 10\,\text{N}$ and $R_C = 10\,\text{N}$

b



Resolving vertically:

$R_C + R_D = 20$

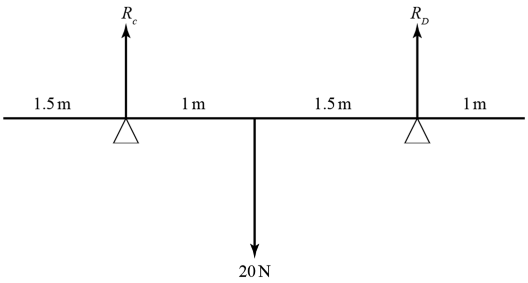
Taking moments about C:

$R_D \times 2 = 20 \times 0.5$

$= 10$

$\Rightarrow R_D = 5\,\text{N}$ and $R_C = 15\,\text{N}$

1 c



Resolving vertically:

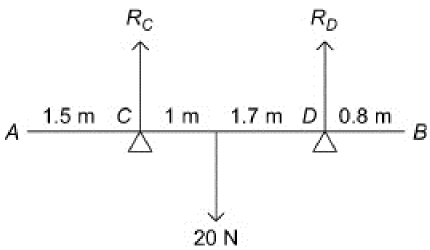
$R_C + R_D = 20$

Taking moments about C:

$$R_D \times 2.5 = 20 \times 1$$
$$= 20$$

$$\Rightarrow R_D = \frac{20}{2.5} = 8\,\text{N} \quad \text{and} \quad R_C = 12\,\text{N}$$

d



Resolving vertically:

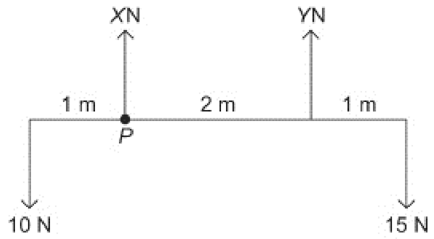
$R_C + R_D = 20$

Taking moments about C:

$$2.7 \times R_D = 20 \times 1$$
$$= 20$$

$$\Rightarrow R_D = \frac{20}{2.7} = 7.4\,\text{N} \quad \text{and} \quad R_C = 12.6\,\text{N}$$

2 a



Resolving vertically:

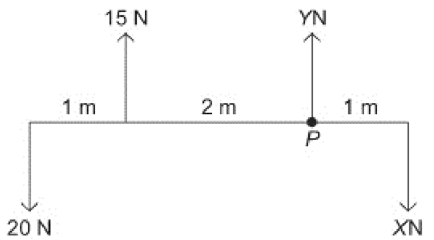
$$X + Y = 10 + 15$$
$$= 25$$

Taking moments about P:

$$15 \times (2+1) = 10 \times 1 + Y \times 2$$
$$45 = 10 + 2Y$$
$$2Y = 35$$
$$Y = 17.5$$

$\Rightarrow X = 7.5$ and $Y = 17.5$

b



Resolving vertically:

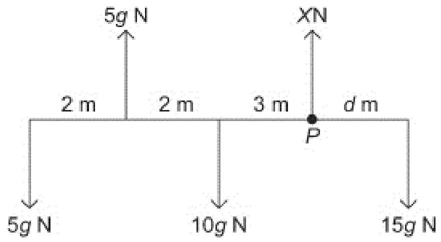
$$15 + Y = 20 + X$$
$$Y - X = 5$$

Taking moments about P:

$$20 \times (2+1) = 15 \times 2 + X \times 1$$
$$60 = 30 + X$$
$$X = 30$$

$\Rightarrow X = 30$ and $Y = 35$

2 c



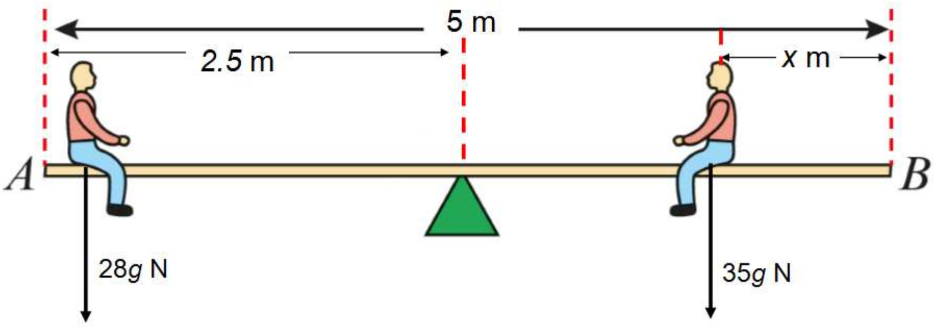
Resolving vertically:

$$5g + X = 5g + 10g + 15g$$
$$= 30g$$
$$\Rightarrow X = 25g = 245$$

Taking moments about P:

$$15g \times d + 5g \times (2+3) = 10g \times 3 + 5g \times (2+2+3)$$
$$15gd + 25g = 30g + 35g$$
$$15d = 40$$
$$d = 2\tfrac{2}{3}$$

3

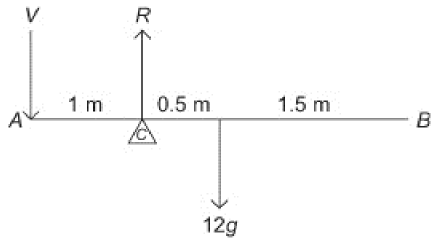


Seesaw is in equilibrium so
clockwise moment about pivot = anticlockwise moment about pivot

$$35g(2.5 - x) = 28g \times 2.5 \qquad \text{(divide both sides by } 7g\text{)}$$
$$5(2.5 - x) = 4 \times 2.5$$
$$5x = 2.5(5 - 4)$$
$$x = \frac{2.5}{5} = 0.5$$

Jack sits 0.5 m from B.

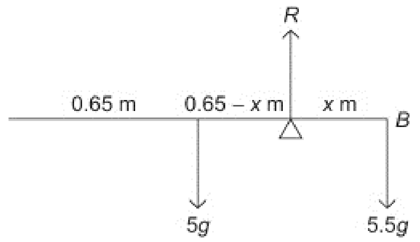
4



Suppose that the force required is V N acting vertically downwards at A.
Taking moments about the pivot (C):
$V \times 1 = 0.5 \times 12g$

$\Rightarrow V = 6g = 59\,\text{N}$ (2 s.f.)

5

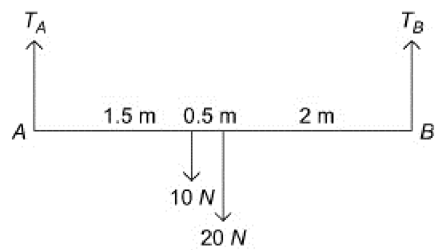


Let the support be x m from the broomhead.
Taking moments about the support:

$$
\begin{aligned}
5.5g \times x &= 5g \times (0.65 - x) \\
5.5x &= 5 \times 0.65 - 5x \\
10.5x &= 3.25 \\
x &= 0.31
\end{aligned}
$$

The support should be 31 cm from the broomhead.

6 a

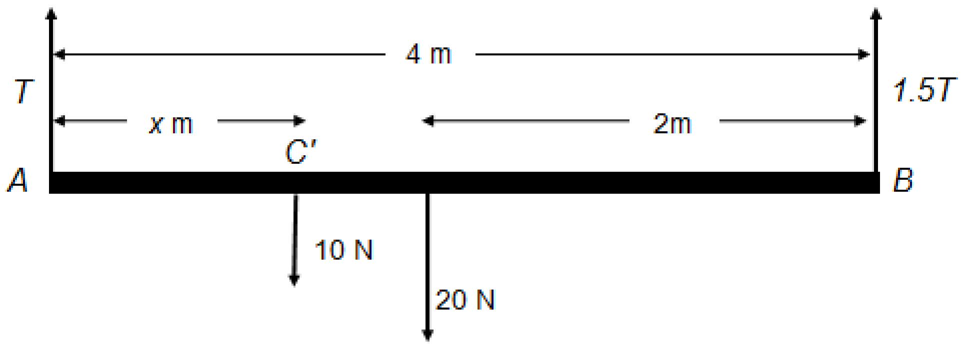


Let the tensions in the two strings be T_A and T_B respectively.
Resolving vertically:
$T_A + T_B = 10 + 20 = 30$
Taking moments about point A:

$$
\begin{aligned}
10 \times 1.5 + 20 \times (1.5 + 0.5) &= 4 \times T_B \\
\Rightarrow 4T_B &= 15 + 40 \\
&= 55
\end{aligned}
$$

$T_B = 13.75\,\text{N}$ and $T_A = 16.25\,\text{N}$

6 b Particle is now at C' where $AC' = x$ m.



Beam is in equilibrium.
Resolving vertically:

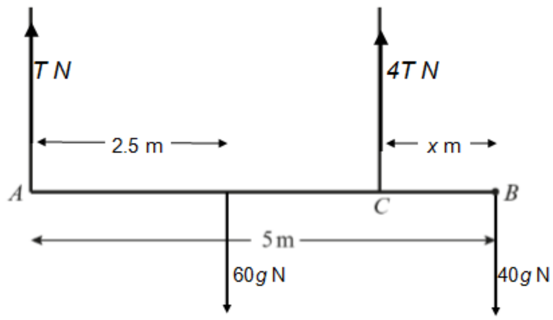
$$T + 1.5T = 10 + 20$$
$$2.5T = 30$$
$$T = 12$$

Taking moments about A:

$$10x + (20 \times 2) = (1.5 \times 12) \times 4$$
$$10x + 40 = 18 \times 4$$
$$10x = 72 - 40$$
$$x = \frac{32}{10} = 3.2$$

The particle is now 3.2 m from A.

7 $BC = x$ m.
Beam is in equilibrium.



a Resolving vertically:

$$4T + T = 40g + 60g$$
$$5T = 100g$$
$$T = 20g$$

So $4T = 80g$

$$4T = 80 \times 9.8 = 784$$

The tension in the wire at C is 784 N.

b Taking moments about B:

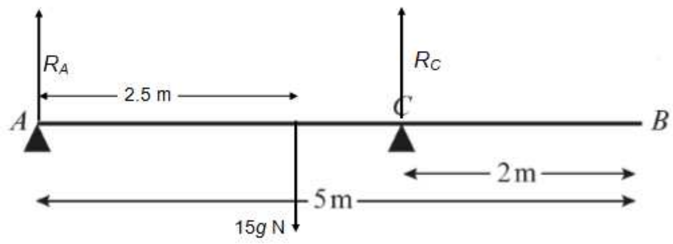
$$(20g \times 5) + 80gx = 60g \times 2.5 \qquad \text{(divide by } 20g\text{)}$$
$$5 + 4x = 7.5$$
$$4x = 2.5$$
$$x = \frac{2.5}{4}$$
$$= 0.625$$

The distance CB is 0.625 m.

8 a Plank is in equilibrium.
Let the reactions at A and C be R_A and R_C respectively.
Taking moments about A:

$$15g \times 2.5 = R_c \times 3$$
$$R_c = 2.5 \times 5g$$
$$R_c = 12.5 \times 9.8$$
$$= 122.5$$

The reaction at C is 122.5 N.



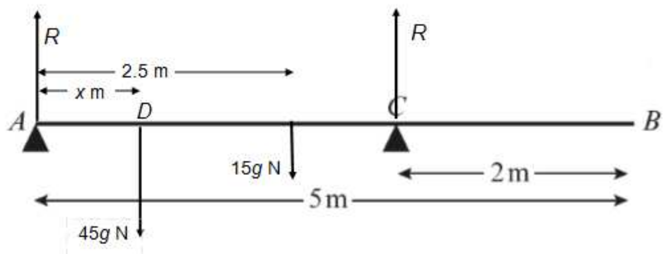
b Let $AD = x$ m
Let $R_A = R_C = R$
Plank remains in equilibrium.
Resolving vertically:

$$2R = 45g + 15g = 60g$$
$$R = 30g$$

Taking moments about A:

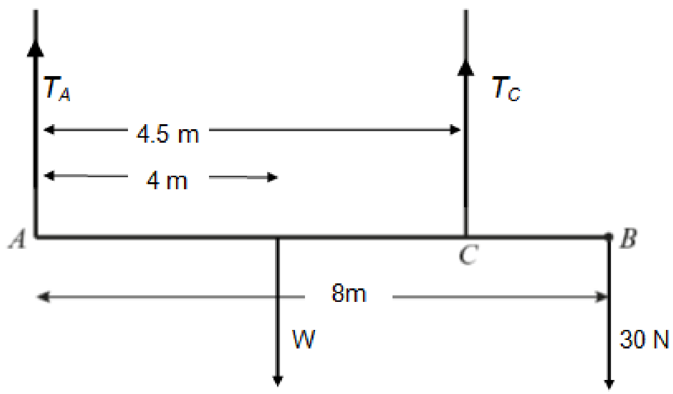
$$45gx + (15g \times 2.5) = 30g \times 3 \qquad \text{(divide by } 15g\text{)}$$
$$3x + 2.5 = 6$$
$$3x = 3.5$$
$$x = \frac{3.5}{3}$$
$$= 1.17$$

The distance AD is 1.17 m (3s.f.).



9 a Beam is in equilibrium.
Let tension in wire at C be T_C
Taking moments about A:

$$4.5T_C = 4W + (8 \times 30)$$
$$\frac{9}{2}T_C = 4W + 240$$
$$9T_C = 8W + 480$$
$$T_C = \frac{8}{9}W + \frac{160}{3} \quad \text{as required.}$$



b Let tension in wire at A be T_A
Resolving vertically:

$$W + 30 = T_A + T_C$$
$$W + 30 = T_A + \frac{8}{9}W + \frac{160}{3}$$
$$9W + 270 = 9T_A + 8W + 480$$
$$W + 270 - 480 = 9T_A$$
$$T_A = \frac{W - 210}{9}$$
$$T_A = \frac{W}{9} - \frac{70}{3}$$

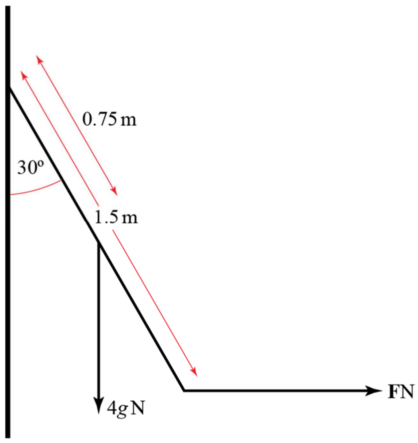
9 c

$$T_C = 12T_A$$
$$\frac{8W}{9} + \frac{160}{3} = \frac{12W}{9} - \frac{12\times 70}{3}$$
$$8W + (160\times 3) = 12W - (12\times 70\times 3)$$
$$480 + 2520 = 12W - 8W$$
$$4W = 3000$$
$$W = 750$$

The weight of the beam is 750 N.

10 The lever is in equilibrium.

Taking moments about point where lever is attached to the wall:



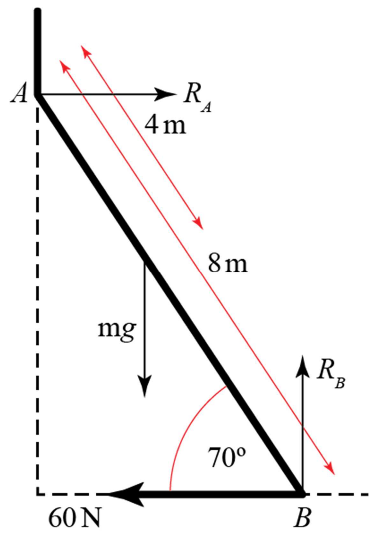
$$5g\times 0.75\sin 30^\circ = F\times 1.5\cos 30^\circ$$
$$F = \frac{5g\times 0.75\sin 30^\circ}{1.5\cos 30^\circ}$$
$$F = \frac{5}{2}g\tan 30^\circ$$
$$F = \frac{5}{2}\times 9.8\tan 30^\circ = 14.1$$

The force F is 14.1 N (3s.f.).

11 a The ladder is in equilibrium.

Resolving horizontally:

The reaction of the ladder on the wall at A = 60 N.

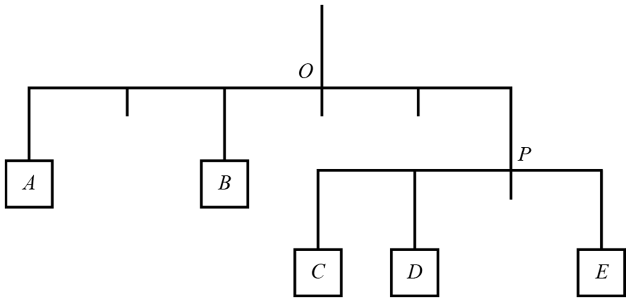


b Taking moments about B:

$$60\times 8\sin 70^\circ = 4mg\cos 70^\circ$$
$$m = \frac{60\times 8\sin 70^\circ}{4g\cos 70^\circ}$$
$$m = \frac{120}{g}\tan 70^\circ$$
$$m = \frac{120}{9.8}\tan 70^\circ$$
$$= 33.6$$

The mass of the ladder is 33.6 kg (3s.f.).

Challenge
Let the masses of the hanging components be A, B, C, D and E kg as shown.



Treating CDE as a single component and taking moments about O:
$(3A + B)g = 2(C + D + E)g$
Since all the numbers are whole, $2(C + D + E)$ is even, so $3A + B$ must be even.
This means that **A & B are either both even or both odd**.
The minimum possible value of $C + D + E = 1 + 2 + 3 = 6$
So $3A + B \geqslant 12$
Maximum value of B is 5
So $3A \geqslant 7$
i.e. $A \geqslant \frac{7}{3} \Rightarrow$ **A cannot be 1 or 2.**

Taking moments about P:
$(2C + D)g = Eg$
Smallest possible value of $2C + D$ is $(2 \times 1) + 2 = 4$
So ***E* must be 4 or 5**

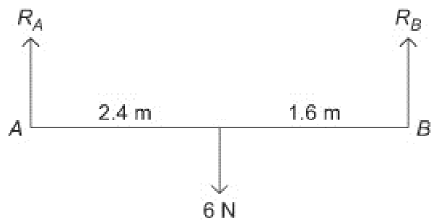
If $E = 4$ then $C = 1$ and $D = 2$
This leaves A & B as 3 and 5.
Either option allowed by rules above.
$2(C + D + E) = 2(1 + 2 + 4) = 14$
since $3 \times 5 > 14$, this means A must be 3 and B must be 5.
To check: $3A + B = (3 \times 3) + 5 = 14$
Therefore this combination works.

However, best to check other possibilities:
If $E = 5$ then either $C = 2$ & $D = 1$ or $C = 2$ & $D = 1$.
First case means A & B are 3 & 4, which **is not allowed** as one odd and one even.
In second case, since A cannot be 2, $A = 4$ and $B = 2$.
Then:
$2(C + D + E) = 2(2 + 1 + 5) = 16$
$3A + B = (3 \times 4) + 2 = 14$
Since these are **not equal**, this combination does not work either.

The masses, from left to right, are: 3 kg, 5 kg, 1 kg, 2 kg and 4 kg.

Moments 4D

1



Resolving vertically:

$6 = R_A + R_B$

Taking moments about A:

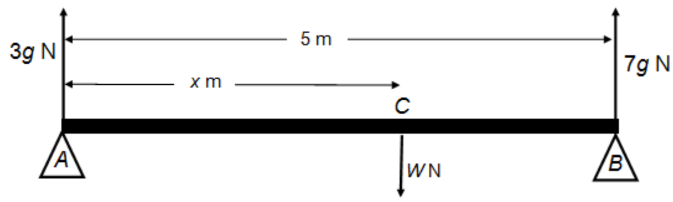
$6 \times 2.4 = 4 \times R_B$

$\Rightarrow R_B = 3.6\text{ N}$

So $R_A = 2.4\text{ N}$

The reactions at A and B are 2.4 N and 3.6 N respectively.

2 Centre of mass is at C, a distance x m from A.
The bar is in equilibrium.



a Resolving vertically:

$W = 3g + 7g$

$= 10g$

The weight of the bar is $10g$ N

b Taking moments about C:

$3gx = 7g(5 - x)$

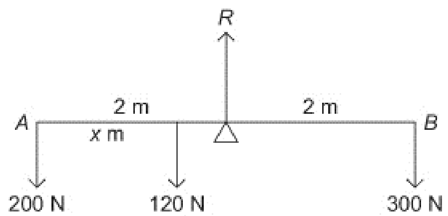
$3gx = 35g - 7gx$

$10x = 35$

$x = 3.5$

The centre of mass is 3.5 m from A.

3



Let the centre of mass be x m from A.
Taking moments about the mid-point:

$120 \times (2 - x) + 200 \times 2 = 300 \times 2$

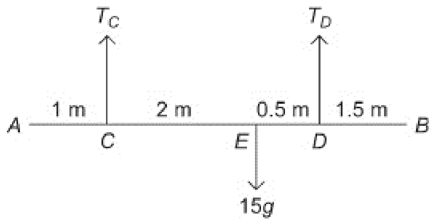
$240 - 120x + 400 = 600$

$120x = 40$

$\Rightarrow x = \frac{40}{120} = \frac{1}{3}$

The centre of mass is $\frac{1}{3}$ m from A.

4 a



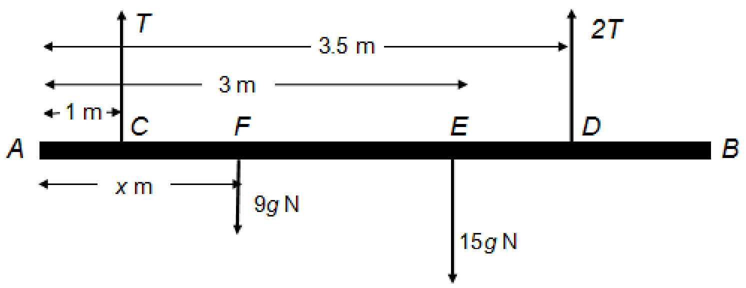
Taking moments about C:

$$T_D \times 2.5 = 15g \times 2$$
$$2.5T_D = 30g$$
$$T_D = 12g$$
$$= 118\text{ N (3 s.f.)}$$

Resolving vertically:

$$T_C + T_D = 15g$$
$$T_C = 3g$$
$$= 29.4\text{ N}$$

b Let distance $AF = x$ m.
The bar is in equilibrium.
Resolve vertically:



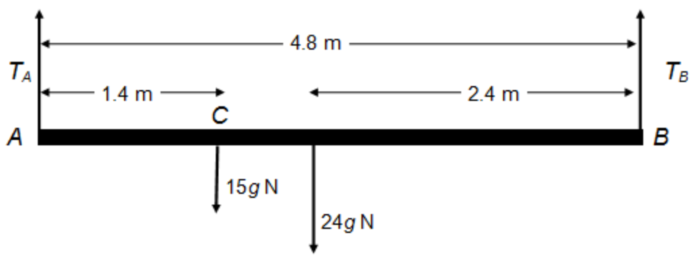
$$T + 2T = 9g + 15g$$
$$3T = 24g$$
$$T = 8g \text{ and } 2T = 16g$$

Taking moments about A:

$$9gx + (15g \times 3) = (8g \times 1) + (16g \times 3.5)$$
$$9x = 8 + 56 - 45$$
$$x = \frac{19}{9}$$
$$= 2.11$$

The distance AF is 2.11 m (3s.f.).

5 a Let the tensions in the ropes be T_A and T_B respectively.
The plank is in equilibrium.
Taking moments about A:

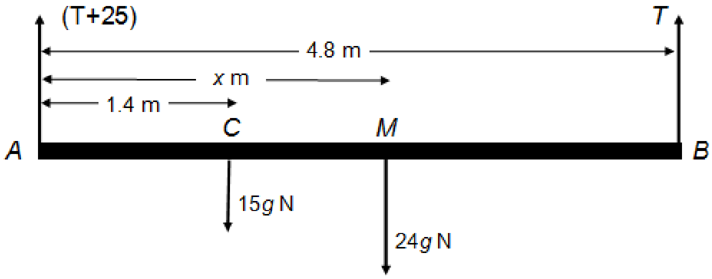


$$T_B \times 4.8 = (1.4 \times 15g) + (2.4 \times 24g)$$
$$4.8T_B = 21g + 57.6g$$
$$T_B = \frac{78.6 \times 9.8}{4.8}$$
$$= 160$$

The tension in the rope at B is 160 N.

5 b Centre of mass is at M, x m from A.
The plank is in equilibrium.
Resolving vertically:

$$T + T + 25 = 15g + 24g$$
$$2T = 39g - 25$$
$$T = \frac{(39 \times 9.8) - 25}{2}$$
$$= 178.6$$



Taking moments about A:

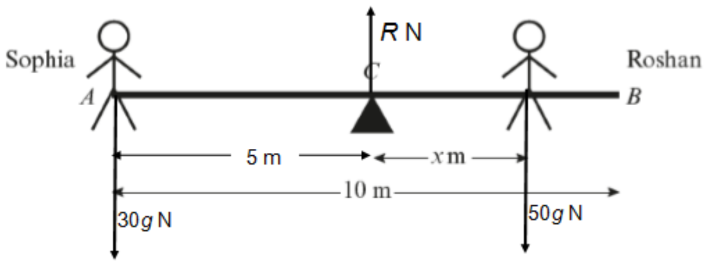
$$(15g \times 1.4) + 24gx = 178.6 \times 4.8$$
$$(147 \times 1.4) + 235.2x = 857.28$$
$$x = \frac{857.28 - 205.8}{235.2}$$
$$= 2.77$$

The centre of mass is 2.77 m from A (3s.f.).

6 a Taking moments about C:

$$30g \times 5 = 50gx$$
$$150g = 50gx$$
$$x = 3$$

The seesaw is in equilibrium when Roshan sits 3 m from C.

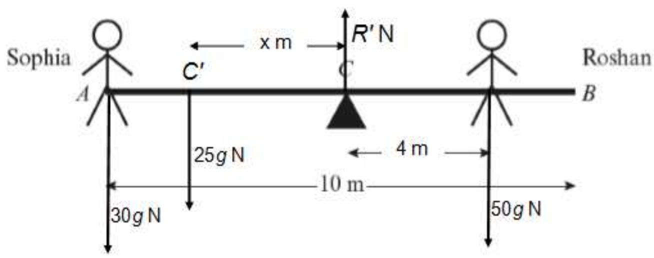


b Modelling the beam as uniform means that the centre of mass of the seesaw is at C, and so weight of the seesaw can be ignored when taking moments about C.

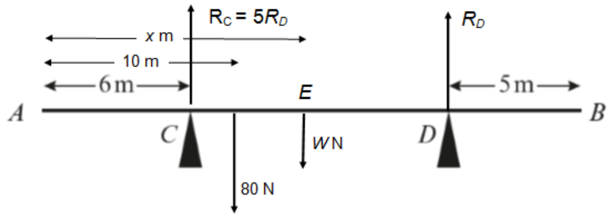
c Centre of mass is at C', y m from C.
Taking moments about C:

$$(30g \times 5) + 25gx = 50g \times 4 \quad \text{(divide by 5g)}$$
$$30 + 5x = 40$$
$$x = \frac{40 - 30}{5} = 2$$

The centre of mass is 2 m to the left of C (towards Sophia).



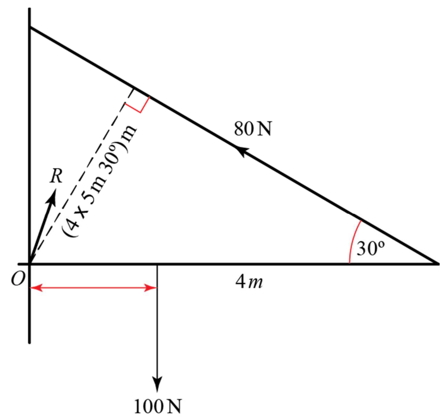
7 The rod is in equilibrium.
Letting $R_D = R$ gives $R_c = 5R$
Resolving vertically:



$$R_c + R_D = 80 + W$$
$$5R + R = 80 + W$$
$$R = \frac{80 + W}{6}$$

Taking moments about A:
$$(6 \times 5R) + 20R = (80 \times 10) + Wx$$
$$50R = 800 + Wx$$
$$50\left(\frac{80 + W}{6}\right) = 800 + Wx \qquad \text{(multiply by 6 and expand)}$$
$$4000 + 50W = 4800 + 6Wx$$
$$(50 - 6x)W = 4800 - 4000 \qquad \text{(divide by 2 and rearrange)}$$
$$W = \frac{400}{25 - 3x} \qquad \text{as required.}$$

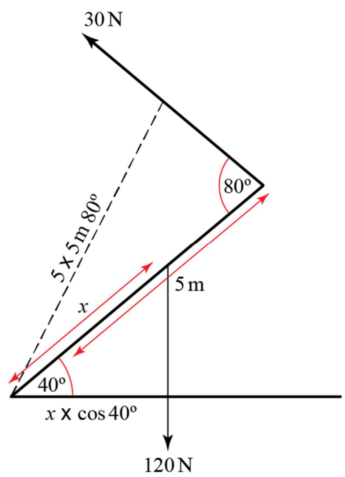
8 The rod is attached to the wall at O.
Let the distance from the wall to the centre of mass be x m.
The rod is in equilibrium.
Taking moments about O:



$$100x = 80 \times 4 \sin 30^\circ$$
$$100x = 160$$
$$x = 1.6$$

The centre of mass of the rod is 1.6 m from the wall.

Challenge
Let the distance from M to the beam's centre of mass be x m.
The beam is in equilibrium.
Taking moments about M:



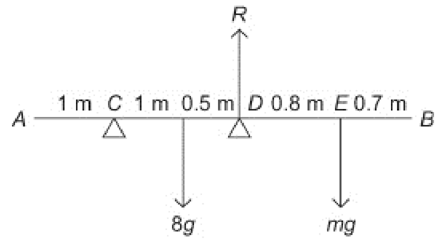
$$120 \times \left(x \cos 40^\circ\right) = 30 \times \left(5 \sin 80^\circ\right)$$
$$4x \cos 40^\circ = 5 \sin 80^\circ$$
$$x = \frac{5 \sin 80^\circ}{4 \cos 40^\circ}$$
$$= 1.61$$

The centre of mass of the beam is 1.61 m from M (3s.f.).

Moments 4E

1 If the rod is about to turn about D then the reaction at C is zero.
Taking moments about point D:

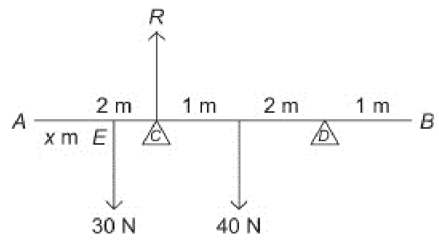
$$8g \times 0.5 = mg \times 0.8$$
$$\Rightarrow m = 5$$



2 If the bar is about to tilt about C then the reaction at D is zero.
Let the distance $AE = x$ m
Taking moments about C:

$$40 \times 1 = 30 \times (2 - x)$$
$$40 = 60 - 30x$$
$$30x = 20$$
$$x = \frac{2}{3}$$

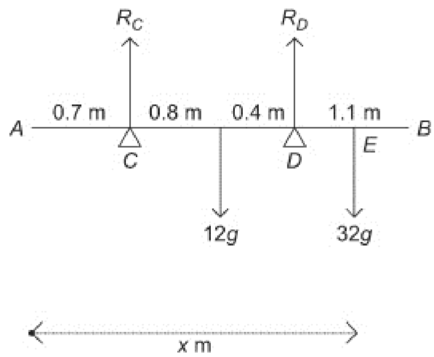
The distance $AE = \frac{2}{3}$ m



3 Let the distance AE be x m.
If the plank is about to tilt about D then $R_C = 0$
Taking moments about D:

$$12g \times 0.4 = 32g \times (x - 1.9)$$
$$12 \times 0.4 = 32x - 32 \times 1.9$$
$$32x = 4.8 + 60.8$$
$$= 65.6$$
$$\Rightarrow x = \frac{65.6}{32}$$
$$= 2.05$$

E is 2.05 m from A



4 a $R(\uparrow)$:

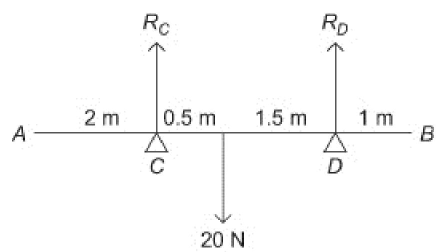
$$R_C + R_D = 20 \qquad \textbf{(1)}$$

Taking moments about C:

$$20 \times 0.5 = R_D \times 2$$
$$R_D = 5\text{ N} \qquad \textbf{(2)}$$

Substituting (**2**) into (**1**):

$$R_C = 20 - 5$$
$$= 15\text{ N}$$



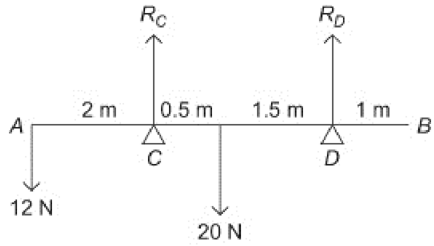
4 b Adding the weight of 12 N:

Taking moments about C:

$$20 \times 0.5 = 12 \times 2 + R_D \times 2$$

$$10 = 24 + 2R_D$$

$\Rightarrow R_D$ is negative, which is impossible, therefore there is an anticlockwise moment about C – rod will tilt.



c Distance AE is x m.

The reactions at the supports are R_C and R_D.

If rod tilts about C, $R_D = 0$.

Taking moments about C:

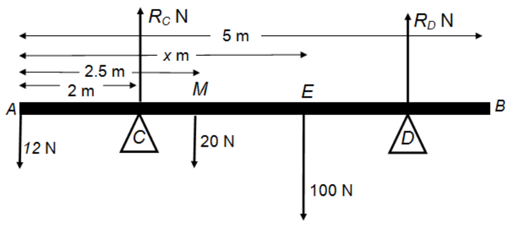
$$12 \times 2 = 20(2.5 - 2) + 100(x - 2)$$

$$24 = 10 + 100x - 200$$

$$x = \frac{200 + 24 - 10}{100}$$

$$= 2.14$$

In this case $AE = 2.14$



If rod tilts about D, $R_C = 0$.

E must be on the other side of D, a distance y m from B.

Taking moments about D:

$$12 \times (5 - 1) + 20(2.5 - 1) = 100y$$

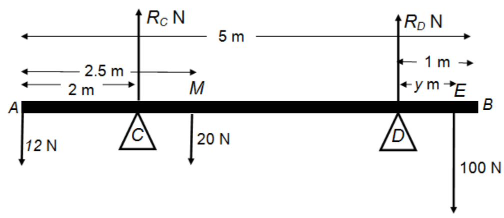
$$48 + 30 = 100y$$

$$y = \frac{78}{100}$$

$$= 0.78$$

In this case $AE = 5 – 1 + 0.78 = 4.78$

The rod will remain in equilibrium if the particle is placed between 2.14 m and 4.78 m from A.



5 The reactions at the supports are R_A N and R_B N.

When the plank tilts, $R_A = 0$ and the man is x m from B.

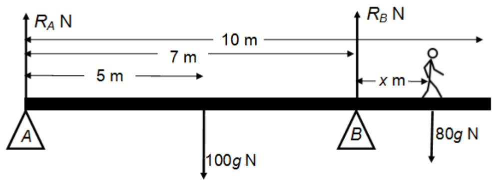
Taking moments about B:

$$100g \times (7 - 5) = 80gx$$

$$x = \frac{200}{80}$$

$$= 2.5$$

The man can walk 2.5 m past B before the plank starts to tip.



6 a Let $ON = x$ m.

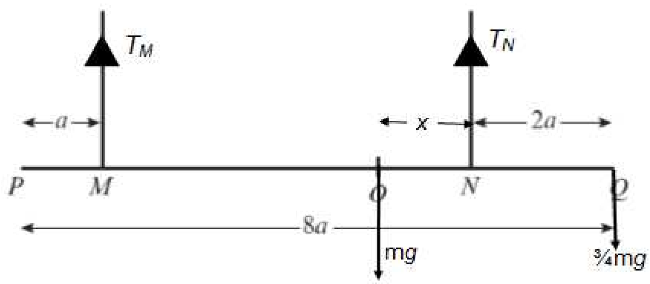
Let the tensions in the two wires be T_M N and T_N N.

Since beam is on the point of tipping about N, $T_M = 0$.

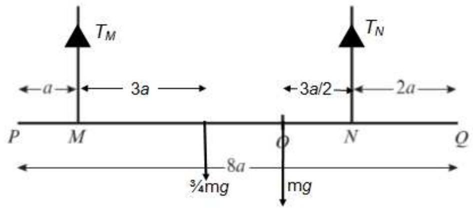
Taking moments about N:

$$mgx = \frac{3}{4}mg \times 2a$$

$$x = \frac{3}{2}a \quad \text{as required.}$$



6 b Taking moments about M:



$$\left(\frac{3}{4}mg\times 3a\right)+mg\left(5-\frac{3}{2}\right)a=T_N\times 5a$$

$$\frac{9}{4}mg+5mg-\frac{3}{2}mg=5T_N$$

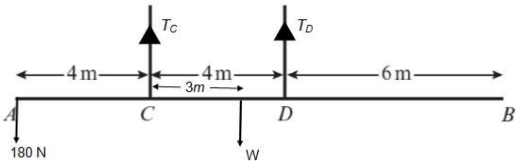
$$\left(\frac{9+20-6}{4}\right)mg=5T_N$$

$$\frac{23}{4}mg=5T_N$$

$$T_N=\frac{23}{20}mg$$

The tension in the wire attached at N is $\frac{23}{20}mg$

7 Let the tensions in the cables be T_C N and T_D N.



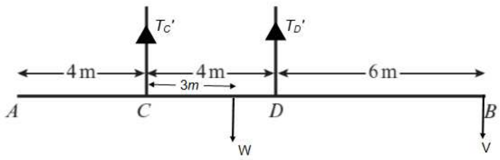
<u>In the first case</u>:
The beam must be on the point of tipping about C, so
$T_D = 0$
(This is because, if $T_C = 0$, there would be a resultant moment around D no matter what the value of W, and the beam would not be in equilibrium.)

Taking moments about C:

$$180\times 4=3W$$

$$W=240$$

<u>In the second case</u>:
When V is at maximum value, the beam will be on the point of tipping around D and $T_C = 0$.



Taking moments about D:

$$W\times 1=V\times 6$$

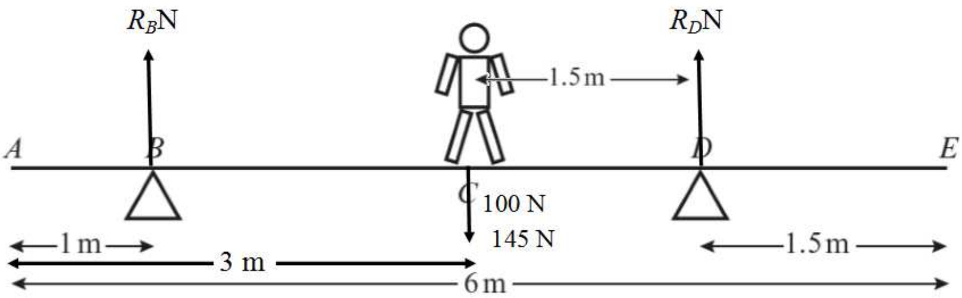
$$V=\frac{240\times 1}{6}$$

$$=40$$

The maximum value of V that keeps the beam level is 40 N.

Moments Mixed exercise 4

1 a



The plank is in equilibrium.
Let the reaction forces at the supports be R_B and R_D.
Considering moments about point D:

$$R_B \times (6 - 1.5 - 1) = (100 + 145) \times (3 - 1.5)$$
$$3.5R_B = 245 \times 1.5$$
$$3.5R_B = 367.5$$
$$R_B = 105$$

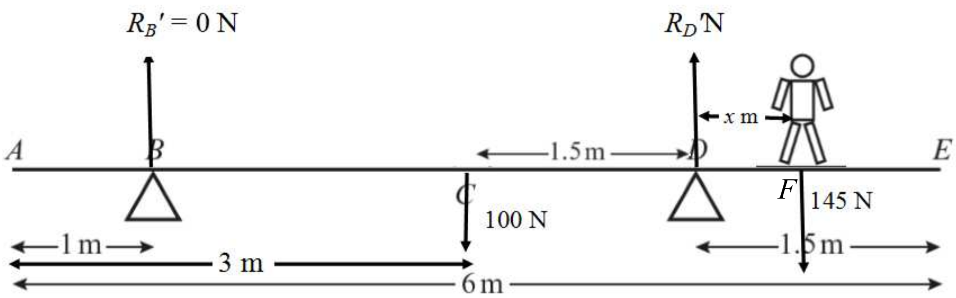
The support at B exerts a force of 105 N on the plank.

b The plank is in equilibrium.
Resolving vertically:

$$R_B + R_D = 100 + 145$$
$$R_D = 245 - 105$$
$$R_D = 140$$

The support at D exerts a force of 140 N on the plank.

c

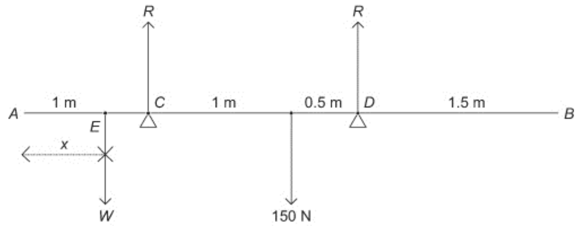


When the plank is on the point of tilting, the new reaction force at support B, $R'_B = 0$ N and plank is again in equilibrium. The child stands a distance x m from support D.
Considering moments about point D:

$$145x = 100 \times (3 - 1.5)$$
$$145x = 150$$
$$x = 1.03$$

The distance DF is 1.03 m.

2



a Since the rod is uniform, the centre of mass is at the mid-point.
Taking moments about A:

$$Wx + 150 \times 2 = R \times 1 + R \times 2.5,$$

$$Wx + 300 = 3.5R \qquad \textbf{(1)}$$

$$R(\uparrow):\ W + 150 = R + R,$$

$$2R = W + 150$$

$$R = \frac{W+150}{2} \qquad \textbf{(2)}$$

Sub (**2**) into (**1**) gives:

$$Wx + 300 = \frac{7}{2} \times \frac{W+150}{2}$$

$$4(Wx + 300) = 7W + 7 \times 150$$

$$4Wx + 1200 = 7W + 1050$$

$$1200 - 1050 = 7W - 4Wx$$

$$W(7 - 4x) = 150$$

$$W = \frac{150}{7-4x}$$

b The range of values of x are:

$$x \geq 0 \text{ and } \frac{150}{7-4x} > 0$$

$$\Rightarrow 7 - 4x > 0$$

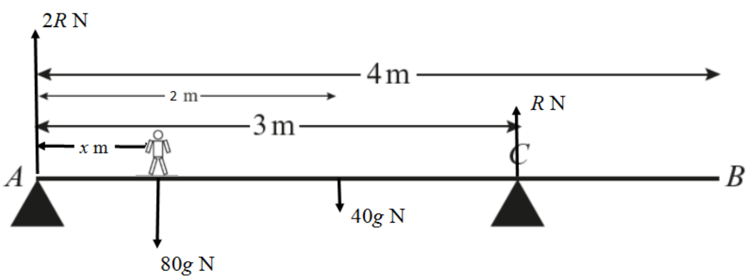
$$4x < 7$$

$$x < \frac{7}{4}$$

$$x < 1.75$$

So $0 \leq x < 1.75$

3 a



The plank is in equilibrium.

3 a Resolving vertically:

$$2R + R = 40g + 80g$$
$$3R = 120 \times 9.8$$
$$3R = 1176$$
$$R = 392$$

The value of R is 392 N.

b Taking moments about A:

$$80gx + (40g \times 2) = 3R$$
$$80g(x+1) = 3 \times 392$$
$$x + 1 = \frac{1176}{80 \times 9.8}$$
$$x + 1 = 1.5$$

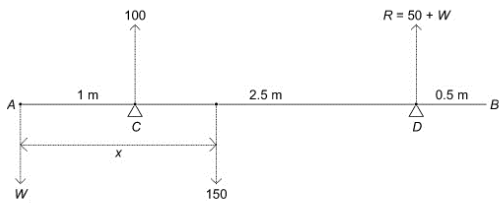
The man stands 0.5 m from A.

c i Assuming the plank is uniform means the weight of the plank acts at its centre of mass: i.e. halfway along the plank.

ii Assuming the plank is a rod means its width can be ignored.

iii Assuming the man is a particle means all his weight acts at the point at which he stands.

4 a



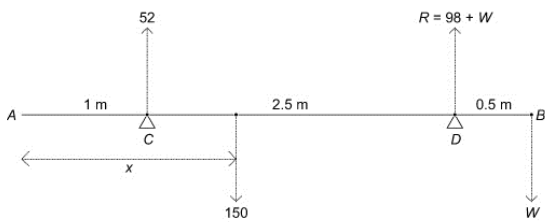
$R(\uparrow)$:

$$100 + R = W + 150$$
$$R = W + 50$$

Taking moments about A:

$$100 \times 1 + (W + 50) \times 3.5 = 150 \times x$$
$$100 + 175 + 3.5W = 150x$$
$$275 + 3.5W = 150x$$
$$550 + 7W = 300x$$

b



$R(\uparrow)$

4 b

$$52 + R = 150 + W$$
$$R = 150 + W - 52$$
$$= 98 + W$$

Taking moments about B:

$$52 \times 3 + (98 + W) \times 0.5 = 150 \times (4 - x)$$
$$156 + 49 + 0.5W = 600 - 150x$$

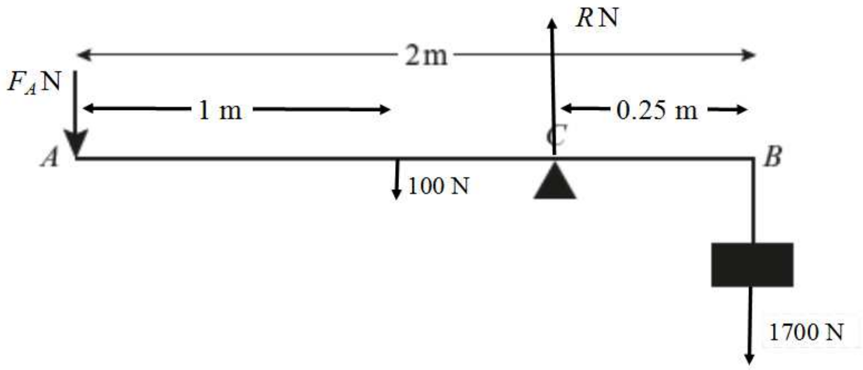
Doubling,

$$410 + W = 1200 - 300x$$
$$W = 790 - 300x$$

c Solving the simultaneous equations obtained in **a** and **b**:

$$\Rightarrow W = 790 - (550 + 7W)$$
$$8W = 790 - 550$$
$$= 240$$
$$\Rightarrow W = 30$$
$$\Rightarrow 410 + 30 = 1200 - 300x$$
$$300x = 760$$
$$x = 2.53 \text{ (3 s.f.)}$$

5 a

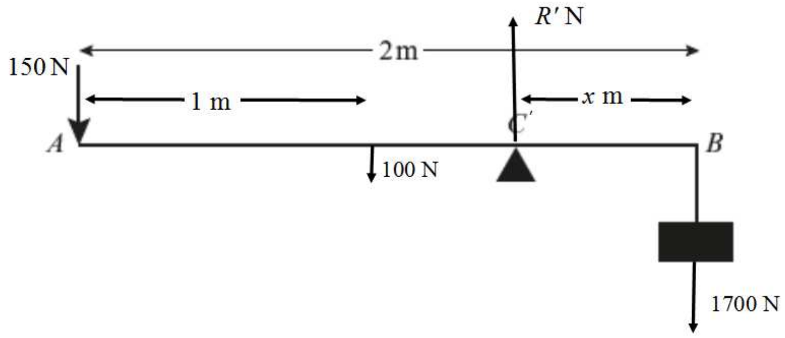


The lever is in equilibrium.
Considering moments about point C:

$$F_A(2 - 0.25) + 100(1 - 0.25) = 1700 \times 0.25$$
$$1.75F_A + 75 = 425$$
$$F_A = \frac{425 - 75}{1.75}$$
$$F_A = 200$$

The force at A is 200 N.

5 b

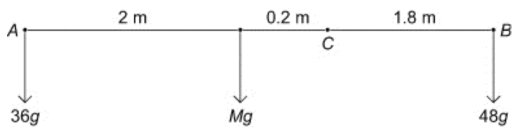


The lever is again in equilibrium. Let x be the distance of the pivot from B.
Considering moments about the new support position C':

$$150(2-x)+100(1-x)=1700x$$
$$300-150x+100-100x=1700x$$
$$400=1700x+250x$$
$$400=1950x$$
$$x=0.205$$

The pivot is now 0.21 m from B (to the nearest cm).

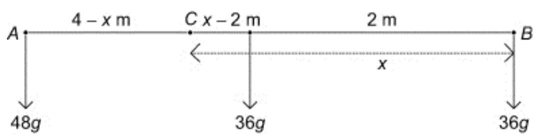
6 a Let the mass of the plank be M. Since the plank is uniform, its centre of mass is at its mid-point.



Taking moments about C:

$$48g\times 1.8 = Mg\times 0.2+36g\times 2.2$$
$$86.4g=0.2Mg+79.2g$$
$$86.4=0.2M+79.2$$
$$0.2M=86.4-79.2$$
$$=7.2$$
$$\Rightarrow M=36\,\text{kg}$$

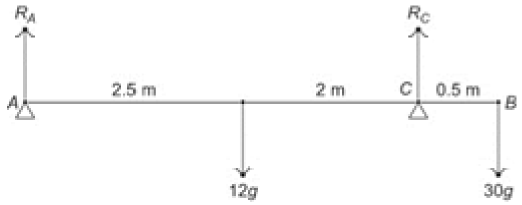
b Let the distance BC be x



Taking moments about C:

$$36gx+36g(x-2)=48g(4-x)$$
$$3x+3(x-2)=4(4-x)$$
$$6x-6=16-4x$$
$$10x=22$$
$$x=2.2\,\text{m}$$

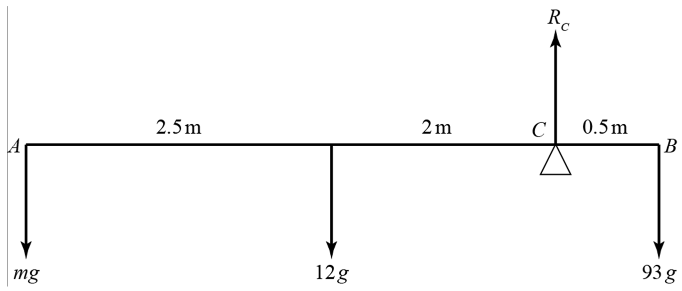
7 a



Taking moments about C:

$$R_A \times 4.5 + 30g \times 0.5 = 12g \times 2$$
$$R_A \times 4.5 = 24g - 15g$$
$$= 9g$$
$$\Rightarrow R_A = 2g$$
$$= 19.6\text{ N}$$

b



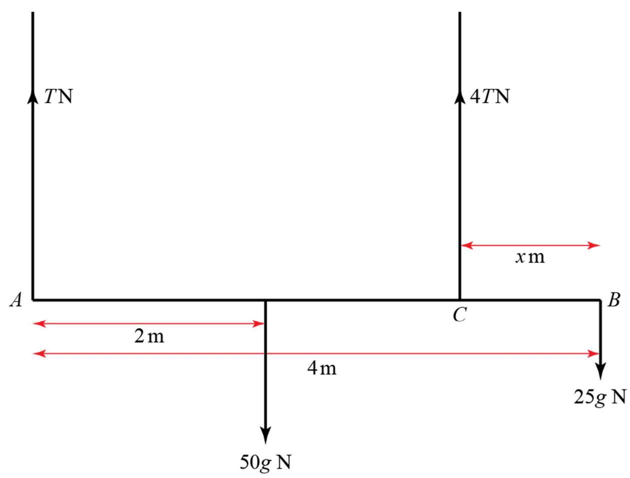
The plank is about to tilt about C
$\Rightarrow$ reaction at $A = 0$
Taking moments about C:

$$mg \times 4.5 + 12g \times 2 = 93g \times 0.5$$
$$4.5m = 93 \times 0.5 - 24$$
$$= 22.5$$
$$m = 5$$

8 The plank is in equilibrium.

Resolving vertically:

$$T + 4T = 50g + 25g$$
$$5T = 75g$$
$$T = 15g$$
$$4T = 60g$$

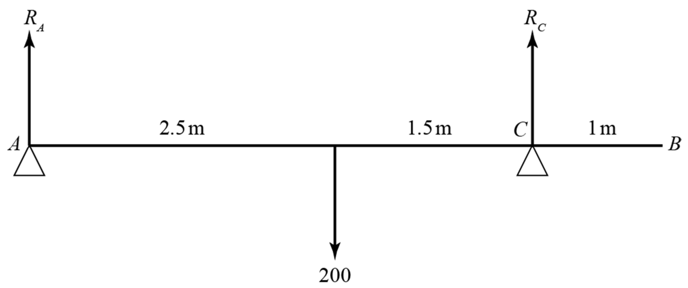


8 Considering moments about B:

$$(50g \times 2) = 60gx + (15g \times 4)$$
$$100g = 60gx + 60g$$
$$100g - 60g = 60gx$$
$$x = \frac{40g}{60g}$$
$$x = 0.666...$$

The distance from B to C is 0.67 m (to the nearest cm).

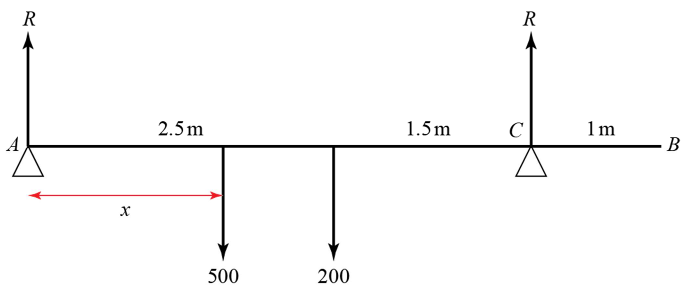
9 a



Taking moments about A:

$$200 \times 2.5 = R_C \times 4$$
$$R_C = 125\,\text{N}$$

b



Let the distance AD be x

$R(\uparrow)$

$$2R = 500 + 200$$
$$= 700$$
$$R = 350\,\text{N}$$

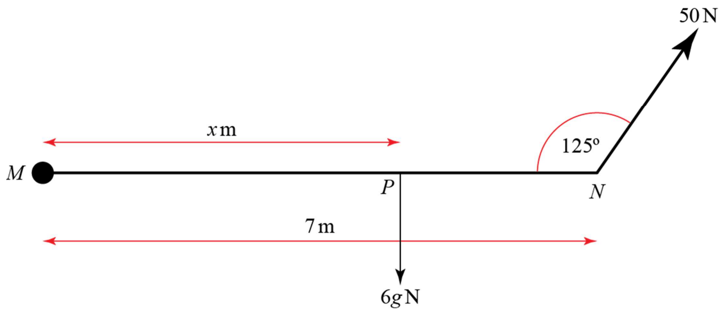
Taking moments about A:

$$R \times 4 = 200 \times 2.5 + 500 \times x$$
$$1400 = 4R$$
$$= 500 + 500x$$
$$900 = 500x$$
$$x = 1.8\,\text{m}$$

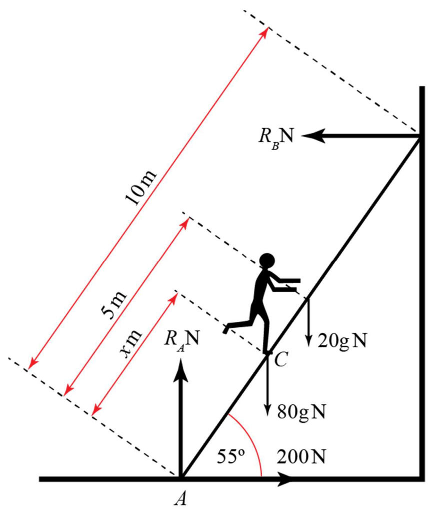
10 Distance $MP = x$ m
The plank is in equilibrium.
Taking moments about M:

$$6gx = 7 \times 50 \sin 55^\circ$$
$$x = \frac{350 \sin 55^\circ}{6 \times 9.8}$$
$$= 4.8759...$$

The distance MP is 4.88 m (3s.f.).



11



The ladder is in equilibrium. Let x be the distance AC.
Resolving horizontally:

$R_B = 200$ N

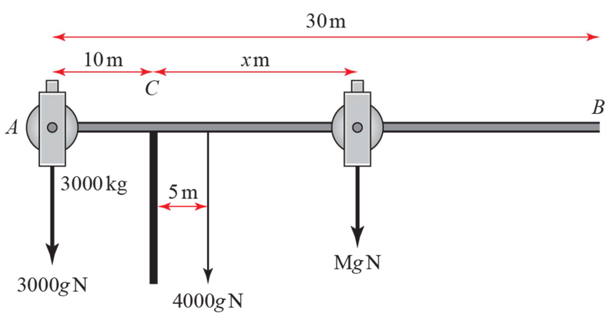
Considering moments about A:

$$(80g \times x \cos 55^\circ) + (20g \times 5 \cos 55^\circ) = 200 \times 10 \sin 55^\circ$$
$$(784 \cos 55^\circ)x = 2000 \sin 55^\circ - 980 \cos 55^\circ$$
$$x = \frac{2000 \sin 55^\circ - 980 \cos 55^\circ}{784 \cos 55^\circ}$$
$$x = 2.3932...$$

The distance AC is 2.39 m (to the nearest cm).

12 a Centre of mass of beam is 5 m from C.
Taking moments about C:

$$3000g \times 10 = (4000g \times 5) + Mgx$$
$$30\,000 = 20\,000 + Mx$$
$$M = \frac{10\,000}{x}$$



12 b Maximum load is when $x = 5$ m:

$$M = \frac{10\,000}{5} = 2000 \text{ kg}$$

Minimum load is when $x = 20$ m:

$$M = \frac{10\,000}{20} = 500 \text{ kg}$$

c It is not very accurate to model the beam as a uniform rod. Since the beam may taper at one end, the centre of mass of the beam may not lie in the middle of the beam.

Challenge

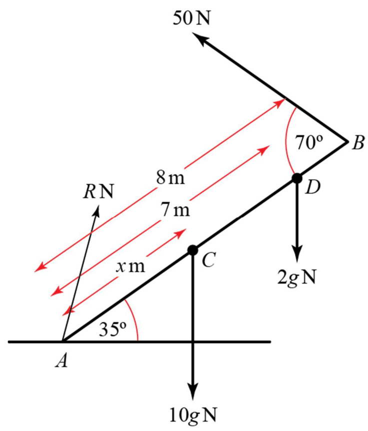
1 Let x be the distance from A to the centre of mass.
The beam is in equilibrium.
Taking moments about A:

$$10g \times x\cos 35^\circ + (2g \times 7\cos 35^\circ) = 8 \times 50\sin 70^\circ$$

$$10gx\cos 35^\circ = 400\sin 70^\circ - 14g\cos 35^\circ$$

$$x = \frac{400\sin 70^\circ - 137.2\cos 35^\circ}{98\cos 35^\circ} = 3.2822...$$

The centre of mass of the beam is 3.28 m from A (3s.f.).



2 a When force is a minimum, system is in limiting equilibrium.
Taking moments about P:

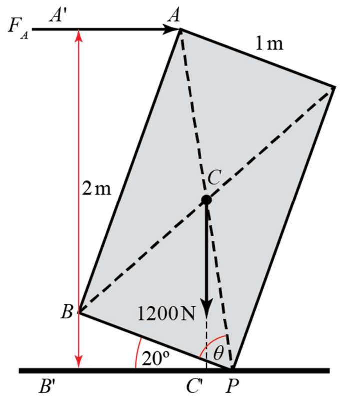
$$F_A \times (A'B') = 1200 \times PC' \qquad \textbf{(1)}$$

Finding $A'B'$:

$$A'B = 2\cos 20^\circ$$

$$BB' = 1\sin 20^\circ$$

$$\therefore A'B' = 2\cos 20^\circ + \sin 20^\circ$$

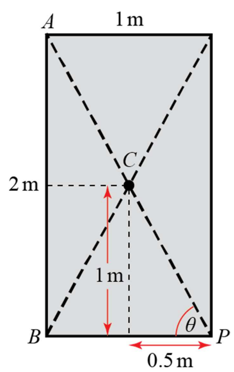


Finding PC':

$$PC' = PC\cos(\theta + 20)$$

$$(PC)^2 = 1^2 + 0.5^2$$

$$PC = \sqrt{1.25}$$



$$\tan\theta = \frac{1}{0.5}$$

$$\theta = 63.434...^\circ$$

$$PC' = \sqrt{1.25} \times \cos(63.4 + 20)^\circ$$

$$PC' = \sqrt{1.25} \times \cos 83.434...^\circ$$

Substituting values for $A'B'$ and PC' into equation **(1)**

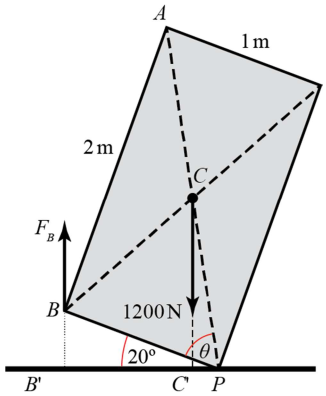
$$F_A \times \left(2\cos 20^\circ + \sin 20^\circ\right) = 1200 \times \sqrt{1.25} \times \cos 83.434...^\circ$$

$$F_A = \frac{1200 \times \sqrt{1.25} \times \cos 83.434...^\circ}{2\cos 20^\circ + \sin 20^\circ}$$

$$F_A = 69.051...$$

A horizontal force of 69.0 N at A will tip the refrigerator (3s.f.).

Challenge
2 b



When force is a minimum, system is in limiting equilibrium.
Taking moments about P:

$$F_B \times (PB') = 1200 \times \sqrt{1.25} \times \cos 83.434...^\circ$$

$$F_B \times 1\cos 20^\circ = 1200 \times \sqrt{1.25} \times \cos 83.434...^\circ$$

$$F_B = \frac{1200 \times \sqrt{1.25} \times \cos 83.434...^\circ}{\cos 20^\circ}$$

$$F_B = 163.25...$$

A vertical force of 163 N at *B* will tip the refrigerator (3s.f.).

Forces and friction 5A

1 a i $12\cos 20° = 11.3\,\text{N}$ (3s.f.)

ii $12\cos 70° = 12\sin 20°$
$= 4.10\,\text{N}$ (3s.f.)

iii $(11.3\mathbf{i} + 4.10\mathbf{j})$ N

b i $5\cos 90° = 0\,\text{N}$

ii $-5\cos 0° = 5\cos 180°$
$= -5\,\text{N}$

iii $-5\mathbf{j}$ N

c i $-8\cos 50° = -5.14\,\text{N}$ (3 s.f.)

ii $8\cos 40° = 6.13\,\text{N}$ (3 s.f.)

iii $(-5.14\mathbf{i} + 6.13\mathbf{j})$ N

d i $-6\cos 50° = -3.86\,\text{N}$ (3 s.f.)

ii $-6\cos 40° = -4.60\,\text{N}$ (3 s.f.)

iii $(-3.86\mathbf{i} - 4.60\mathbf{j})$ N

2 a i $8\cos 60° - 6 = -2\,\text{N}$

ii $8\cos 30° - 0 = 6.93\,\text{N}$ (3 s.f.)

b i $6\cos 40° + 5\cos 45° = 8.13\,\text{N}$ (3 s.f.)

ii $10 + 6\cos 50° - 5\cos 45° = 10.3\,\text{N}$ (3 s.f.)

c i $P\cos\alpha + Q - R\cos(90° - \beta) = P\cos\alpha + Q - R\sin\beta$

ii $P\cos(90° - \alpha) - R\cos\beta = P\sin\alpha - R\cos\beta$

3 a Using the cosine rule:

$R^2 = 25^2 + 35^2 - (2 \times 25 \times 35 \cos 80^\circ)$

$R^2 = 1850 - 303.88...$

$R = 39.320...$

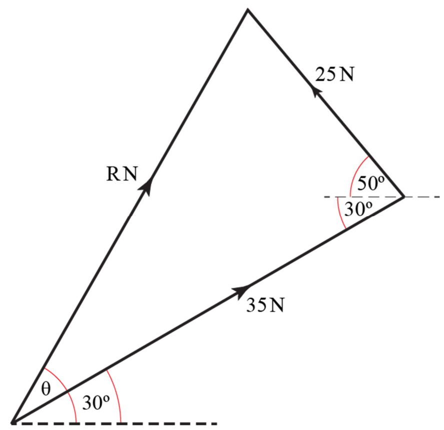
Using the sine rule:

$\frac{\sin(\theta - 30^\circ)}{25} = \frac{\sin 80^\circ}{39.320...}$

$\sin(\theta - 30^\circ) = \frac{25 \sin 80^\circ}{39.320...}$

$(\theta - 30^\circ) = 38.765...^\circ$

$\theta = 68.765...^\circ$



The resultant force has a magnitude of 39.3 N (3s.f.) and acts at 68.8° above the horizontal (3s.f.).

b Using the cosine rule:

$R^2 = 20^2 + 15^2 - (2 \times 20 \times 15 \cos 105^\circ)$

$R^2 = 780.29...$

$R = 27.933...$

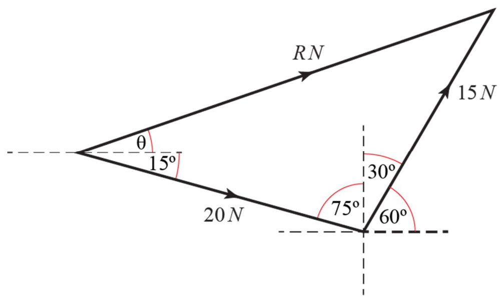
Using the sine rule:

$\frac{\sin(15^\circ + \theta)}{15} = \frac{\sin 105^\circ}{27.933...}$

$\sin(15^\circ + \theta) = \frac{15 \sin 105^\circ}{27.933...}$

$(15^\circ + \theta) = 31.244...^\circ$

$\theta = 16.244...^\circ$



The resultant force has a magnitude of 27.9 N (3s.f.) and acts at 16.2° above the horizontal (3s.f.).

c Using the cosine rule then the sine rule, as before:

$R^2 = 5^2 + 2^2 - (2 \times 5 \times 2 \cos 5^\circ)$

$R^2 = 9.0761...$

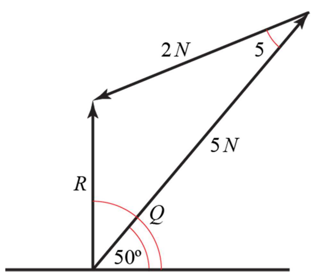
$R = 3.0126...$

$\frac{\sin(\theta - 50^\circ)}{2} = \frac{\sin 5^\circ}{3.0126...}$

$\sin(\theta - 50^\circ) = \frac{2 \sin 5^\circ}{3.0126...}$

$(\theta - 50^\circ) = 3.3169...^\circ$

$\theta = 53.316...^\circ$



The resultant force has a magnitude of 3.01 N (3s.f.) and acts at 53.3° above the horizontal (3s.f.).

4 a Resolving horizontally:

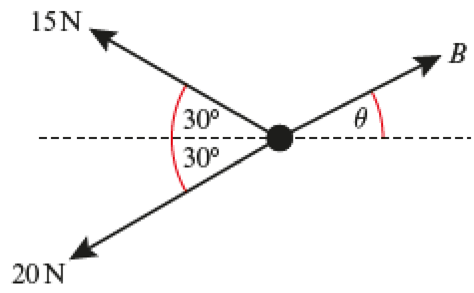
$B\cos\theta = 15\cos 30^\circ + 20\cos 30^\circ$

$B\cos\theta = 35\frac{\sqrt{3}}{2}$ **(1)**

Resolving vertically:

$B\sin\theta = -15\sin 30^\circ + 20\sin 30^\circ$

$B\sin\theta = \frac{5}{2}$ **(2)**



$\mathbf{(2)} \div \mathbf{(1)} \Rightarrow$

$\frac{B\sin\theta}{B\cos\theta} = \frac{5}{2} \times \frac{2}{35\sqrt{3}}$

$\tan\theta = \frac{1}{7\sqrt{3}}$

$\theta = 4.7150...$

$\mathbf{(1)}^2 + \mathbf{(2)}^2 \Rightarrow$

$B^2\left(\cos^2\theta + \sin^2\theta\right) = \left(35\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{5}{2}\right)^2$

$B^2 = \frac{(35^2 \times 3) + 25}{4}$

$B = \frac{\sqrt{3700}}{2}$

$= 30.413...$

B has a magnitude of 30.4 N (3s.f.) and acts at 4.72 to the horizontal (3s.f.).

b Resolving horizontally:

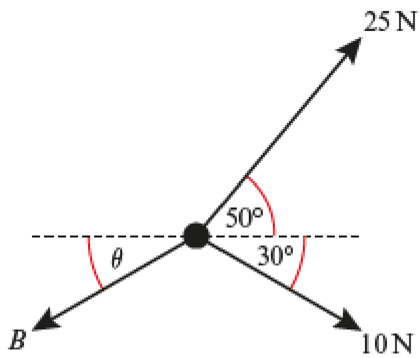
$B\cos\theta = 25\cos 50^\circ + 10\cos 30^\circ$

$B\cos\theta = 24.729...$ **(1)**

Resolving vertically:

$B\sin\theta = -10\sin 30^\circ + 25\sin 50^\circ$

$B\sin\theta = 14.151...$ **(2)**



$\mathbf{(2)} \div \mathbf{(1)} \Rightarrow$

$\frac{B\sin\theta}{B\cos\theta} = \frac{14.151...}{24.729...}$

$\tan\theta = 0.57224...$

$\theta = 29.779...$

$\mathbf{(1)}^2 + \mathbf{(2)}^2 \Rightarrow$

$B^2\left(\cos^2\theta + \sin^2\theta\right) = 14.151...^2 + 24.729...^2$

$B = \sqrt{811.77...}$

$= 28.491...$

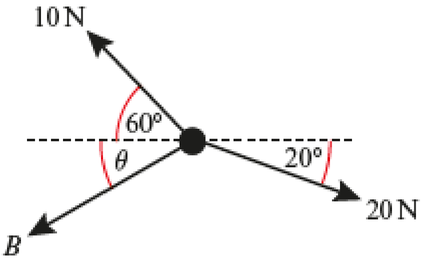
B has a magnitude of 28.5 N (3s.f.) and acts at 29.8°below the horizontal (3s.f.).

4 c Resolving horizontally:

$$B\cos\theta = 20\cos 20^\circ - 10\cos 60^\circ$$
$$B\cos\theta = 13.793... \qquad \mathbf{(1)}$$

Resolving vertically:

$$B\sin\theta = 10\sin 60^\circ - 20\sin 20^\circ$$
$$B\sin\theta = 1.8198... \qquad \mathbf{(2)}$$



$\mathbf{(2)} \div \mathbf{(1)} \Rightarrow$

$$\frac{B\sin\theta}{B\cos\theta} = \frac{1.8195...}{13.793...}$$
$$\tan\theta = 0.13193...$$
$$\theta = 7.5157...$$

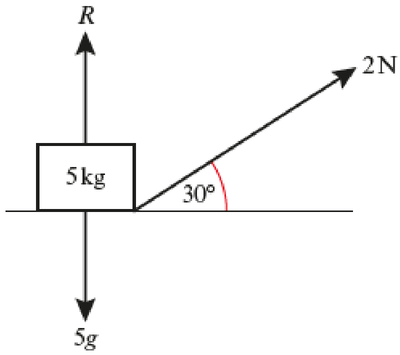
$\mathbf{(1)}^2 + \mathbf{(2)}^2 \Rightarrow$

$$B^2\left(\cos^2\theta + \sin^2\theta\right) = 1.8195...^2 + 13.793...^2$$
$$B = \sqrt{193.55...}$$
$$= 13.912...$$

B has a magnitude of 13.9 N (3s.f.) and acts at 7.52° below the horizontal (3s.f.).

5 a Using Newton’s second law and resolving horizontally:

$$F = ma$$
$$2\cos 30^\circ = 5a$$
$$2\frac{\sqrt{3}}{2} = 5a$$
$$a = \frac{\sqrt{3}}{5}$$



The box accelerates at $\frac{\sqrt{3}}{5}$ ms^{-2}

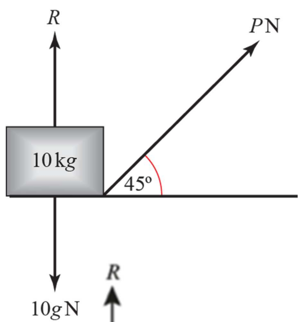
b Resolving vertically:

$$5g = R + 2\sin 30^\circ$$
$$R = (5\times 9.8) - 1$$
$$R = 48$$

The normal reaction of the box with the floor is 48 N.

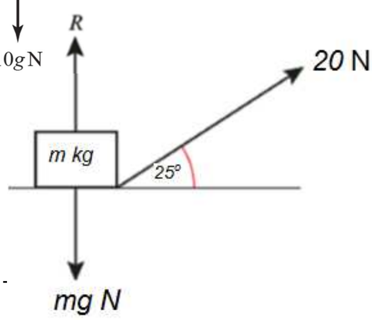
6 Using Newton’s second law and resolving horizontally:

$$F = ma$$
$$P\cos 45^\circ = 10\times 2$$
$$P = \frac{20}{\cos 45^\circ}$$
$$P = 20\sqrt{2}$$



The force P is $20\sqrt{2}$ N.

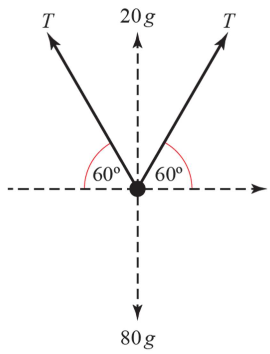
7 Using Newton’s second law and resolving horizontally:



$$F = ma$$
$$20\cos 25^\circ = 0.5m$$
$$m = 2 \times 20\cos 25^\circ$$
$$m = 36.252...$$

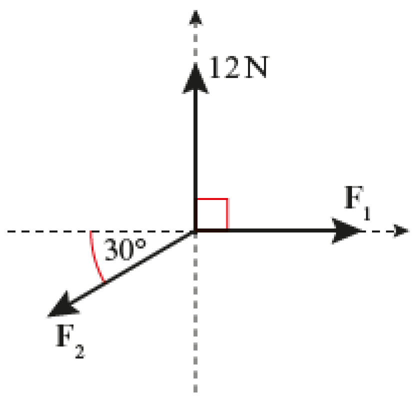
The mass of the box is 36.3 kg.

8 Resolving vertically:



$$20g + 2T\sin 60^o = 80g$$
$$2T\sin 60^o = 80g - 20g$$
$$2T\frac{\sqrt{3}}{2} = 60g$$
$$T = \frac{60g}{\sqrt{3}} = 20\sqrt{3}g \quad \text{as required.}$$

9 Resolving vertically:



$$2 = 12 - F_2\sin 30^\circ$$
$$F_2 = \frac{12-2}{\sin 30^\circ}$$
$$F_2 = 20$$

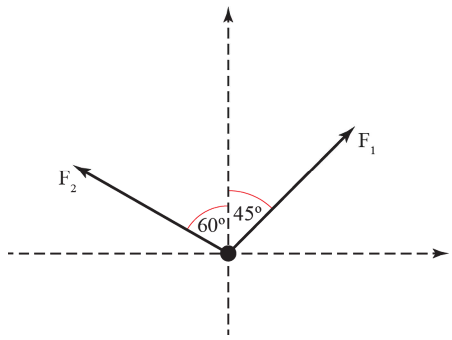
Resolving horizontally:

$$2\sqrt{3} = F_1 - F_2\cos 30^\circ$$
$$F_1 = 2\sqrt{3} + 20\cos 30^\circ$$
$$F_1 = 2\sqrt{3} + \frac{20\sqrt{3}}{2}$$
$$F_1 = 12\sqrt{3}$$

The forces F_1 and F_2 are $12\sqrt{3}$ N and 20 N respectively.

Challenge

Resolving vertically:



$$5 = F_1 \cos 45^\circ + F_2 \cos 60^\circ$$

$$5 = \frac{F_1}{\sqrt{2}} + \frac{F_2}{2}$$

$$\frac{F_1}{\sqrt{2}} = 5 - \frac{F_2}{2} \qquad \textbf{(1)}$$

Resolving horizontally:

$$3 = F_1 \sin 45^\circ - F_2 \sin 60^\circ$$

$$3 = \frac{F_1}{\sqrt{2}} - \frac{F_2\sqrt{3}}{2} \qquad \textbf{(2)}$$

Substituting $\frac{F_1}{\sqrt{2}} = 5 - \frac{F_2}{2}$ from (**1**), in (**2**):

$$3 = 5 - \frac{F_2}{2} - \frac{F_2\sqrt{3}}{2}$$

$$2 = \frac{F_2}{2} + \frac{F_2\sqrt{3}}{2}$$

$$4 = (\sqrt{3}+1)F_2$$

$$F_2 = \frac{4}{\sqrt{3}+1}$$

$$F_2 = \frac{4(\sqrt{3}-1)}{3-1}$$

$$F_2 = 2\sqrt{3} - 2$$

Substituting $F_2 = 2\sqrt{3} - 2$ in (**1**):

$$\frac{F_1}{\sqrt{2}} = 5 - \left(\frac{2\sqrt{3}-2}{2}\right)$$

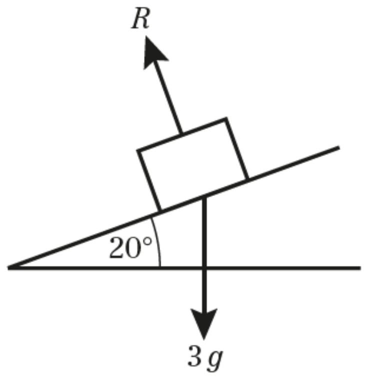
$$\frac{F_1}{\sqrt{2}} = 6 - \sqrt{3}$$

$$F_1 = 6\sqrt{2} - \sqrt{6}$$

The forces F_1 and F_2 are $6\sqrt{2} - \sqrt{6}$ N and $2\sqrt{3} - 2$ N respectively.

Forces and friction 5B

1 a



b R(↖):

$$R = 3g\cos 20^\circ$$
$$= 3 \times 9.8 \cos 20^\circ$$
$$= 27.626...$$

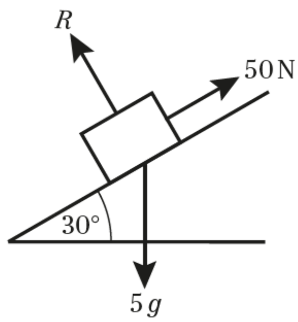
The normal reaction between the particle and the plane is 27.6 N (3s.f.).

c Using Newton's second law of motion and R(↙):

$$F = ma$$
$$3g\sin 20^\circ = 3a$$
$$a = 9.8 \times \sin 20^\circ$$
$$= 3.3517...$$

The acceleration of the particle is 3.35 ms^{-2} (3s.f.).

2 a



b R(↖):

$$R = 5g\cos 30^\circ$$
$$= 5 \times 9.8 \cos 30^\circ$$
$$= \frac{49\sqrt{3}}{2}$$
$$= 42.44..$$

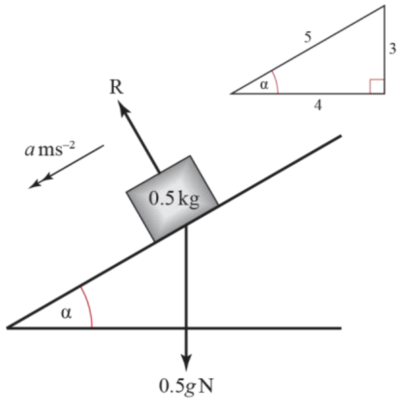
The normal reaction between the particle and the plane is 42.4 N.

c Using Newton's second law of motion and R(↙):

$$F = ma$$
$$50 - 5g\sin 30^\circ = 5a$$
$$a = 10 - \left(9.8 \times \frac{1}{2}\right) = 5.1$$

The acceleration of the particle is 5.1 ms^{-2}

3 $\tan\alpha = \frac{3}{4}$ so $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$



a R(↖):

$$R = 0.5g\cos\alpha$$
$$= 0.5\times9.8\times\frac{4}{5}$$
$$= 3.92$$

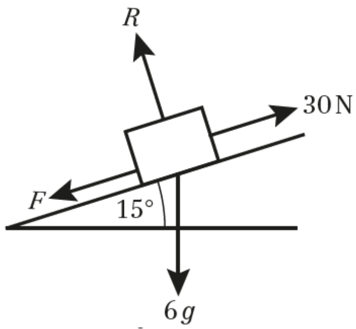
The normal reaction is 3.92 N.

b Using Newton's second law of motion and R(↙):

$$F = ma$$
$$0.5g\sin\alpha = 0.5a$$
$$a = 9.8\times\frac{3}{5}$$
$$= 5.88$$

The acceleration of the particle is 5.88 ms^{-2}

4 a

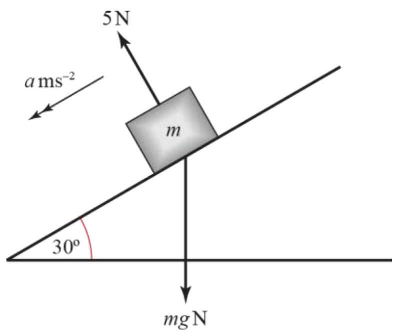


b Since mass is moving at constant speed, the resultant force parallel to the slope is zero.
R(↗):

$$30 = F + 6g\sin15^\circ$$
$$F = 30 - (6\times9.8\sin15^\circ)$$
$$= 14.781...$$

The resistance due to friction is 14.8 N (3s.f.).

5 a R(↖):



$$5 = mg\cos30^\circ$$
$$m = \frac{5}{9.8\times\frac{\sqrt{3}}{2}}$$
$$= 0.58913...$$

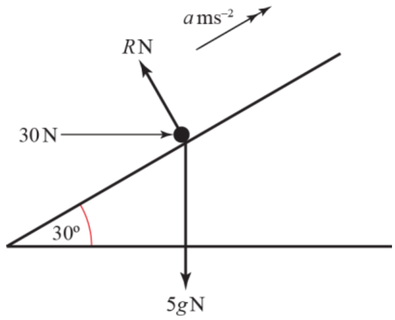
The mass of the particle is 0.589 kg (3s.f.).

b Using Newton's second law of motion and R(↙):

$$F = ma$$
$$mg\sin30^\circ = ma$$
$$a = 9.8\times\frac{1}{2} = 4.9$$

The acceleration of the particle is 4.9 ms^{-2}

6 Using Newton's second law of motion and R(↗):



$$F = ma$$

$$30\cos 30^\circ - 5g\sin 30^\circ = 5a$$

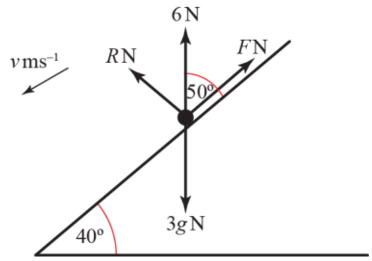
$$6\cos 30^\circ - g\sin 30^\circ = a$$

$$a = 6\frac{\sqrt{3}}{2} - \left(9.8\times\frac{1}{2}\right)$$

$$= 0.29615...$$

The acceleration of the particle is 0.296 ms^{-2} (3s.f.).

7 Since mass is moving at constant speed, the resultant force parallel to the slope is zero.
We are not told whether the mass is moving up or down the slope.
However, since the force acting down the slope is greater than the force acting up the slope, the particle must be moving down the slope.
Hence, friction acts up the slope to balance the forces.



R(↗):

$$F + 6\cos 50^\circ = 3g\sin 40^\circ$$

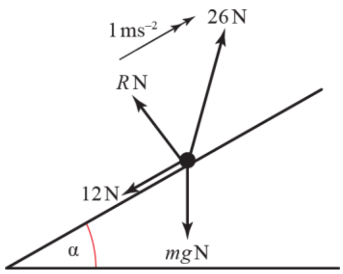
$$F = (3\times 9.8\sin 40^\circ) - 6\cos 50^\circ$$

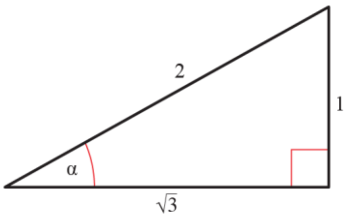
$$F = 15.041...$$

The frictional force is 15.0 N (3s.f.).

8 $\tan\alpha = \frac{1}{\sqrt{3}}$ so $\sin\alpha = \frac{1}{2}$ and $\cos\alpha = \frac{\sqrt{3}}{2}$

Using Newton's second law of motion and R(↗):





$$F = ma$$

$$26\cos 45^\circ - mg\sin\alpha - 12 = m\times 1$$

$$\frac{26}{\sqrt{2}} - \frac{9.8m}{2} - 12 = m$$

$$13\sqrt{2} - 12 = (4.9+1)m$$

$$m = \frac{13\sqrt{2}-12}{5.9} = 1.0821...$$

The mass of the particle is 1.08 kg (3s.f.).

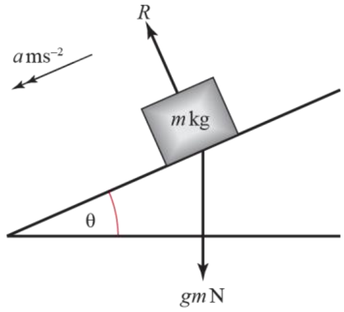
Challenge

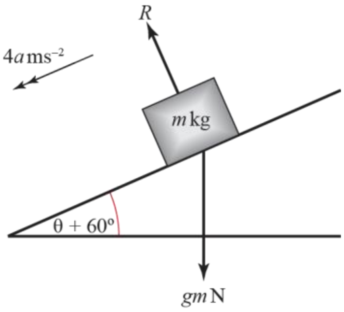
a Using $F = ma$ and R(↙) for the plane inclined at θ to the horizontal:

$mg\sin\theta = ma$ **(1)**

Using $F = ma$ and R(↙) for the plane inclined at $(\theta + 60^\circ)$ to the horizontal:

$mg\sin(\theta + 60^\circ) = 4ma$ **(2)**





Substituting **(1)** into **(2)** gives:

$mg\sin(\theta + 60^\circ) = 4mg\sin\theta$

$$4\sin\theta = \sin(\theta + 60^\circ)$$

$$4\sin\theta = \sin\theta\cos 60^\circ + \cos\theta\sin 60^\circ$$

$$4\sin\theta = \frac{1}{2}\sin\theta + \frac{\sqrt{3}}{2}\cos\theta$$

$$\frac{7}{2}\sin\theta = \frac{\sqrt{3}}{2}\cos\theta$$

$$\tan\theta = \frac{\sqrt{3}}{7}$$

b $\theta = \tan^{-1}\dfrac{\sqrt{3}}{7} = 13.9^\circ$

Forces and friction 5C

1 a i $R(-)$

$$R - 5g = 0$$
$$R = 5g$$
$$= 49 \text{ N}$$
$$\therefore F_{MAX} = \frac{1}{7} \times 49$$
$$= 7 \text{ N}$$

Since the driving force is only 3 N, the friction will only need to be 3 N to prevent the block from slipping, so $F = 3$ N .

ii Since driving force is equal to frictional force, body remains at rest in equilibrium.

b i $F_{MAX} = 7$ N (from part **a**), and driving force is 7 N, so friction will need to be at its maximum value to prevent the block from slipping, i.e. $F = 7$ N .

ii F is equal to the driving force of 7 N, so the body remains at rest in limiting equilibrium.

c i $F_{MAX} = 7$ N (from part **a**), and driving force is 12 N, so friction will be at its maximum value of 7 N.

ii Since the driving force is greater than the frictional force, there is a resultant force and the body accelerates.

iii $R(\rightarrow)$

$$F = ma$$
$$12 - 7 = 5a$$
$$a = 1 \text{ms}^{-2}$$

Body accelerates at 1ms^{-2}

d i $R(-)$

$$R - 14 - 5g = 0$$
$$R = 63 \text{ N}$$
$$\therefore F_{MAX} = \mu R$$
$$= \frac{1}{7} \times 63$$
$$= 9 \text{ N}$$

Since the driving force is only 6 N, the friction will only need to be 6 N to prevent the block from slipping, so $F = 6$ N .

ii Since driving force is equal to frictional force, body remains at rest in equilibrium.

e i $F_{MAX} = 9$ N (from part **d**), and driving force is 9 N, so friction will need to be at its maximum value to prevent the block from slipping, i.e. $F = 9$ N .

ii F is equal to the driving force of 9 N, so the body remains at rest in limiting equilibrium.

1 f i $F_{MAX} = 9\text{ N}$ (from part **d**), and driving force is 12 N, so friction will be at its maximum value of 9 N.

ii Since the driving force is greater than the frictional force, there is a resultant force and the body accelerates.

iii $R(\rightarrow)$

$$F = ma$$
$$12 - 9 = 5a$$
$$a = 0.6\text{ ms}^{-2}$$

Body accelerates at 0.6 m s^{-2}

g i $R(\uparrow)$

$$R + 14 - 5g = 0$$
$$R = 35\text{ N}$$
$$\therefore F_{MAX} = \mu R$$
$$= \tfrac{1}{7} \times 35$$
$$= 5\text{ N}$$

Since the driving force is only 3 N, the friction will only need to be 3 N to prevent the block from slipping, so $F = 3\text{ N}$.

ii Since driving force is equal to frictional force, body remains at rest in equilibrium.

h i $F_{MAX} = 5\text{ N}$ (from part **g**), and driving force is 5 N, so friction will need to be at its maximum value to prevent the block from slipping, i.e. $F = 5\text{ N}$.

ii F is equal to the driving force of 5 N, so the body remains at rest in limiting equilibrium.

i i $F_{MAX} = 5\text{ N}$ (from part **g**), and driving force is 6 N, so friction will be at its maximum value of 5 N

ii Since the driving force is greater than the frictional force, there is a resultant force and the body accelerates.

iii $R(\rightarrow)$

$$F = ma$$
$$6 - 5 = 5a$$
$$a = 0.2\text{ m s}^{-2}$$

Body accelerates at 0.2 m s^{-2}

1 j i $R(\uparrow)$

$$R + 14\sin 30° - 5g = 0$$
$$R = 42 \text{ N}$$
$$\therefore F_{MAX} = \mu R$$
$$= \frac{1}{7} \times 42$$
$$= 6 \text{ N}$$

Considering horizontal forces:

Driving force $- F_{MAX} = 14\cos 30° - 6 > 0$, so $F = F_{MAX} = 6$ N

ii Since the driving force is greater than the frictional force, there is a resultant force and the body accelerates.

iii $R(\rightarrow)$

$$F = ma$$
$$14\cos 30° - 6 = 5a$$
$$a = 1.22 \text{ m s}^{-2} \text{ (3 s.f.)}$$

Body accelerates at 1.22 m s^{-2} (3 s.f.)

k i $R(\uparrow)$

$$R + 28\sin 30° - 5g = 0$$
$$R = 35 \text{ N}$$
$$\therefore F_{MAX} = \mu R$$
$$= \frac{1}{7} \times 35$$
$$= 5 \text{ N}$$

Considering horizontal forces:

Driving force $- F_{MAX} = 28\cos 30° - 5 > 0$, so $F = F_{MAX} = 5$ N

ii Since the driving force is greater than the frictional force, there is a resultant force and the body accelerates.

iii $R(\rightarrow)$

$$F = ma$$
$$28\cos 30° - 5 = 5a$$
$$a = 3.85 \text{ m s}^{-2} \text{ (3 s.f.)}$$

Body accelerates at 3.85 m s^{-2} (3 s.f.)

1 l i $R(-)$

$$R - 56\cos 45° - 5g = 0$$
$$\therefore R = 88.6 \text{ N} \ \ (3 \text{ s.f.})$$
$$\therefore F_{MAX} = \mu R$$
$$= \frac{1}{7} \times 88.6$$
$$= 12.657 \text{ N}$$

Considering horizontal forces:

Driving force $- F_{MAX} = 56\sin 45° - 12.657 > 0$, so $F = F_{MAX} = 12.7$ N (3 s.f.)

ii Since the driving force is greater than the frictional force, there is a resultant force and the body accelerates.

iii $R(\rightarrow)$

$$F = ma$$
$$56\sin 45° - 12.657 = 5a$$
$$5a = 26.941$$
$$a = 5.388 \text{ m s}^{-2}$$

So the acceleration is 5.39 m s^{-2} (3 s.f.)

2 a $R(-)$

$$R + 20\sin 30° - 10g = 0$$
$$R = 88 \text{ N}$$

$R(\rightarrow)$

$$F = ma$$
$$20\cos 30° - \mu \times 88 = 10 \times 1$$
$$\mu = 0.083 \ (2 \text{ s.f.})$$

b $R(-)$

$$R + 20\cos 30° - 10g = 0$$
$$R = 80.679... \text{N}$$

$R(\rightarrow)$

$$F = ma$$
$$20\cos 60° - \mu \times 80.679 = 10 \times 0.5$$
$$\mu = 0.062 \ \ (2 \text{ s.f.})$$

c $R(-)$

$$R - 20\sqrt{2}\sin 45° - 10g = 0$$
$$R = 118 \text{N}$$

$R(\rightarrow)$

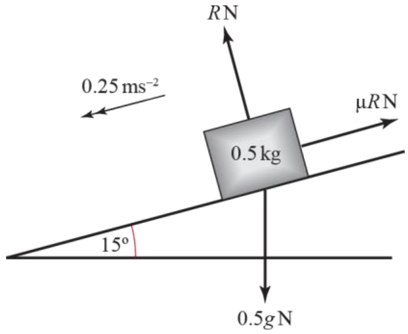
$$20\sqrt{2}\cos 45° - \acute{m}\ \ 118 = 10\,\acute{}\ \ 0.5$$
$$m = 0.13 \ (2 \text{ s.f.})$$

3 R(↖):

$R = 0.5g\cos 15^\circ$

$= 0.5 \times 9.8\cos 15^\circ$

$= 4.7330...$



Using Newton's second law of motion and R(↙):

$$F = ma$$

$$0.5g\sin 15^\circ - \mu R = 0.5 \times 0.25$$

$$\mu R = (0.5 \times 9.8\sin 15^\circ) - 0.125$$

$$\mu = \frac{1.2682... - 0.125}{4.7330...}$$

$$= 0.24153...$$

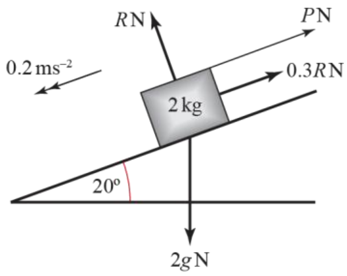
The coefficient of friction is 0.242 (3s.f.).

4 R(↖):

$R = 2g\cos 20^\circ$

$= 2 \times 9.8\cos 20^\circ$

$= 18.418...$



Using Newton's second law of motion ($F = ma$) and R(↙):

$$2g\sin 20^\circ - 0.3R - P = 2 \times 0.2$$

$$(2 \times 9.8\sin 20^\circ) - (0.3 \times 18.418...) - 0.4 = P$$

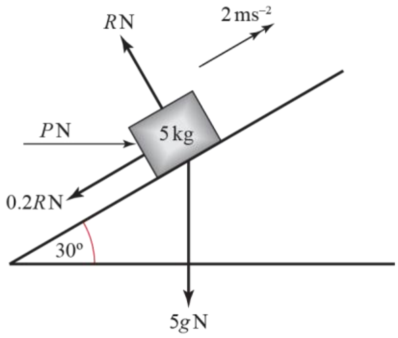
$$P = 0.7782...$$

The force P is 0.778 N (3s.f.).

5 R(↖):

$R = 5g\cos 30^\circ + P\sin 30^\circ$

$= \frac{49\sqrt{3}}{2} + \frac{P}{2}$



Using Newton's second law of motion and R(↗):

$P\cos 30^\circ - 5g\sin 30^\circ - 0.2R = 5 \times 2$

$$P\cos 30^\circ = 10 + 5g\sin 30^\circ + \frac{1}{5}\left(\frac{P}{2} + \frac{49\sqrt{3}}{2}\right)$$

$$\left(\frac{\sqrt{3}}{2} - \frac{1}{10}\right)P = 10 + \frac{5 \times 9.8}{2} + \frac{49\sqrt{3}}{10}$$

$$\left(5\sqrt{3} - 1\right)P = 100 + 245 + 49\sqrt{3}$$

$$P = \frac{429.8704896}{7.6602...} = 56.117...$$

The force P is 56.1 N (3s.f.).

6 Resolving vertically:

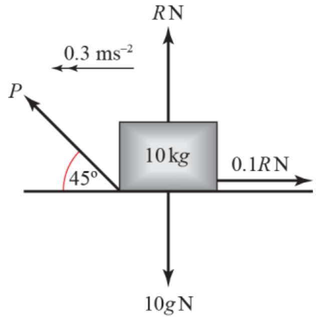
$R + P\sin 45^\circ = 10g$

$P\sin 45^\circ = 10g - R$ **(1)**

Resolving horizontally and using $F = ma$:

$P\cos 45^\circ - 0.1R = 10 \times 0.3$

$P\cos 45^\circ = 3 + 0.1R$ **(2)**



Since $\sin 45^\circ = \cos 45^\circ$, we can equate (**1**) and (**2**):

$10g - R = 3 + 0.1R$

$1.1R = 10g - 3$

$R = \dfrac{(10 \times 9.8) - 3}{1.1}$

$= 86.3636...$

Sub $R = 86.36$ into (**1**):

$P\sin 45^\circ = 10g - 86.36$

$P = \dfrac{(10 \times 9.8) - 86.36}{\sin 45^\circ} = 16.45...$

The force P is 16.5 N (3s.f.).

7 **a** $v = 0$ ms^{-1}, $u = 30$ ms^{-1}, $t = 20$ s, $a = ?$

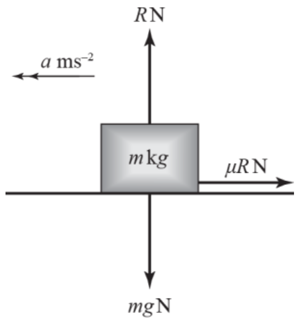
$v = u + at$

$0 = 30 + 20a$

$a = -\dfrac{20}{30} = -\dfrac{2}{3}$

Resolving vertically:

$R = mg$



Since the wheels lock up, the force which causes the deceleration is the maximum frictional force between the wheels and the track.

Resolving horizontally and using Newton's second law:

$-\mu R = -\dfrac{2}{3}m$

$-\mu mg = -\dfrac{2}{3}m$

$\mu g = \dfrac{2}{3}$

$\mu = \dfrac{2}{3g}$

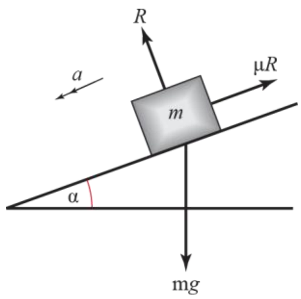
7 b Suppose there is an added constant resistive force of air resistance, A, where $A > 0$
Resolving horizontally and using Newton's second law:

$$\mu mg + A = \frac{2}{3}m$$

$$\mu = \frac{2}{3g} - \frac{A}{mg} < \frac{2}{3g}$$

So the coefficient of friction found by the second model is less than the coefficient of friction found by the first model.

Challenge



R(↖):
$R = mg\cos\alpha$

Using Newton's second law of motion and R(↙):

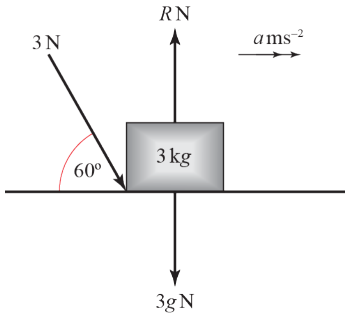
$$mg\sin\alpha - \mu R = ma$$

$$mg\sin\alpha - \mu mg\cos\alpha = ma$$

$$g(\sin\alpha - \mu\cos\alpha) = a$$

Since m does not appear in this expression, a is independent of m.

Forces and friction Mixed exercise 5



1 a Resolving vertically:

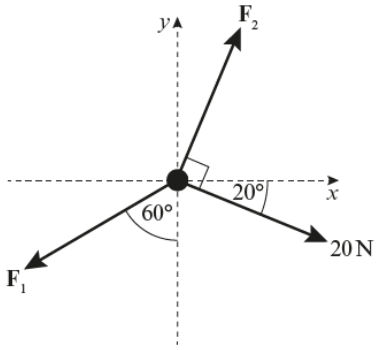
$$R = 3g + 3\sin 30^o$$
$$= (3\times 9.8) + 3\frac{\sqrt{3}}{2}$$
$$= 31.998...$$

The normal reaction of the floor on the box is 32.0 N (3s.f.).

b Resolving horizontally and using $F = ma$:

$$3\cos 60^o = 3a$$
$$a = 0.5$$

The acceleration of the box is 0.5 ms^{-2}



2 Resolving vertically (**j** components):

$$F_2\cos 20^\circ - F_1\cos 60^\circ - 20\sin 20^\circ = 2$$
$$F_2\cos 20^\circ - F_1\cos 60^\circ = 2 + 20\sin 20^\circ$$
$$F_2\cos 20^\circ - F_1\cos 60^\circ = 8.8404... \qquad \textbf{(1)}$$

Resolving horizontally (**i** components):

$$F_2\sin 20^\circ + 20\cos 20^\circ - F_1\sin 60^\circ = 3$$
$$20\cos 20^\circ - 3 + F_2\sin 20^\circ = F_1\sin 60^\circ$$
$$\frac{20\cos 20^\circ - 3 + F_2\sin 20^\circ}{\sin 60^\circ} = F_1$$
$$18.237 + 0.39493F_2 = F_1 \qquad \textbf{(2)}$$

Substituting value for F_1 from (**2**) into (**1**):

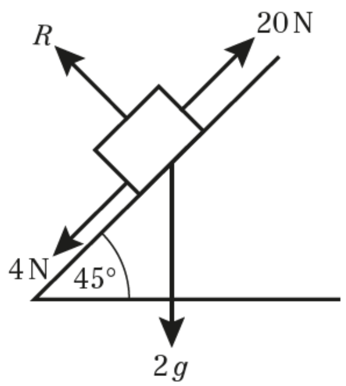
$$8.8404 = F_2\cos 20^\circ - (18.237 + 0.39493F_2)\cos 60^\circ$$
$$8.8404 = 0.93969F_2 - 9.1185 - 0.19746F_2$$
$$8.8404 + 9.1185 = (0.93969 - 0.19746)F_2$$
$$F_2 = \frac{17.958...}{0.7422...} = 24.196...$$

Substituting $F_2 = 24.196...$ into (**2**)

$$F_1 = 18.237 + (0.39493\times 24.196)$$
$$F_1 = 27.792...$$

The forces F_1 and F_2 are 27.8 N and 24.2 N respectively (both to 3s.f.).

3 a



3 b R(↖):

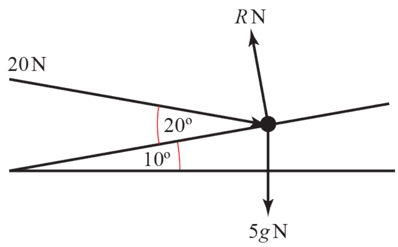
$$R = 2g\cos 45^\circ$$
$$= 2\times 9.8\cos 45^\circ$$
$$= 13.859...$$

The normal reaction between the particle and the plane is 13.9 N (3s.f.).

c Using Newton's second law of motion and R(↗):

$$F = ma$$
$$20 - 4 - 2g\sin 45^\circ = 2a$$
$$a = 8 - g\sin 45^\circ$$
$$a = 1.0703...$$
$$a = 1.1\text{ ms}^{-2}\text{ (2 s.f.)}$$

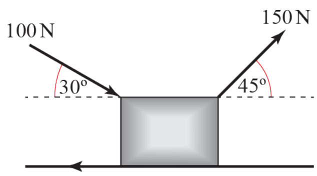
4 Using Newton's second law of motion and R(↗):



$$F = ma$$
$$20\cos 20^\circ - 5g\sin 10^\circ = 5a$$
$$4\cos 20^\circ - g\sin 10^\circ = a$$
$$a = 2.0570...$$

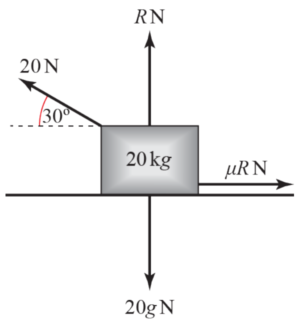
The acceleration of the particle is 2.06 ms^{-2} (3 s.f.) up the slope.

5 Since the box is moving at constant speed, the horizontal component of the resultant force is zero.
Resolving horizontally:



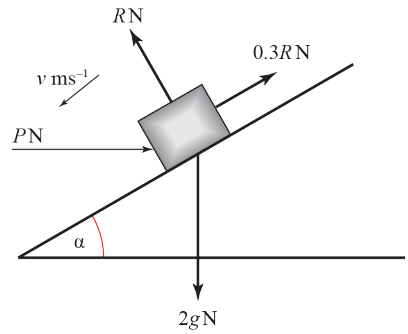
$$F = 150\cos 45^\circ + 100\cos 30^\circ$$
$$F = \frac{150}{\sqrt{2}} + \frac{100\sqrt{3}}{2}$$
$$F = \frac{50}{2}(3\sqrt{2} + 2\sqrt{3})$$
$$F = 25(3\sqrt{2} + 2\sqrt{3})\text{ N as required.}$$

6 Resolving vertically:
$R + 20\sin 30° = 20g$
So $R = 20g - 10$
Resolving horizontally:



$$20\cos 30^\circ = \mu R$$
$$20\cos 30^\circ = (20g - 10)$$
$$\mu = \frac{20\cos 30^\circ}{20g - 10}$$
$$\mu = \frac{20\sqrt{3}}{20\times 9.8 - 10} = \frac{5\sqrt{3}}{93}$$

7 $\tan\alpha = \frac{3}{4}$ so $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$



R(↖):

$$R = 2g\cos\alpha + P\sin\alpha$$

$$R = \frac{8}{5}g + \frac{3}{5}P$$

Particle moving at a constant velocity means that forces parallel to the slope are balanced.

R(↗):

$$P\cos\alpha + 0.3R = 2g\sin a$$

$$\frac{4}{5}P + \frac{3}{10}\left(\frac{8}{5}g + \frac{3}{5}P\right) = \frac{6}{5}g$$

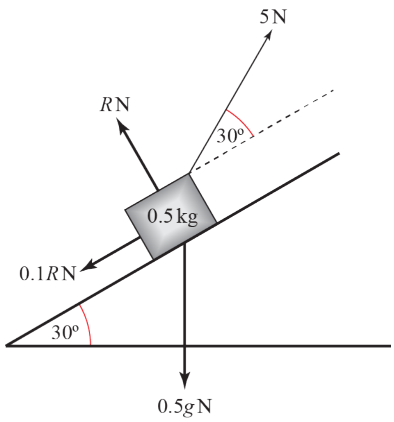
$$40P + 3(8g + 3P) = 60g$$

$$40P + 9P = 60g - 24g$$

$$P = \frac{36\times 9.8}{49} = 7.2$$

The force P is 7.2 N.

8 R(↖):



$$R + 5\sin 30^\circ = 0.5g\cos 30^\circ$$

$$R = \frac{g\sqrt{3}}{4} - \frac{5}{2}$$

Using Newton's second law of motion and R(↗):

$$ma = F$$

$$0.5a = 5\cos 30^\circ - 0.1R - 0.5g\sin 30^\circ$$

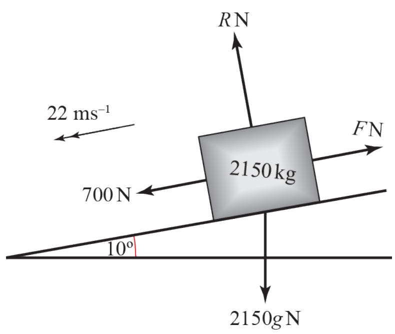
$$a = 5\sqrt{3} - \frac{1}{5}\left(\frac{g\sqrt{3}}{4} - \frac{5}{2}\right) - \frac{g}{2}$$

$$a = 5\sqrt{3} + \frac{1}{2} - \left(\frac{\sqrt{3}}{20} + \frac{1}{2}\right)\times 9.8$$

$$a = 3.4115...$$

The acceleration of the particle is 3.41 ms^{-2} (3s.f.).

9 a Since the car is travelling at constant speed, the resultant force parallel to the road is zero.



R(↗):

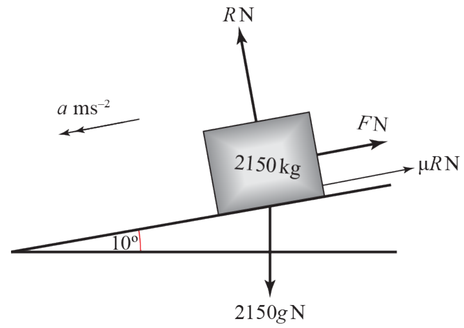
$$F = 700 + 2150g\sin 10^\circ$$

$$F = 700 + (2150\times 9.8\sin 10^\circ)$$

$$F = 4358.7...$$

The air resistance, F, is 4400 N (2s.f.).

9 b R(↖):



$R = 2150g\cos 10^\circ$
$= 2150 \times 9.8\cos 10^\circ$
$= 20750$

$R(\swarrow)$:

$u = 22\text{ ms}^{-1}$, $v = 0\text{ ms}^{-1}$, $s = 40\text{ m}$, $a = ?$

$v^2 = u^2 + 2as$

$0 = 22^2 + (2a \times 40)$

$a = -\dfrac{22^2}{80} = -6.05$

A negative value for acceleration indicates *deceleration* (or acceleration *up* the slope).

R(↗):

$$F = ma$$
$$F + \mu R - 2150g\sin 10^\circ = 2150a$$
$$4358.7 + 20750\mu - 2150g\sin 10^\circ = 2150 \times 6.05$$ (taking F from part **a**)
$$\mu = \frac{13007.5 - 700}{20750}$$
$$= 0.59313...$$

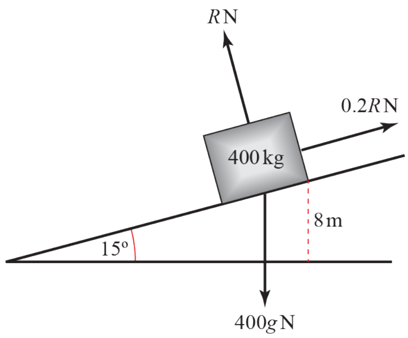
The coefficient of friction is 0.59 (2s.f.).

c e.g. The force due to air resistance will reduce as the car slows.
If the skid causes the tyres to heat, the value of μ is also likely to vary.

Challenge

At the instant the cable breaks, the boat is still moving up the slope with speed 5 ms^{-1}
At this time, friction acts down the slope and the boat will decelerate to instantaneous rest on the slipway.

When the boat is about to accelerate back down the slipway to the water, the forces acting on the boat are as shown:



To show that the boat will slide back down the slipway, we need to show that the component of weight acting down the slope is greater than the limiting friction:

$R(\nwarrow): R = 400g\cos 15^\circ$

Calculate magnitude of limiting frictional force acting up the slope:

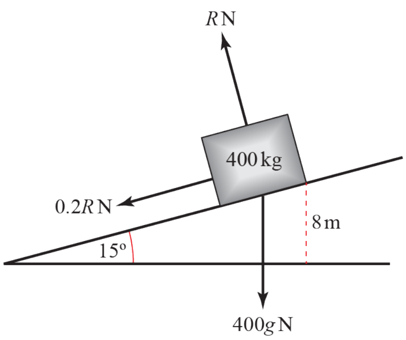
$$\begin{aligned} F_{MAX} &= \mu R \\ &= 0.2 \times 400g\cos 15^\circ \\ &= 80g\cos 15^\circ \\ &= 757.28... \end{aligned}$$

Component of weight acting down the slope $= 400g\sin 15^o$
$= 1014.5...$

$F_{MAX} = 757 \text{ N} < 1015 \text{ N} = \text{Weight down slope}$

Hence, boat will slide back down into the water.

While the boat continues to move up the slope after the cable breaks, μR continues to act down the slope.



Between the time when the cable breaks and the boat comes to instantaneous rest on the slipway, $R(\swarrow)$

$$F = ma$$

$$400g\sin 15^o + 80g\cos 15^o = 400a$$

$$a = \left(\sin 15^o + 0.2\cos 15^o\right)g$$

$$a = 4.4296...$$

Challenge (cont.)

$R(\nearrow)$: $u = 5\text{ ms}^{-1}$, $v = 0$, $a = -4.430\text{ ms}^{-2}$, $t = t_1$

$$v = u + at$$
$$0 = 5 - 4.43t_1$$
$$t_1 = \frac{5}{4.43} = 1.1287...$$

During time t_1 seconds, the boat has moved a distance s_1 up the slope.

$$s = \frac{v^2 - u^2}{2a}$$
$$s_1 = \frac{25^2}{2 \times 4.430} = 2.8219\text{ m}$$

Once the boat has come to rest, frictional force acts up the slope (see the first diagram).
$R(\swarrow)$:

$$F = ma$$
$$400g\sin 15^o - 80g\cos 15^o = 400a$$
$$a = \left(\sin 15^o - 0.2\cos 15^o\right)g$$
$$a = 0.64321$$

Distance along slipway from where cable breaks to the sea is 8 m:
Total distance travelled by boat to reach the water
= 8 + 2.8219 = 10.8219 m

$R(\swarrow)$: $u = 0\text{ ms}^{-1}$, $a = 0.6432\text{ ms}^{-2}$, $s = 10.8219\text{ m}$, $t = t_2$

$$s = ut + \frac{1}{2}at^2$$
$$10.8219 = 0t_2 + \frac{1}{2}0.6432\left(t_2\right)^2$$
$$10.8219 = 0.3216\left(t_2\right)^2$$
$$t_2 = \sqrt{\frac{10.8219}{0.3216}}$$
$$t_2 = 5.8008...$$

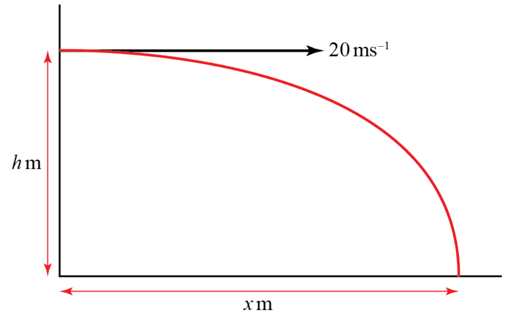
The boat returns to the sea t seconds after the cable snaps, where

$$t = t_1 + t_2$$
$$= 1.13 + 5.80$$
$$= 6.9\text{ s} \quad \text{(2 s.f.)}$$

Projectiles 6A

In this exercise, the positive direction is considered to be downwards.

1 a R(↓): $u_y = 0$, $t = 5$ s, $a = g = 9.8$ ms^{-2}, $s = h$



$$s = ut + \tfrac{1}{2}at^2$$
$$h = 0 + \frac{1}{2}\times 9.8\times 5^2$$
$$= 122.5$$

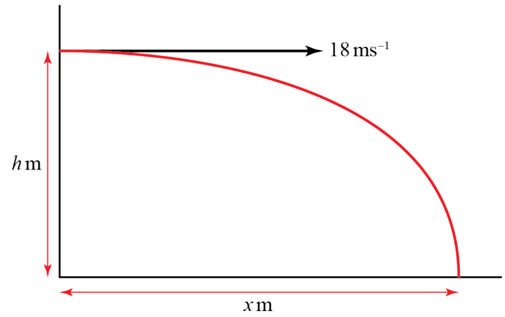
The height h is 122.5 m.

b R(→): $u_x = 20$ ms^{-1}, $t = 5$ s, $s = x$

$$s = vt$$
$$x = 20\times 5$$
$$= 100$$

The particle travels a horizontal distance of 100 m.

2 a R(→): $u_x = 18$ ms^{-1}, $t = 2$ s, $s = x$



$$s = vt$$
$$x = 18\times 2$$
$$= 36$$

R(↓): $u_y = 0$, $t = 2$ s, $a = g = 9.8$ ms^{-2}, $s = y$

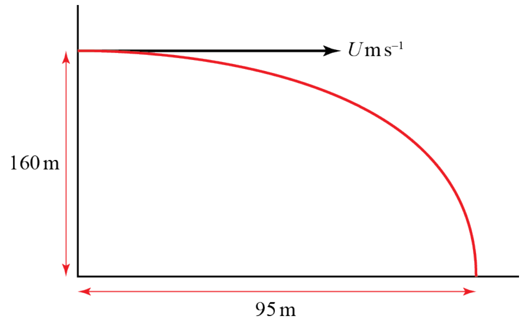
$$s = ut + \frac{1}{2}at^2$$
$$h = 0 + \frac{1}{2}\times 9.8\times 2^2$$
$$= 19.6$$

The horizontal and vertical components of the displacement are 36 m and 19.6 m respectively.

b $d^2 = 36^2 + 19.6^2$

$$d = \sqrt{1680.16} = 40.989...$$

The distance from the starting point is 41.0 m (3s.f.).



3 R(↓): $u_y = 0$, $a = g = 9.8\ \mathrm{ms^{-2}}$, $s = 160$ m, $t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$160 = 0 + \frac{1}{2}\times 9.8\times t^2$$

$$t^2 = \frac{160}{4.9}$$

$$t = \pm\frac{40}{7}$$

The negative root can be ignored.

R(→): $u_x = U$, $t = \dfrac{40}{7}$ s, $s = 95$ m

$$s = vt$$

$$95 = U\times\frac{40}{7}$$

$$U = \frac{7\times 95}{40} = 16.625$$

The projection speed is $16.6\ \mathrm{ms^{-1}}$ (3s.f.).

4 $R(\downarrow)$

$$u = 0,\quad s = 16,\quad a = 9.8,\quad t = ?$$

$$s = ut + \frac{1}{2}at^2$$

$$16 = 0 + 4.9t^2$$

$$t^2 = \frac{16}{4.9} = 3.265...$$

$$t = 1.807$$

Let the speed of the projection be $\boldsymbol{u}\,\mathbf{m\,s^{-1}}$

$R(\rightarrow)$

$$s = ut$$

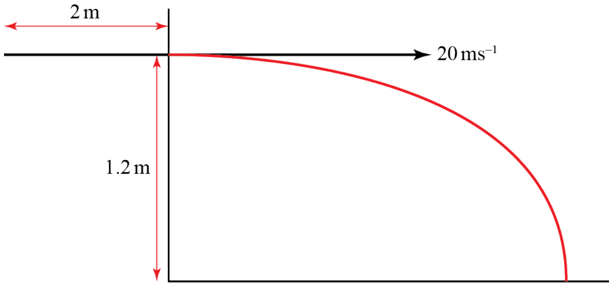
$$140 = u\times 1.807...$$

$$u = \frac{140}{1.807...}$$

$$= 77.475$$

The speed of projection of the particle is $77.5\,\mathrm{m\,s^{-1}}$ (3 s.f.)

5 Whilst particle is on the table:



$R(\rightarrow)$

$s = vt$

$2 = 20 \times t$

$t = 0.1$

Once particle leaves the table:

$R(\downarrow)$ $u_y = 0$, $a = g = 9.8$ ms^{-2}, $s = 1.2$ m, $t = ?$

$s = ut + \frac{1}{2}at^2$

$1.2 = 0 + \frac{1}{2} \times 9.8 \times t^2$

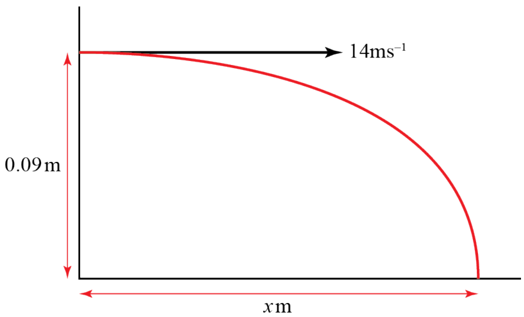
$t^2 = \frac{1.2}{4.9}$

$t = \pm 0.49487...$

The negative root can be ignored.

The total time the particle takes to reach the floor is 0.1 + 0.49 = 0.59 s (2s.f.).

6 $R(\downarrow)$ $u_y = 0$, $a = g = 9.8$ ms^{-2}, $s = 9$ cm $= 0.09$ m, $t = ?$



$s = ut + \frac{1}{2}at^2$

$0.09 = 0 + \frac{1}{2} \times 9.8 \times t^2$

$t^2 = \frac{0.09}{4.9}$

$t = \pm 0.13552...$

The negative root can be ignored.

$R(\rightarrow)$: $u_x = 14$ ms^{-1}, $t = 0.13552...$ s, $s = x$

$s = vt$

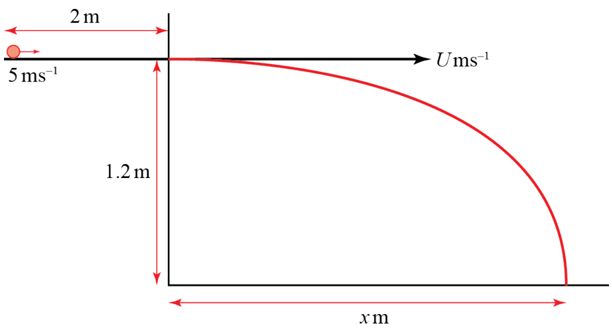
$x = 14 \times 0.13552...$

$x = 1.8973...$

The dart is thrown from a point 1.90 m (3s.f.) from the board.

7 **a** Once particle leaves the surface:

$R(\downarrow)$ $u_y = 0$, $a = g = 9.8$ ms^{-2}, $s = 1.2$ m, $t = ?$



$s = ut + \frac{1}{2}at^2$

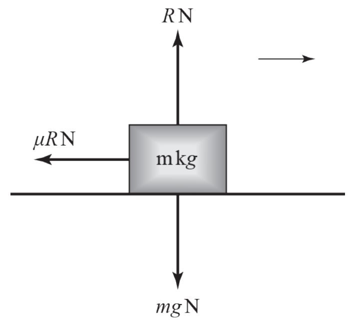
$1.2 = 0 + \frac{1}{2} \times 9.8 \times t^2$

$t^2 = \frac{1.2}{4.9}$

$t = \pm 0.49487...$

Total travel time is 1.0 s, so particle is in contact with the surface for 1.0 – 0.49 = 0.51 s (2s.f.).

7 b Considering forces acting on particle while on surface:



$R(\downarrow)$: $R = mg$

$$R(\rightarrow): \quad F = ma$$
$$-\mu R = ma \quad \text{since } F = F_{MAX}$$
$$-\mu mg = ma$$
$$a = -\mu g \qquad (\mathbf{1})$$

Use equations of motion to calculate the acceleration of the particle whilst on the surface:

$s = 2$ m, $u = 5$ ms^{-1}, $t = 0.50513\ldots$ s, $a = ?$

$$s = ut + \frac{1}{2}at^2$$
$$2 = (5 \times 0.50513...) + \left(\frac{1}{2} \times a \times 0.50513...^2\right)$$
$$0.12757... \times a = 2 - 2.5256...$$
$$a = \frac{-0.52564...}{0.12757...}$$
$$a = -4.1201... \qquad (\mathbf{2})$$

Substitute (**2**) in (**1**):

$$-4.1201... = -\mu g$$
$$-4.1201... = -9.8 \times \mu$$
$$\mu = 0.42042...$$

The coefficient of friction is 0.42 (2s.f.).

c While particle is on the surface: $s = 2$ m, $u = 5$ ms^{-1}, $t = 0.50513\ldots$ s, $v = U$

$$s = \frac{1}{2}(u + v)t$$
$$2 = \frac{1}{2}(5 + U)0.50513\ldots$$
$$5 + U = \frac{4}{0.50513...}$$
$$U = 7.9187... - 5 = 2.9187...$$

Considering horizontal motion of particle once it has left the surface:

R($\rightarrow$): $u_x = U = 2.9187\ \ldots$ ms^{-1}, $t = 0.495$ s, $s = x$

$$s = vt$$
$$x = 2.9187... \times 0.495$$
$$x = 1.4447...$$

The total distance travelled = 1.4447… + 2 = 3.44 (3 s.f.)

Projectiles 6B

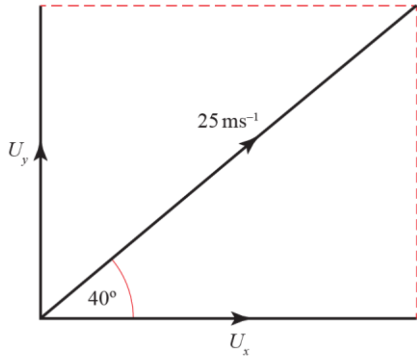
1 a Components of velocity (3s.f.):

$u_x = 25\cos 40^\circ$

$= 19.2 \text{ ms}^{-1}$

$u_y = 25\sin 40^\circ$

$= 16.1 \text{ ms}^{-1}$



b $\mathbf{u} = (19.2\mathbf{i} + 16.1\mathbf{j}) \text{ ms}^{-1}$

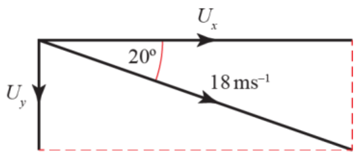
2 a Components of velocity (3s.f.):

$u_x = 18\cos 20^\circ$

$= 16.9 \text{ ms}^{-1}$

$u_y = -18\sin 20^\circ$

$= -6.15 \text{ ms}^{-1}$



b $\mathbf{u} = (16.9\mathbf{i} - 6.15\mathbf{j}) \text{ ms}^{-1}$

3 a $\tan\alpha = \frac{5}{12}$ so $\sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$

Components of velocity (3s.f.):

$u_x = 35\cos\alpha$

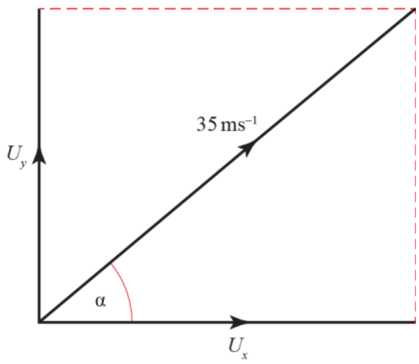
$= 35 \times \frac{12}{13}$

$= 32.3 \text{ ms}^{-1}$

$u_y = 35\sin\alpha$

$= 35 \times \frac{5}{13}$

$= 13.5 \text{ ms}^{-1}$



b $\mathbf{u} = (32.3\mathbf{i} + 13.5\mathbf{j}) \text{ ms}^{-1}$

4 a $\tan\alpha = \frac{7}{24}$ so $\sin\alpha = \frac{7}{25}$ and $\cos\alpha = \frac{24}{25}$

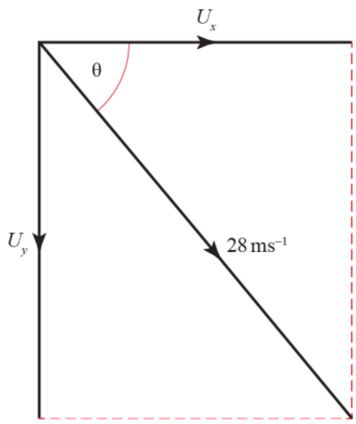
Components of velocity (3s.f.):

$u_x = 28\cos\theta$

$= 26.9 \text{ ms}^{-1}$

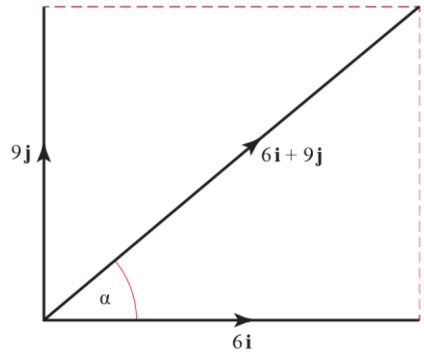
$u_y = -28\sin\theta$

$= -7.8 \text{ ms}^{-1}$



b $\mathbf{u} = (26.9\mathbf{i} - 7.8\mathbf{j}) \text{ ms}^{-1}$

5 Speed is magnitude of velocity:



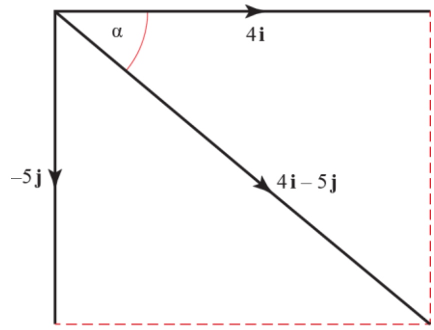
$$|\mathbf{u}| = \sqrt{6^2 + 9^2}$$
$$= 10.816...$$

The initial speed of the particle is 10.8 ms^{-1} (3 s.f.).

$$\tan\alpha = \frac{9}{6}$$
$$\alpha = 56.309...$$

Particle is projected at an angle of 56.3° above the horizontal (3 s.f.).

6 Speed is magnitude of velocity:



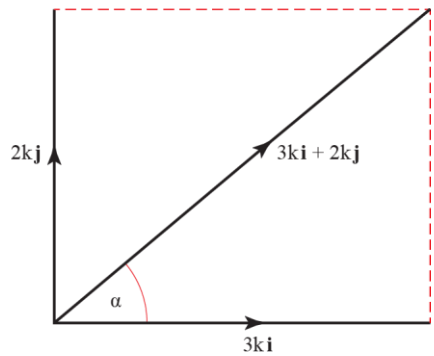
$$|\mathbf{u}| = \sqrt{4^2 + 5^2}$$
$$= 6.4031...$$

The initial speed of the particle is 6.40 ms^{-1} (3 s.f.).

$$\tan\alpha = \frac{5}{4}$$
$$\alpha = 51.340...$$

Particle is projected at an angle of 51.3° below the horizontal (3 s.f.).

7 **a** Let the angle of projection be α



$$\tan\alpha = \frac{2k}{3k} = \frac{2}{3}$$
$$\Rightarrow \alpha = 33.690...$$

The angle of projection is 33.7° (3s.f.).

b Speed = magnitude of velocity, so:

$$\left(3\sqrt{13}\right)^2 = (3k)^2 + (2k)^2$$
$$9 \times 13 = 9k^2 + 4k^2$$
$$117 = 13k^2$$
$$k^2 = 9$$
$$k = \pm 3$$

Projectiles 6C

Unless otherwise stated, the positive direction is upwards.

1 Resolving the initial velocity vertically:

$R(\uparrow),\ u_y = 35\sin 60°$

$u = 35\sin 60°,\quad v = 0,\quad a = -9.8,\quad t = ?$

$$v = u + at$$
$$0 = 35\sin 60° - 9.8t$$
$$t = \frac{35\sin 60°}{9.8}$$
$$= 3.092...$$

The time the particle takes to reach its greatest height is 3.1 s (2 s.f.).

2 Resolving the initial velocity vertically:

$R(\uparrow),\ u_y = 18\sin 40°$

$u = 18\sin 40°,\quad a = -9.8,\quad t = 2,\quad s = ?$

$$s = ut + \frac{1}{2}at^2$$
$$= 18\sin 40° \times 2 - 4.9 \times 2^2$$
$$= 3.540...$$

The height of the ball above the ground 2 s after projection is $(5 + 3.5)\,\text{m} = 8.5\,\text{m}$ (2 s.f.).

3 Taking the downwards direction as positive.
Resolving the initial velocity horizontally and vertically:

$R(\rightarrow)\ u_x = 32\cos 10°$

$R(\uparrow)\ u_y = 32\sin 10°$

a $R(\uparrow)$

$u = 32\sin 10°,\quad a = -9.8,\quad t = 2.5,\quad s = ?$

$$s = ut + \frac{1}{2}at^2$$
$$= 32\sin 10° \times 2.5 + 4.9 \times 2.5^2$$
$$= 44.517...$$

The stone is projected from 44.5 m above the ground.

b $R(\rightarrow)$

$u = 32\cos 10°,\quad t = 2.5,\quad s = ?$

$$s = vt$$
$$= 2.5 \times 32\cos 10°$$
$$= 78.785...$$

The stone lands 78.8 m away from the point on the ground vertically below where it was projected from.

4 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 150\cos 10°$

$R(\uparrow)\ u_y = 150\sin 10°$

a $R(\uparrow)$

$u = 150\sin 10°,\quad v = 0,\quad a = -9.8,\quad t = ?$

$$v = u + at$$
$$0 = 150\sin 10° - 9.8t$$
$$t = \frac{150\sin 10°}{9.8}$$
$$= 2.657...$$

The time taken to reach the projectile's highest point is 2.7 s (2 s.f.).

b First, resolve vertically to find the time of flight:

$R(\uparrow)\ u = 150\sin 10°,\quad s = 0,\quad a = -9.8,\quad t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$0 = 150t\sin 10° - 4.9t^2$$
$$0 = t(150\sin 10° - 4.9t)$$
$$t = 0 \text{ s} \quad \text{or} \quad t = \frac{150\sin 10°}{4.9}$$
$$= 5.316... \text{ s}$$

[Note that, alternatively, you can consider the symmetry of the projectile's path:
The time of flight is twice as long as the time it takes to reach the highest point, that is

$$t = 2.657... \times 2$$
$$= 5.315 \text{ s}]$$

$R(\rightarrow)$

$u = 150\cos 10°,\ t = 5.315,\ s = ?$

$$s = ut$$
$$= 150\cos 10° \times 5.315$$
$$= 785.250...$$

The range of the projectile is 790 m (2 s.f.).

5 Resolving the initial velocity horizontally and vertically:

$R(\rightarrow)\ u_x = 20\cos 45° = 10\sqrt{2}$

$R(\uparrow)\ u_y = 20\sin 45° = 10\sqrt{2}$

a $R(\uparrow)$

$u = 10\sqrt{2},\ v = 0,\ a = -9.8,\ s = ?$

$$v^2 = u^2 + 2as$$
$$0 = 200 - 19.6s$$
$$s = \frac{200}{19.6}$$
$$= 10.204...$$

The greatest height above the plane reached by the particle is 10 m (2 s.f.).

b To find the time taken to move from O to X, first find the time of flight:

$R(\uparrow)$

$u = 10\sqrt{2},\ s = 0,\ a = -9.8,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$0 = 10\sqrt{2}t - 4.9t^2$$
$$0 = t\left(10\sqrt{2} - 4.9t\right)$$
$$t = \frac{10\sqrt{2}}{4.9} \qquad \text{(ignore } t = 0\text{)}$$
$$= 2.886...\text{ s}$$

$R(\rightarrow)$

$u = 10\sqrt{2},\ t = 2.886..,\ s = ?$

$$s = ut$$
$$= 10\sqrt{2} \times 2.886...$$
$$= 40.86...$$

$\Rightarrow OX = 41\text{ m}\ \ (2\text{ s.f.})$

6 $\sin\theta = \frac{4}{5} \Rightarrow \cos\theta = \frac{3}{5}$

Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 24\cos\theta = 14.4$

$R(\uparrow)\ u_y = 24\sin\theta = 19.2$

a $R(\uparrow)$

$u = 19.2,\ \ s = 0,\ \ a = -9.8,\ \ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = 19.2t - 4.9t^2$$

$$= t(19.2 - 4.9t)$$

$$t = \frac{19.2}{4.9} \qquad \text{(ignore } t = 0\text{)}$$

$$= 3.918...$$

The time of flight of the ball is 3.9 s (2 s.f.).

b $R(\rightarrow)$

$u = 14.4,\ t = 3.918,\ s = ?$

$$s = ut$$

$$= 14.4 \times 3.918...$$

$$= 56.424...$$

$AB = 56\,\text{m}$ (2 s.f.)

7 Resolving the initial velocity vertically,

$u_y = 21\sin\alpha$

$R(\uparrow)$: $u = 21\sin\alpha,\ v = 0,\ a = -9.8,\ s = 15$

$$v^2 = u^2 + 2as$$

$$0 = (21\sin\alpha)^2 - 2 \times 9.8 \times 15$$

$$441\sin^2\alpha = 294$$

$$\sin^2\alpha = \frac{294}{441} = \frac{2}{3}$$

$$\sin\alpha = \sqrt{\frac{2}{3}} = 0.816$$

$$\alpha = 54.736°$$

$$= 55° \quad \text{(nearest degree)}$$

8 **a** $R(\rightarrow)$

$u = 12,\ t = 3,\ s = ?$

$$\begin{aligned} s &= ut \\ &= 12 \times 3 \\ &= 36 \end{aligned}$$

$R(\uparrow)$

$u = 24,\ a = -g,\ t = 3,\ s = ?$

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ &= 24 \times 3 - 4.9 \times 9 \\ &= 27.9 \end{aligned}$$

The position vector of P after 3 s is $(36\mathbf{i} + 27.9\mathbf{j})$ m

b $R(\rightarrow)$ $u_x = 12$, throughout the motion

$R(\uparrow)$ $v = u + at$

$$v_y = 24 - 9.8 \times 3 = -5.4$$

Let the speed of P after 3 s be V m s^{-1}

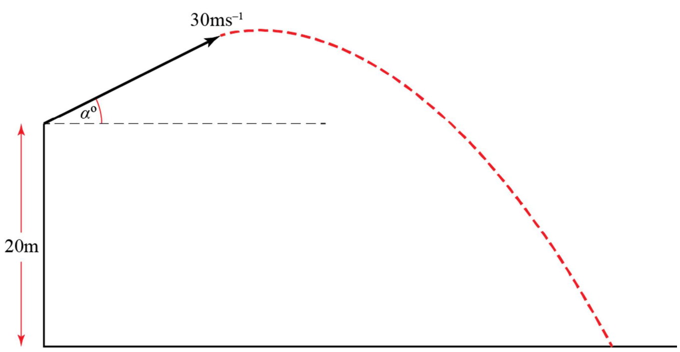
$$\begin{aligned} V^2 &= u_x^2 + v_y^2 \\ &= 12^2 + (-5.4)^2 \\ &= 173.16 \\ V &= \sqrt{173.16} \\ &= 13.159... \end{aligned}$$

The speed of P after 3 s is 13 m s^{-1} (2 s.f.).

9 Let α be the angle of projection above the horizontal. Resolving the initial velocity horizontally and vertically.

$R(\rightarrow)$ $u_x = 30\cos\alpha$

$R(\uparrow)$ $u_y = 30\sin\alpha$



a $R(\uparrow)$

$u = 30\sin\alpha,\ \ s = -20,\ \ a = -9.8,\ \ t = 3.5$

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ -20 &= 30\sin\alpha \times 3.5 - 4.9 \times 3.5^2 \\ \sin\alpha &= \frac{4.9 \times 3.5^2 - 20}{30 \times 3.5} \\ &= 0.381190... \\ \alpha &= 22.407...^\circ \end{aligned}$$

The angle of projection of the stone is 22° (2 s.f.) above the horizontal.

9 b $R(\rightarrow)$

$u = 30\sin 22.407...°, \quad t = 3.5, \quad s = ?$

$$s = ut$$
$$= 30\sin 22.407...° \times 3.5$$
$$= 97.072...$$

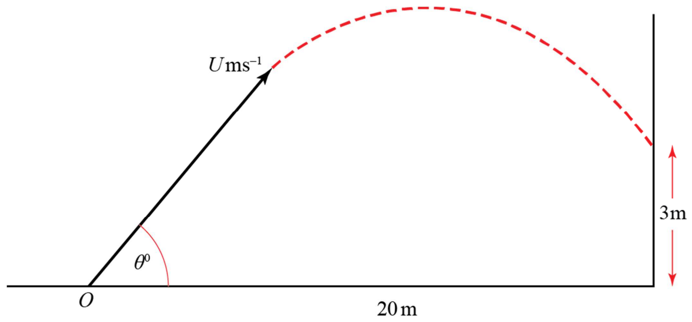
The horizontal distance from the window to the point where the stone hits the ground is 97 m (2 s.f.).

10 $\tan\theta = \frac{3}{4} \Rightarrow \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}$

Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = U\cos\theta = \frac{4U}{5}$

$R(\uparrow)\ u_y = U\sin\theta = \frac{3U}{5}$



a $R(\rightarrow)$

$u = \frac{4U}{5}, \quad s = 20, \quad t = ?$

$$s = ut$$
$$20 = \frac{4tU}{5}$$
$$t = \frac{25}{U} \qquad \textbf{(1)}$$

$R(\uparrow)$

$u = \frac{3U}{5}, \quad s = 3, \quad a = -g, \quad t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$3 = \frac{3U}{5} \times t - 4.9t^2 \qquad \textbf{(2)}$$

Substituting $t = \frac{25}{U}$ from (**1**) into (**2**):

$$3 = \frac{3U}{5} \times \frac{25}{U} - 4.9 \times \frac{25^2}{U^2}$$
$$3 = 15 - \frac{3062.5}{U^2}$$
$$\Rightarrow U^2 = \frac{3062.5}{12}$$
$$= 255.208...$$
$$U = 15.975...$$
$$= 16 \ \ (2 \text{ s.f.})$$

10 b $R(\rightarrow)$

$$t = \frac{25}{U}$$
$$= \frac{25}{15.975...}$$
$$= 1.5649...$$

The time from the instant the ball is thrown to the instant that it strikes the wall is 1.6 s (2 s.f.).

11 a Resolve vertically for motion between A and B:

$R(\uparrow)$

$u = 4u,\ \ s = 20 - 12 = -8,\ \ a = -g,\ \ t = 4$

$$s = ut + \frac{1}{2}at^2$$
$$-8 = 4u \times 4 - 4.9 \times 4^2$$
$$u = \frac{4.9 \times 4^2 - 8}{16}$$
$$= 4.4$$

b Resolve horizontally for motion between A and B:

$R(\rightarrow)$

$u = 5u = 5 \times 4.4 = 22,\ \ t = 4,\ \ s = k$

$$s = ut$$
$$k = 22 \times 4$$
$$= 88$$

c $u_x = 22\ \text{ms}^{-1}$ throughout the motion.

Resolve vertically to find v_y at C:

$R(\uparrow)$

$u = 4 \times 4.4,\ \ a = -g,\ \ s = -20,\ \ v = ?$

$$v^2 = u^2 + 2as$$
$$v_y^2 = (4 \times 4.4)^2 + 2 \times (-9.8) \times (-20)$$
$$= 16 \times 4.4^2 + 392$$
$$= 701.76$$

Let θ be angle that the path of P makes with the x-axis as it reaches C.

$$\tan\theta = \frac{v_y}{u_x}$$
$$= \frac{\sqrt{701.76}}{22}$$
$$= 1.204...$$
$$\theta = 50.291...$$

The angle the path of P makes with the x-axis as it reaches C is $50°$ (2 s.f.).

12 Take downwards as the positive direction.
Resolving the initial velocity horizontally and vertically:

$R(\rightarrow)\ u_x = 30\cos 15°$

$R(\uparrow)\ u_y = 30\sin 15°$

a $R(\downarrow)$

$u = 30\sin 15°,\quad s = 14,\quad a = 9.8,\quad t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$14 = 30t\sin 15° + 4.9t^2$$

$$4.9t^2 + 30t\sin 15° - 14 = 0$$

Using the formula for solving the quadratic,

$$t = \frac{-30\sin 15°\sqrt{(900\sin^2 15 + 4\times 14\times 4.9)}}{9.8}$$

$= 1.074...$ (the negative solution can be ignored)

The time the particle takes to travel from A to B is 1.1 s (2 s.f.).

b $R(\rightarrow)$

$u = 30\cos 15°,\quad t = 1.074..,\quad s = ?$

$$s = ut$$

$$= (30\cos 15°)\times 1.074$$

$$= 31.136...$$

$$AB^2 = 14^2 + (31.136...)^2$$

$$= 1165.456...$$

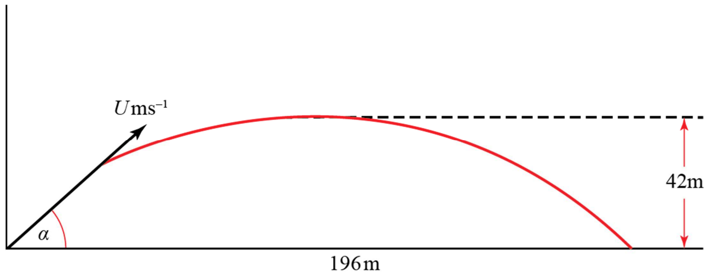
$$AB = 34.138...$$

The distance AB is 34 m (2 s.f.).

13 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = U\cos\alpha$

$R(\uparrow)\ u_y = U\sin\alpha$



To get one equation in U and α, resolve vertically when particle reaches its maximum height of 42 m:

$R(\uparrow)$

$u = U\sin\alpha,\quad a = -g,\quad s = 42,\quad v = 0$

$$v^2 = u^2 + 2as$$

$$0 = U^2\sin^2\alpha - 2g \times 42$$

$$U^2\sin^2\alpha = 84g \qquad \textbf{(1)}$$

To get a second equation in U and α, we must resolve both horizontally and vertically to find expressions for t when the particle hits the ground. We can then equate these expressions and eliminate t:

$R(\rightarrow)$

$u = U\cos\alpha,\quad s = 196,\quad t = ?$

$$s = ut$$

$$196 = U\cos\alpha \times t$$

$$t = \frac{196}{U\cos\alpha} \qquad (*)$$

$R(\uparrow)$

$u = U\sin\alpha,\quad a = -g,\quad s = 0,\quad t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = Ut\sin\alpha - \frac{1}{2}gt^2$$

$$= t\left(U\sin\alpha - \frac{1}{2}gt\right)$$

$$\frac{1}{2}gt = U\sin\alpha \qquad (\text{ignore } t = 0)$$

$$t = \frac{2U\sin\alpha}{g} \qquad (**)$$

(*) = (**):

$$\frac{196}{U\cos\alpha} = \frac{2U\sin\alpha}{g}$$

$$U^2\sin\alpha\cos\alpha = 98g \qquad \textbf{(2)}$$

Now we have two equations in U and α, (**1**) and (**2**), that we can solve simultaneously.

(**1**) ÷ (**2**):

13 (cont.)

$$\frac{U^2 \sin^2 \alpha}{U^2 \sin \alpha \cos \alpha} = \frac{84g}{98g}$$

$$\tan \alpha = \frac{6}{7}$$

$$\alpha = 40.6° \quad (3 \text{ s.f.})$$

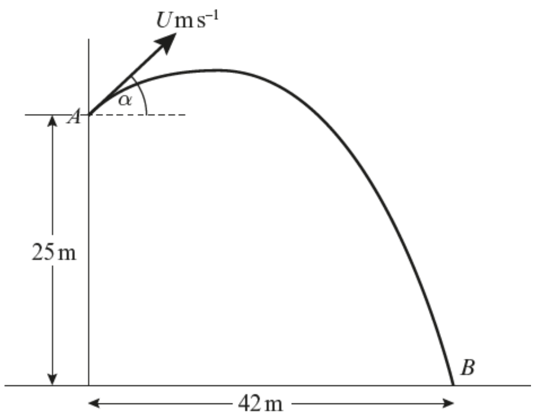
Sub $\alpha = 40.6°$ in (1):

$U \sin 40.6° = \sqrt{84g}$ (discard the negative square root as U is a scalar, so must be positive)

$$U = \frac{\sqrt{84 \times 9.8}}{\sin 40.6°}$$

$$= 44 \quad (2 \text{ s.f.})$$

14 $\tan \alpha = \frac{5}{12}$ so $\sin \alpha = \frac{5}{13}$ and $\cos \alpha = \frac{12}{13}$



R(→): $u_x = U \cos \alpha = \frac{12}{13} U$

R(↑): $u_y = U \sin \alpha = \frac{5}{13} U$

a Resolve horizontally to find time at which particle hits the ground:

R(→): $v = u_x = \frac{12}{13} U$ ms^{-1}, $s = 42$ m, $t = ?$

$$s = vt$$

$$42 = \frac{12}{13} Ut$$

$$t = \frac{13 \times 42}{12U}$$

$$= \frac{91}{2U}$$

Resolve vertically with $t = \frac{91}{2U}$:

R(↑): $u_y = \frac{5}{13} U$, $t = \frac{91}{2U}$, $a = g = -10$, $s = -25$

$$s = ut + \frac{1}{2} at^2$$

$$-25 = \left(\frac{5}{13} U \times \frac{91}{2U}\right) + \frac{1}{2}\left(-10 \times \left(\frac{91}{2U}\right)^2\right)$$

$$-25 = \frac{35}{2} - 5\left(\frac{91}{2U}\right)^2$$

14 a (cont.)

$$\frac{85}{2} = 5\left(\frac{91}{2U}\right)^2$$

$$\frac{85}{10} = \left(\frac{91}{2U}\right)^2$$

$$85 \times 4U^2 = 10 \times 91^2$$

$$U = \sqrt{\frac{82810}{340}}$$

$$= 15.606...$$

The speed of projection is 15.6 ms^{-1} (3s.f.).

b From **a**:

$$t = \frac{91}{2U}$$

$$= \frac{91}{2 \times 15.606...}$$

$$= 2.9154...$$

The object takes 2.92 s (3s.f.) to travel from A to B.

c At 12.4 m above the ground:

$v_x = u_x = \frac{12}{13}U$ ms^{-1} and

v_y is found by resolving vertically with $s = -25 + 12.4 = -12.6$ m

R(↑): $u_y = \frac{5}{13}U$, $a = g = -10$, $s = -12.6$ m, $v = v_y$

$$v^2 = u^2 + 2as$$

$$v_y{}^2 = \left(\frac{5}{13}U\right)^2 + 2(-10)(-12.6)$$

$$v_y{}^2 = \left(\frac{5}{13}U\right)^2 + 252$$

The speed at 12.4 m above the ground is given by:

$$v^2 = v_x^2 + v_y^2$$

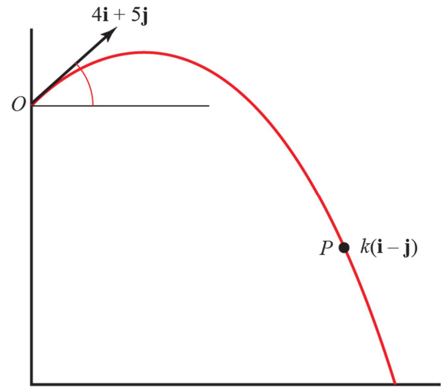
$$v^2 = \left(\frac{12}{13}U\right)^2 + \left(\frac{5}{13}U\right)^2 + 252$$

$$v^2 = U^2 + 252$$

$$v = \sqrt{15.606...^2 + 252}$$

$$v = 22.261...$$

The speed of the object when it is 12.4 m above the ground is 22.3 ms^{-1} (3s.f).



15 a First, resolve horizontally to find the time at which object reaches P:

R(→): $v = u_x = 4$, $s = k$, $t = ?$

$$s = vt$$
$$k = 4t$$
$$t = \frac{k}{4}$$

Now resolve vertically at the instant when object reaches P:

R(↑): $u = u_y = 5$, $t = \frac{k}{4}$, $a = g = -9.8$, $s = -k$

$$s = ut + \frac{1}{2}at^2$$
$$-k = \frac{5k}{4} + \frac{1}{2}\left(-9.8 \times \frac{k^2}{16}\right)$$
$$\frac{9}{4} = 4.9\frac{k}{16} \qquad \text{(We have divided through by } k, \text{ since } k > 0)$$
$$k = \frac{4 \times 9}{4.9}$$
$$k = 7.3469...$$

The value of k is 7.35 (3s.f.).

15 b i At P:

$$v_x = u_x = 4\,\text{ms}^{-1}$$

v_y is found by resolving vertically with $s = -k = -7.3469...$

R(↑): $u_y = 5$, $a = g = -9.8$, $s = -k$, $v = v_y$

$$v^2 = u^2 + 2as$$
$$v_y^{\,2} = 5^2 + 2(-9.8)(-k)$$
$$v_y^{\,2} = 25 + 19.6k$$

The speed at P is given by:

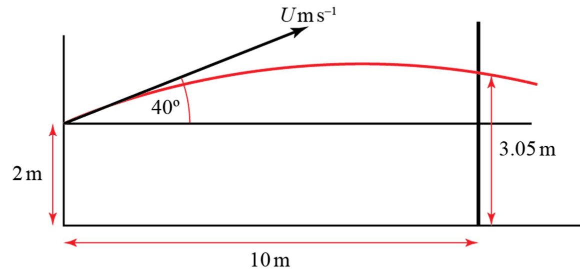
$$v^2 = v_x^2 + v_y^2$$
$$v^2 = 4^2 + 25 + 19.6k$$
$$v^2 = 41 + (19.6 \times 7.3469...)$$
$$v = \sqrt{185}$$
$$v = 13.601...$$

The speed of the object at P is 13.6 ms^{-1} (3s.f.).

15 b ii The object passes through P at an angle α where:

$$\cos\alpha = \frac{v_x}{v} \qquad \left(\text{alternatively, } \tan\alpha = \frac{v_y}{v_x} \text{ or } \sin\alpha = \frac{v_y}{v}\right)$$
$$\cos\alpha = \frac{4}{\sqrt{185}}$$
$$\alpha = 72.897...$$

The object passes through P travelling at an angle of 72.9° below the horizontal (to 3s.f.).



16 a Let U be the speed at which the basketball is thrown. Resolve horizontally to find, in terms of U, the time at which the ball reaches the basket:

R(→): $v = u_x = U\cos 40^\circ$, $s = 10$, $t = ?$

$$s = vt$$
$$10 = Ut\cos 40^\circ$$
$$t = \frac{10}{U\cos 40^\circ}$$

Now resolve vertically at the instant when the ball passes through the basket:

R(↑): $u = u_y = U\sin 40^\circ$, $t = \dfrac{10}{U\cos 40^\circ}$ s, $a = g = -9.8$, $s = 3.05 - 2 = 1.05$

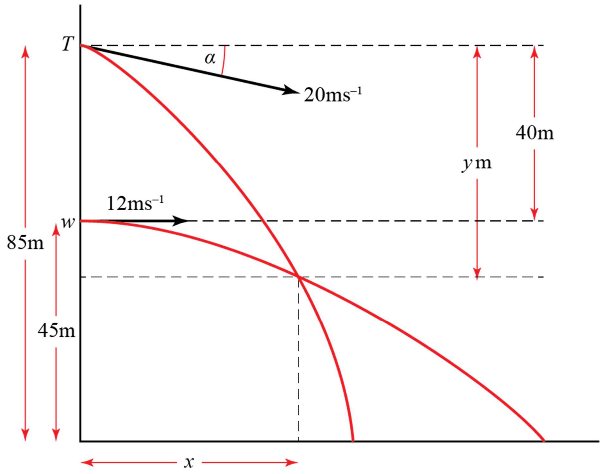
$$s = ut + \frac{1}{2}at^2$$
$$1.05 = \frac{10U\sin 40^\circ}{U\cos 40^\circ} + \frac{1}{2}\left(-9.8\times\left(\frac{10}{U\cos 40^\circ}\right)^2\right)$$
$$1.05 = 10\tan 40^\circ - \frac{490}{(U\cos 40^\circ)^2}$$
$$(U\cos 40^\circ)^2 = \frac{490}{10\tan 40^\circ - 1.05}$$
$$U^2 = \frac{490}{(10\tan 40^\circ - 1.05)(\cos 40^\circ)^2}$$
$$U = 10.665...$$

The player throws the ball at 10.7 ms^{-1} (3s.f.).

b By modelling the ball as a particle, we can ignore the effects of air resistance, the weight of the ball and any energy or path changes caused by the spin of the ball.

Challenge
Let the positive direction be downwards.

The stone thrown from the top of the tower is T, and that from the window is W.
Let u_{T_x} denote the horizontal component of the initial velocity of T, and u_{W_y} denote the vertical component of the initial velocity of W, etc.



The stones collide at time t at a horizontal distance x m from the tower.

For T, R(→): $v = u_{T_x} = 20\cos\alpha$ ms^{-1}, $s = x$, $t = t$

For W, R(→): $v = u_{W_x} = 12$ ms^{-1}, $s = x$, $t = t$

$$s = vt$$

$$x = u_{T_x}t = u_{W_x}t$$

$$20\cos\alpha = 12$$

$$\cos\alpha = \frac{3}{5} \Rightarrow \sin\alpha = \frac{4}{5}$$

For T, R(↓): $u = u_{T_y} = 20\sin\alpha = 16$ ms^{-1}, $a = g$, $s = s_{T_y}$, $t = t$

For W, R(↓): $u = u_{W_y} = 0$, $a = g$, $s = s_{T_W} = s_{T_y} - 40$, $t = t$

$$s_{W_y} = s_{T_y} - 40$$

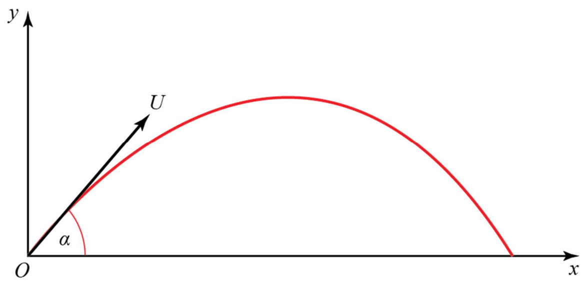
$$u_{W_y}t + \frac{1}{2}gt^2 = u_{T_y}t + \frac{1}{2}gt^2 - 40 \qquad \left(\text{since } s = ut + \frac{1}{2}at^2\right)$$

$$0 = 16t - 40 \qquad \left(\text{subtracting } \frac{1}{2}gt^2 \text{ from each side in line above, and sub values for } u\right)$$

$$t = \frac{40}{16} = 2.5$$

The stones collide after 2.5 s of flight.

Projectiles 6D



1 At maximum height, h, the vertical component of velocity, $v_y = 0$

R(↑): $u = u_y = U\sin\alpha$, $a = -g$, $s = h$, $v = 0$

$$v^2 = u^2 + 2as$$

$$0 = U^2\sin^2\alpha - 2gh$$

$$2gh = U^2\sin^2\alpha$$

$$h = \frac{U^2\sin^2\alpha}{2g} \quad \text{as required.}$$

2 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 21\cos\alpha$

$R(\uparrow)\ u_y = 21\sin\alpha$

a Resolve horizontally and vertically at the point (x, y):

$R(\rightarrow)$

$u = u_x = 21\cos\alpha,\ s = x,\ t = ?$

$$s = ut$$

$$x = t \times 21\cos\alpha$$

$$t = \frac{x}{21\cos\alpha}$$

$R(\uparrow)$

$u = u_y = 21\sin\alpha,\ s = y,\ t = \dfrac{x}{21\cos\alpha},\ a = -g$

$$s = ut + \frac{1}{2}at^2$$

$$y = 21\sin\alpha\left(\frac{x}{21\cos\alpha}\right) - 4.9\left(\frac{x}{21\cos\alpha}\right)^2$$

$$= x\tan\alpha - \frac{4.9x^2}{441\cos^2\alpha}$$

$$= x\tan\alpha - \frac{x^2}{90\cos^2\alpha} \quad \text{as required.}$$

2 **b** $\dfrac{1}{\cos^2\alpha} \equiv \sec^2\alpha \equiv 1+\tan^2\alpha$

Hence $\dfrac{x^2}{90\cos^2\alpha} \equiv \dfrac{x^2}{90}\left(1+\tan^2\alpha\right)$

Evaluating $y = x\tan\alpha - \dfrac{x^2}{90\cos^2\alpha}$ when $y = 8.1,\ x = 36$ gives:

$$8.1 = 36\tan\alpha - \frac{36^2}{90}\left(1+\tan^2\alpha\right)$$
$$8.1 = 36\tan\alpha - 14.4\left(1+\tan^2\alpha\right)$$
$$0 = 144\tan^2\alpha - 360\tan\alpha + 225$$
$$0 = 16\tan^2\alpha - 40\tan\alpha + 25$$
$$0 = \left(4\tan\alpha - 5\right)^2$$
$$\frac{5}{4} = \tan\alpha$$

3 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = U\cos\alpha$

$R(\uparrow)\ u_y = U\sin\alpha$

a We find time of flight by setting $s_y = 0$

$R(\uparrow)$: $s = 0,\ u = U\sin\alpha,\ a = -g,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$0 = Ut\sin\alpha - \frac{1}{2}gt^2$$
$$= t\left(U\sin\alpha - \frac{1}{2}gt\right)$$

$\frac{1}{2}gt = U\sin\alpha$ (ignore $t = 0$, which corresponds to the point of projection)

$t = \dfrac{2U\sin\alpha}{g}$ as required

b We find range by considering horizontal motion when $t = \dfrac{2U\sin\alpha}{g}$

$R(\rightarrow)$: $s = R,\ v = U\cos\alpha,\ t = \dfrac{2U\sin\alpha}{g}$

$$s = vt$$
$$R = U\cos\alpha \times \frac{2U\sin\alpha}{g}$$
$$R = \frac{U^2 \times 2\sin\alpha\cos\alpha}{g}$$

Using the trigonometric identity $\sin 2\alpha = 2\sin\alpha\cos\alpha$, it follows that

$R = \dfrac{U^2\sin 2\alpha}{g}$, as required

3 c The greatest possible value of $\sin 2\alpha$ is 1, which occurs when

$$2\alpha = 90°$$
$$\Rightarrow \alpha = 45°$$

Hence, for a fixed U, the greatest possible range is when $\alpha = 45°$

d $R = \dfrac{U^2 \sin 2\alpha}{g} = \dfrac{2U^2}{5g}$

$$\Rightarrow \sin 2\alpha = \frac{2}{5}$$
$$2\alpha = 23.578°, 156.422°$$
$$\alpha = 11.79°,\ 78.21°$$

The two possible angles of elevation are $12°$ and $78°$ (nearest degree).

4 First find the time it took the firework to reach max. height.

R(↑): initial velocity = v, final velocity = 0, $a = -g$, $t = ?$

$$v = u + at$$
$$0 = v - gt$$
$$t = \frac{v}{g}$$

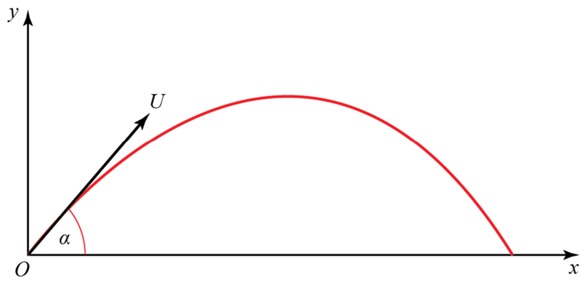
The two parts of the firework will take the same time to fall as the firework did to climb.
Considering the horizontal motion of one part of the firework as it falls:

R(→): $u = 2v$, $t = \dfrac{v}{g}$, $s = ?$

$$s = ut$$
$$s = 2v \times \frac{v}{g}$$
$$s = \frac{2v^2}{g}$$

The other part travels the same distance in the opposite direction, so the two parts land $\dfrac{2v^2}{g} + \dfrac{2v^2}{g} = \dfrac{4v^2}{g}$ m apart.

5 a Considering horizontal motion, first find time at which $s = x$:



R(→): $u_x = U\cos\alpha,\ s = x,\ t = ?$

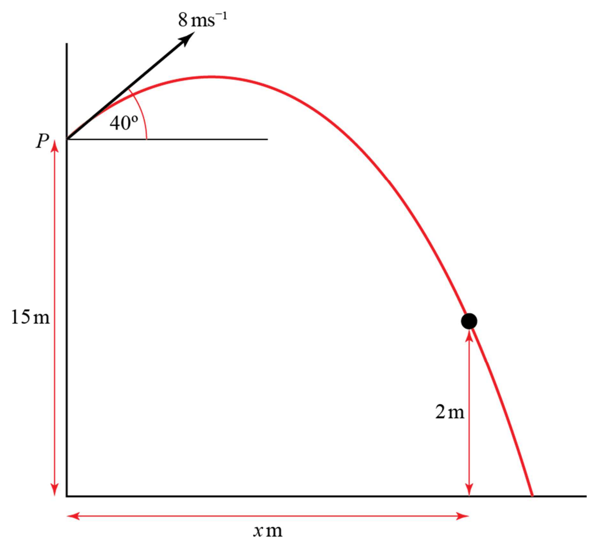
$$s = ut$$
$$x = (U\cos\alpha)\times t$$
$$t = \frac{x}{U\cos\alpha}$$

Now consider vertical motion with $t = \frac{x}{U\cos\alpha}$ to find y:

R(↑): $u_y = U\sin\alpha$, $a = -g$, $t = \frac{x}{U\cos\alpha}$, $s = y$

$$s = ut + \frac{1}{2}at^2$$
$$y = U\sin\alpha\times\frac{x}{U\cos\alpha} - \frac{1}{2}g\left(\frac{x}{U\cos\alpha}\right)^2$$
$$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha} \quad \text{as required.}$$

b $U = 8\text{ ms}^{-1}$, $\alpha = 40°$, $y = -13$ m



Substituting these values into the equation derived in **a**:

$$y = x\tan\alpha - \frac{gx^2}{2U^2\cos^2\alpha}$$
$$-13 = x\tan 40^\circ - \frac{9.8x^2}{2\times 8^2\cos^2 40^\circ}$$
$$-13 = 0.8391x - \frac{9.8x^2}{128\times 0.5868}$$
$$-13 = 0.8391x - 0.1305x^2$$
$$0 = 0.1305x^2 - 0.8391x - 13$$

Using the formula for the roots of a quadratic equation:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x = \frac{0.8391\pm\sqrt{0.8391^2-(4\times 0.1305\times(-13))}}{2\times 0.1305}$$
$$x = \frac{0.8391\pm 2.737}{0.2609}$$

$x = 13.702...$ or $x = -7.2714...$ negative root can be ignored as behind point of projection

The stone is 2 m above sea level at 13.7 m from the end of the pier (to 3 s.f.).

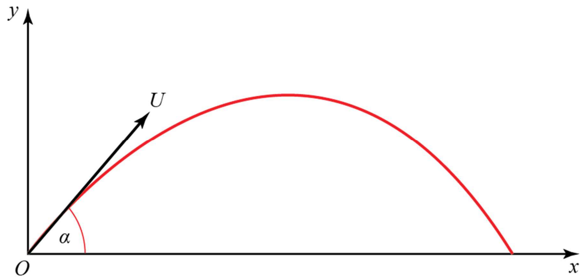
6 a Considering horizontal motion, first find time at which $s = x$:

R(→): $u_x = U\cos\alpha$, $s = x$, $t = ?$

$s = ut$

$x = (U\cos\alpha)\times t$

$t = \dfrac{x}{U\cos\alpha}$



Now consider vertical motion with $t = \dfrac{x}{U\cos\alpha}$ to find y:

R(↑): $u_y = U\sin\alpha$, $a = -g$, $t = \dfrac{x}{U\cos\alpha}$, $s = y$

$s = ut + \frac{1}{2}at^2$

$y = U\sin\alpha \times \dfrac{x}{U\cos\alpha} - \dfrac{1}{2}g\left(\dfrac{x}{U\cos\alpha}\right)^2$

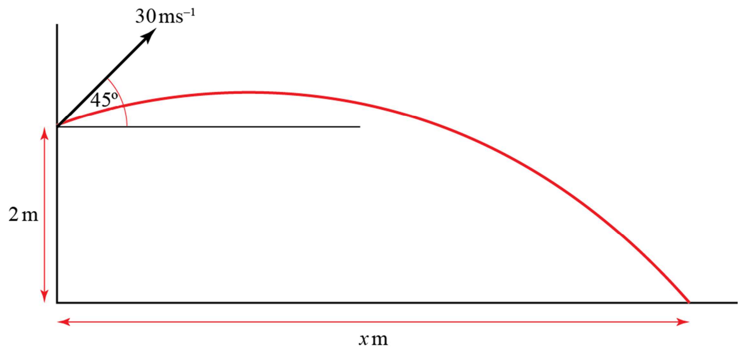
$y = x\tan\alpha - \dfrac{gx^2}{2U^2\cos^2\alpha}$

$y = x\tan\alpha - \dfrac{gx^2}{2U^2}\left(\dfrac{1}{\cos^2\alpha}\right)$

but $\dfrac{1}{\cos^2\alpha} \equiv \sec^2\alpha \equiv 1+\tan^2\alpha$ so

$y = x\tan\alpha - \dfrac{gx^2}{2U^2}\left(1+\tan^2\alpha\right)$ as required.

6 b $U = 30\text{ ms}^{-1}$, $\alpha = 45°$, $y = -2$ m



Substituting these values into the equation derived in **a**:

$y = x\tan\alpha - \dfrac{gx^2}{2U^2}\left(1+\tan^2\alpha\right)$

$-2 = x\tan 45° - \dfrac{9.8x^2}{2\times 30^2}\left(1+\tan^2 45°\right)$

$-2 = x - \dfrac{9.8x^2}{900}$

$0 = \dfrac{9.8x^2}{900} - x - 2$

$x = \dfrac{1\pm\sqrt{1^2-(4\times 0.0109\times(-2))}}{2\times 0.0109}$ (using the quadratic formula)

$x = \dfrac{1\pm 1.043}{0.0218}$

$x = 93.794...$ or $x = -1.9582...$ negative root can be ignored as behind point of projection

The javelin lands 93.8 m from P (to 3s.f.).

6 c As shown in part **a**:

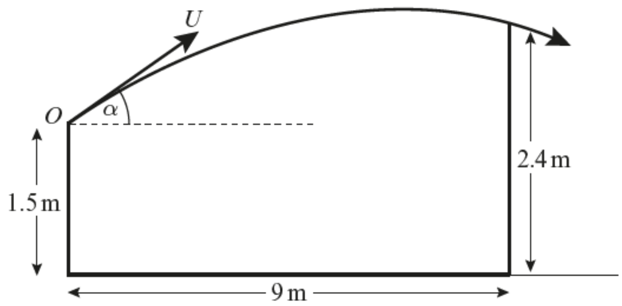
time of flight, $t = \dfrac{x}{U\cos\alpha}$

$U = 30\text{ ms}^{-1}$, $\alpha = 45^{\text{o}}$, $x = 93.79$ m

$\therefore t = \dfrac{93.79}{30\cos 45^\circ} = 4.42$

The javelin lands after 4.4 s.

7 a R(→): $u_x = U\cos\alpha\text{ ms}^{-1}$, $s = 9$ m



$s = vt$

$9 = U\cos\alpha \times t$

$t = \dfrac{9}{U\cos\alpha}$

R(↑): $u_y = U\sin\alpha$, $a = -g$,

$s = 2.4 - 1.5 = 0.9$ m, $t = \dfrac{9}{U\cos\alpha}$

$s = ut + \frac{1}{2}at^2$

$0.9 = U\sin\alpha \times \dfrac{9}{U\cos\alpha} - \dfrac{1}{2}g\left(\dfrac{9}{U\cos\alpha}\right)^2$

$0.9 = 9\tan\alpha - \dfrac{81g}{2U^2\cos^2\alpha}$ as required.

7 b $\alpha = 30^o \Rightarrow \tan\alpha = \dfrac{1}{\sqrt{3}}$, $\cos\alpha = \dfrac{\sqrt{3}}{2}$ and $\sin\alpha = \dfrac{1}{2}$

Substituting these values into the equation above:

$0.9 = \dfrac{9}{\sqrt{3}} - \dfrac{4\times 81g}{2U^2 \times 3}$

$4.296 = \dfrac{529.2}{U^2}$

$U^2 = \dfrac{529.2}{4.296}$

$U = 11.098...$

When ball passes over the net:

R(→): $v_x = u_x$

$u_x = U\cos 30^\circ$

$= 11.10\cos 30^\circ$

$= 9.6117...$

R(↑): $u_y = U\sin 30^\circ$, $a = -g$, $s = 0.9$ m, $v = ?$

$v^2 = u^2 + 2as$

${v_y}^2 = \left(11.10 \times \frac{1}{2}\right)^2 + 2(-9.8)(0.9)$

${v_y}^2 = 30.79 - 17.64 = 13.154...$

7 b The speed at P is given by:

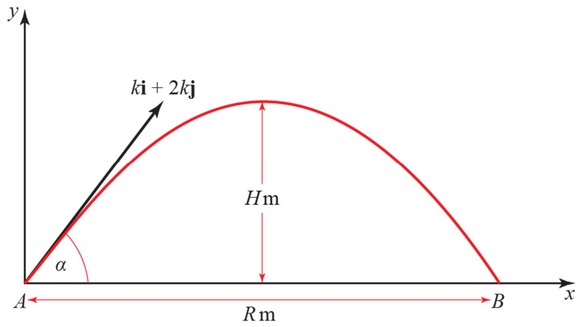
$$v^2 = v_x^2 + v_y^2$$

$$v^2 = 9.612^2 + 13.15$$

$$v = \sqrt{105.5} = 10.273...$$

The ball passes over the net at a speed of 10.3 ms^{-1} (3s.f).

8 a R(→): $u_x = k\ \text{ms}^{-1}$, $s = x$



$$s = vt$$

$$x = kt$$

$$t = \frac{x}{k}$$

R(↑): $u_y = 2k\ \text{ms}^{-1}$, $a = -g$, $s = y$, $t = \dfrac{x}{k}$

$$s = ut + \frac{1}{2}at^2$$

$$y = \frac{2kx}{k} - \frac{1}{2}g\left(\frac{x}{k}\right)^2$$

$$y = 2x - \frac{gx^2}{2k^2} \qquad \text{as required.}$$

8 b i When $x = R$, $y = 0$

Substituting these values into the equation derived in **a**:

$$0 = 2R - \frac{gR^2}{2k^2}$$

$$\frac{gR^2}{2k^2} = 2R$$

$$R^2 = \frac{2R \times 2k^2}{g}$$

$$R = \frac{4k^2}{g}$$

(The equation also gives a value of $R = 0$. This can be ignored, as it represents the value of x when the object is projected.)

Therefore, the distance AB is $\dfrac{4k^2}{g}$ m.

8 b ii When $y = H$, $x = \dfrac{R}{2} = \dfrac{2k^2}{g}$

Substituting these values into the equation derived in **a**:

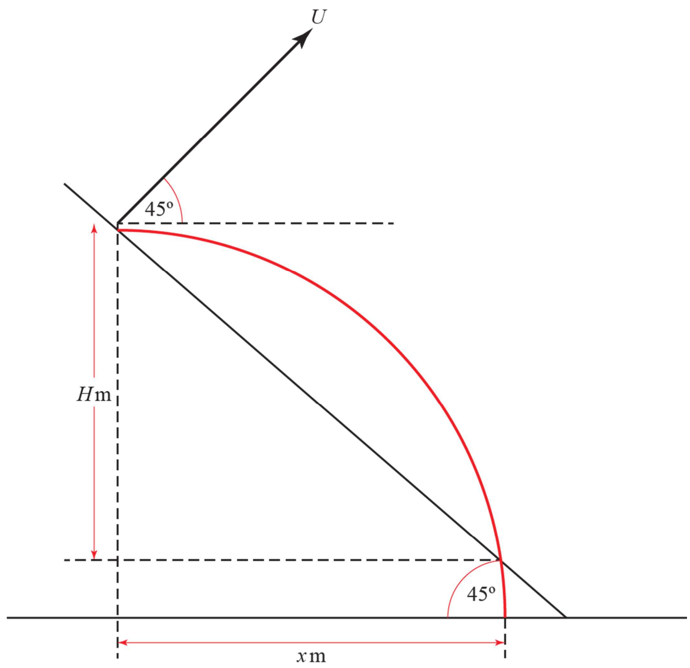
$$H = 2 \times \frac{2k^2}{g} - \frac{g}{2k^2}\left(\frac{2k^2}{g}\right)^2$$

$$H = \frac{4k^2}{g} - \frac{2k^2}{g}$$

$$H = \frac{2k^2}{g}$$

The maximum height reached is $\dfrac{2k^2}{g}$ m.

Challenge



If the point where the stone lands is taken as
$x = x$, $y = 0$,
and stone is projected from a height h m above the hill, then
the equation for the hill is:

$y = h - x$

and, when $y = 0$

$x = h$

For the stone, $y = -h$

Using the equation for the trajectory of a projectile:

$$y = x\tan\alpha - \frac{gx^2}{2U^2}\left(1+\tan^2\alpha\right)$$

$$-h = x\tan 45^\circ - \frac{gx^2}{2U^2}\left(1+\tan^2 45^\circ\right)$$

$$-h = x - \frac{gx^2}{U^2}$$

But, from above, $x = h$ so:

$$-x = x - \frac{gx^2}{U^2}$$

$$\frac{gx^2}{U^2} = 2x$$

Ignoring the solution $x = 0$:

$$\frac{gx}{U^2} = 2$$

$$x = \frac{2U^2}{g}$$

Therefore, the distance as measured along the slope of the hill, d, is given by:

$$\cos 45^\circ = \frac{x}{d}$$

$$d = \frac{x}{\cos 45^\circ}$$

$$d = \frac{2U^2}{\frac{1}{\sqrt{2}}g} = \frac{2\sqrt{2}U^2}{g} \qquad \text{as required.}$$

Projectiles Mixed exercise 6

1 **a** Resolving the initial velocity vertically

$R(\uparrow)\ u_y = 42\sin 45°$

$= 21\sqrt{2}$

$u = 21\sqrt{2},\ v = 0,\ a = -9.8,\ s = ?$

$$v^2 = u^2 + 2as$$

$$0^2 = \left(21\sqrt{2}\right)^2 - 2\times 9.8\times s$$

$$s = \frac{\left(21\sqrt{2}\right)^2}{2\times 9.8} = \frac{882}{19.6} = 45$$

The greatest height above the plane reached by P is 45 m.

b $R(\uparrow)$

$u = 21\sqrt{2},\ s = 0,\ a = -9.8,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = 21\sqrt{2}t - 4.9t^2$$

$$t \neq 0$$

$$t = \frac{21\sqrt{2}}{4.9} = 6.0609...$$

The time of flight of P is 6.1 s (2 s.f.).

2 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 21$

$R(\uparrow)\ u_y = 0$

Resolve horizontally to find the time of flight:

$R(\rightarrow):\ s = 56,\ u = 21,\ t = ?$

$$s = ut$$

$$56 = 21\times t$$

$$t = \frac{56}{21} = \frac{8}{3}$$

Resolve vertically with $t = \frac{8}{3}$ s to find h

$R(\downarrow):\ u = 0,\ s = h,\ a = 9.8,\ t = \frac{8}{3}$

$$s = ut + \tfrac{1}{2}at^2$$

$$h = 0 + 4.9\left(\frac{8}{3}\right)^2 = 34.844$$

$$h = 35\ (2\text{ s.f.})$$

3 a $\tan\theta = \frac{4}{3} \Rightarrow \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$

Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 15\cos\alpha = 15 \times \frac{3}{5} = 9$

$R(\uparrow)\ u_y = 15\sin\alpha = 15 \times \frac{4}{5} = 12$

$R(\rightarrow):\ u = 9,\ t = 4,\ s = ?$

$$\begin{aligned} s &= ut \\ &= 9 \times 4 \\ &= 36 \end{aligned}$$

The horizontal distance between the point of projection and the point where the ball hits the lawn is 36 m.

b Let the vertical height above the lawn from which the ball was thrown be h m

$R(\uparrow):\ u = 12,\ \ s = -h,\ a = -9.8,\ t = 4$

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ -h &= 12 \times 4 - 4.9 \times 4^2 \\ &= -30.4 \\ \Rightarrow h &= 30.4 \end{aligned}$$

The vertical height above the lawn from which the ball was thrown is 30 m (2 s.f.).

4 a Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 40\cos 30° = 20\sqrt{3}$

$R(\uparrow)\ u_y = 40\sin 30° = 20$

First, resolve vertically to find the time of flight:

$R(\uparrow):\ u = 20,\ \ s = 0,\ a = -9.8,\ t = ?$

$$\begin{aligned} s &= ut + \frac{1}{2}at^2 \\ 0 &= 20t - 4.9t^2 \\ 0 &= t(20 - 4.9t) \\ t \neq 0 &\Rightarrow t = \frac{20}{4.9} \end{aligned}$$

Now resolve horizontally with $t = \frac{20}{4.9}$ to find distance AB

$R(\rightarrow):\ u = v = 20\sqrt{3},\ t = \frac{20}{4.9},\ s = ?$

$$\begin{aligned} s &= ut \\ &= 20\sqrt{3} \times \frac{20}{4.9} = 141.39... \end{aligned}$$

$AB = 140$ (2 s.f.)

4 b $R(\uparrow)$: $u = 20,\ v = v_y,\ a = -9.8,\ s = 15$

$$v^2 = u^2 + 2as$$

$$v_y^2 = 20^2 - 2\times 9.8 \times 15 = 106$$

$$V^2 = u_x^2 + v_y^2 = \left(20\sqrt{3}\right)^2 + 106 = 1306$$

$$V = \sqrt{1306} = 36.138...$$

The speed of the projectile at the instants when it is 15 m above the plane is $36\,\mathrm{m\,s^{-1}}$ (2 s.f.)

5 a Taking components of velocity horizontally and vertically:

$R(\rightarrow)$ $u_x = U\cos\theta$

$R(\uparrow)$ $u_y = U\sin\theta$

First resolve vertically to find time of flight:

$R(\uparrow)$: $u = U\sin\theta,\ a = -g,\ s = 0,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = (U\sin\theta)\times t - \frac{1}{2}gt^2$$

$$0 = t\left(U\sin\theta - \frac{1}{2}gt\right)$$

$$t = \frac{2u\sin\theta}{g} \qquad \text{(since } t = 0 \text{ corresponds to launch)}$$

Let the range be R. Resolve horizontally with $t = \dfrac{2u\sin\theta}{g}$ to find R:

$R(\rightarrow)$: $u = U\cos\theta,\ s = R,\ t = \dfrac{2u\sin\theta}{g}$

$$s = vt$$

$$R = U\cos\theta \times \frac{2U\sin\theta}{g}$$

$$= \frac{2U\sin\theta\cos\theta}{g}$$

Using the identity $\sin 2\theta = 2\sin\theta\cos\theta$

$$R = \frac{U^2\sin 2\theta}{g}$$

b R is a maximum when $\sin 2\theta = 1$, that is when $\theta = 45°$

The maximum range of the projectile is $\dfrac{U^2}{g}$

c $R = \dfrac{U^2\sin 2\theta}{g} = \dfrac{2U^2}{3g}$

$$\Rightarrow \sin 2\theta = \frac{2}{3}$$

$$2\theta = 41.81°,\ (180 - 41.81)°$$

$$\theta = 20.9°,\ 69.1°,\ \text{(nearest } 0.1°\text{)}$$

6 Taking components horizontally and vertically

$R(\rightarrow)$ $u_x = 40\cos 30° = 20\sqrt{3}$

$R(\uparrow)$ $u_y = 40\sin 30° = 20$

a $R(\uparrow)$: $u = 20,\ v = 0,\ a = -g,\ t = ?$

$$v = u + at$$
$$0 = 20 - 9.8t$$
$$t = \frac{20}{9.8} = 2.0408...$$

The time taken by the ball to reach its greatest height above A is 2.0 s (2 s.f.)

b Resolve vertically with $s = 15.1$ m to find time of flight.

$R(\uparrow)$: $u = 20,\ s = 15.1,\ a = -g,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$
$$15.1 = 20t - 4.9t^2$$
$$4.9t^2 - 20t + 15.1 = 0$$
$$(t-1)(4.9t - 15.1) = 0$$

On the way down the time must be greater than the result in part **a**, so $t \neq 1$

$$\Rightarrow t = \frac{15.1}{4.9} = 3.0816...$$

The time taken for the ball to travel from A to B is 3.1s (2 s.f.)

c $R(\uparrow)$: $u = 20,\ a = -g,\ t = \frac{15.1}{4.9},\ v = v_y$

$$v_y = u + at$$
$$v_y = 20 - 9.8 \times \frac{15.1}{4.9}$$
$$= -10.2$$

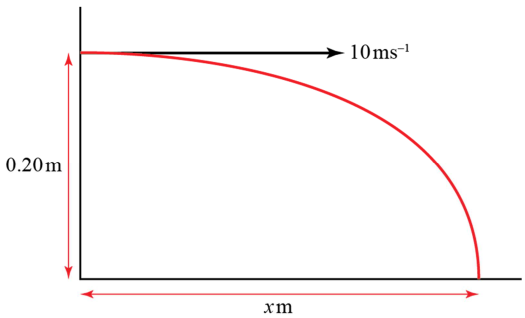
$R(\rightarrow)$ $v_x = u_x = 20\sqrt{3}$

Hence:

$$V^2 = u_x^2 + v_y^2$$
$$= \left(20\sqrt{3}\right)^2 + (-10.2)^2$$
$$= 1304.04$$
$$V = \sqrt{1304.04} = 36.111...$$

The speed with which the ball hits the hoarding is 36 m s^{-1} (2 s.f.).

7 **a** Let downwards be the positive direction.



First, resolve vertically to find the time of flight:
R(↓): $u = u_y = 0$, $a = g = 10\text{ ms}^{-2}$, $s = 20\text{ cm} = 0.20\text{ m}$, $t = ?$

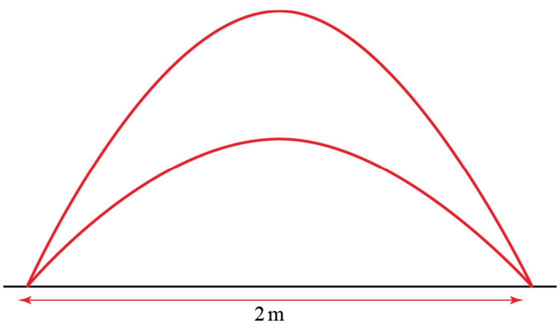
$$s = ut + \frac{1}{2}at^2$$
$$0.2 = 0 + \frac{1}{2}\times 10\times t^2$$
$$t^2 = \frac{0.2}{5}$$
$$t = 0.2$$

Let the horizontal distance to the target be x m.
R(→): $v = u_x = 10\text{ ms}^{-1}$, $t = 0.2$ s, $s = x$

$$s = vt$$
$$x = 10\times 0.2$$
$$x = 2$$

The target is 2 m from the point where the ball was thrown.

b Using the equation



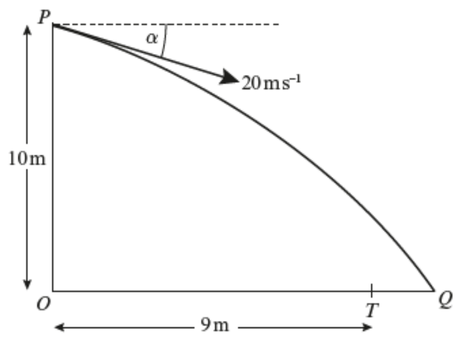
$$\text{Range} = \frac{U^2\sin 2\alpha}{10}$$

gives:

$$2 = 10\sin 2\alpha$$
$$\sin 2\alpha = 0.2$$
$$2\alpha = 11.536... \Rightarrow \alpha = 5.7684...$$
or
$$2\alpha = 168.46... \Rightarrow \alpha = 84.231...$$

For the ball to pass through the hole the boy must throw the ball at 5.77° or 84.2 ° above the horizontal (both angles to 3s.f.).

8 Let downwards be the positive direction.



$\tan\alpha = \frac{3}{4}$ so $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

a R(↓): $u_y = 20\sin\alpha = 12\text{ ms}^{-1}$, $a = g = 10\text{ ms}^{-2}$, $s = 10$ m, t = ?

$$s = ut + \frac{1}{2}at^2$$
$$10 = 12t + \frac{1}{2}10t^2$$
$$0 = 5t^2 + 12t - 10$$
$$t = \frac{-12\pm\sqrt{144-(4\times 5\times(-10))}}{10}$$
$$t = 0.65472...\ \text{ or } -3.0547$$

The negative answer does not apply, so the time taken to travel PQ is 0.65 s (2s.f.).

8 b First, find OQ:

R(→): $v = u_x = 20\cos\alpha = 16$, $s = 10$, $t = 0.65472\ldots$

$s = vt$

$OQ = 16 \times 0.65472...$

$= 10.475...$

Next find TQ:

$TQ = OQ - 9$

$= 10.475... - 9$

$= 1.475...$

The distance TQ is 1.5 m (2s.f.).

c First, resolve horizontally to find the time at which the ball passes through A

R(→): $v_x = u_x = 20\cos\alpha = 16$, $s = 9$, $t = ?$

$s = vt$

$9 = 16 \times t$

$t = 0.5625$

Then resolve vertically with $t = 0.5625$ to find vertical speed of ball as it passes through A

R(↓): $u_y = 20\sin\alpha = 12$, $a = g = 10$, $v_y = ?$

$v = u + at$

$v_y = 12 + (10 \times 0.5625)$

$v_y = 17.625$

The speed of ball at A is given by:

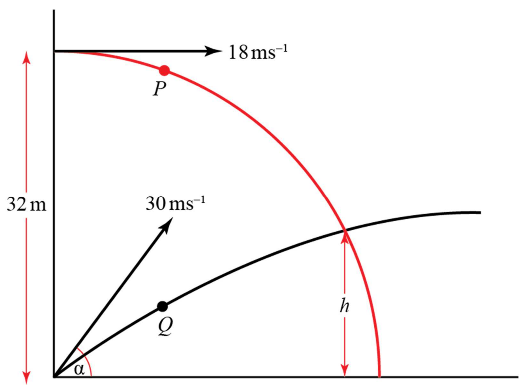
$v^2 = v_x^2 + v_y^2$

$v^2 = 16^2 + 17.625^2$

$v = \sqrt{566.64...} = 23.804...$

The speed of the ball at A is 23.8 ms^{-1} (3s.f.).

9 Let u_{P_x} denote the horizontal component of the initial velocity of P, and u_{Q_y} denote the vertical component of the initial velocity of Q, etc.



a For P: R(→): $v = u_{P_x} = 18$

$v = u_{Q_x} = 30\cos\alpha$

Since the balls eventually collide, these two speeds must be the same, so:

$30\cos\alpha = 18$

$\cos\alpha = \frac{18}{30} = \frac{3}{5}$ as required.

9 b Since $\cos\alpha = \frac{3}{5} \Rightarrow \sin\alpha = \frac{4}{5}$

Suppose the balls collide at a height h above the ground.

Resolve the vertical motion of both P and Q to find two equations for h in terms of t. We can then equate the two to solve for t.

For P, R(↓): $u = u_{P_y} = 0$, $a = g$, $s = 32 - h$, $t = t$

$$s = ut + \frac{1}{2}at^2$$

$$32 - h = 0 + \frac{1}{2}gt^2$$

$$h = 32 - \frac{1}{2}gt^2 \qquad \textbf{(1)}$$

For Q, R(↑): $u = u_{Q_y} = 30\sin\alpha = 24$, $a = -g$, $s = h$, $t = t$

$$s = ut + \frac{1}{2}at^2$$

$$h = 24t - \frac{1}{2}gt^2 \qquad \textbf{(2)}$$

(1) = **(2)**:

$$32 - \frac{1}{2}gt^2 = 24t - \frac{1}{2}gt^2$$

$$24t = 32$$

$$t = \frac{32}{24} = \frac{4}{3}$$

The balls collide after $\frac{4}{3}$ s of flight.

Challenge

The vertical motion of the golf ball is unaffected by the motion of the ship and, therefore, the time of flight is given by the usual equation for the time of flight of a projectile:

$$T = \frac{2v\sin\alpha}{g} = \frac{2v\sin 60^\circ}{g}$$

The absolute path of the ball is a parabola, and the horizontal component of the velocity is, as usual, constant.

However, the ball's horizontal speed relative to the ship is not constant: the ball appears to decelerate at the same rate as the ship is accelerating and the path appears to be non-symmetrical.

Therefore, considering the horizontal motion of the ball:

R(→): $s = 250$ m, a $= -1.5$ ms^{-2}, $t = T = \dfrac{2v\sin 60^\circ}{g}$ s, $u = v_x = v\cos 60^\circ$ ms^{-1}

$$s = ut + \frac{1}{2}at^2$$

$$250 = v\cos 60^\circ\left(\frac{2v\sin 60^\circ}{g}\right) - \frac{1.5}{2}\left(\frac{2v\sin 60^\circ}{g}\right)^2$$

$$250 = \frac{v^2 \times 2\cos 60^\circ \sin 60^\circ}{g} - \frac{3v^2 \times \sin^2 60^\circ}{g^2}$$

$$250g^2 = \left(g\sin 120^\circ - 3\sin^2 60^\circ\right)v^2$$

$$v^2 = \frac{250 \times 9.8^2}{\left(\frac{\sqrt{3}}{2} \times 9.8\right) - \left(3 \times \frac{3}{4}\right)}$$

$$v = \sqrt{3849.5...} = 62.044...$$

The initial speed of the golf ball is 62 ms^{-1} (to 2s.f.).

[Note that the equation above can be written:

$$250 + \frac{3}{4}\left(\frac{2v\sin 60^\circ}{g}\right)^2 = \frac{v^2 \sin 120^\circ}{g}$$

The additional term on the LHS is the distance covered by the ship during the time of flight of the ball, and the RHS is the usual equation for the range of a projectile.]

Applications of forces 7A

Give exact answers using $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$ or give decimal answers using your calculator.

1 a i $Q - 5\cos 30° = 0$

ii $P - 5\sin 30° = 0$

iii $Q = 5\cos 30° = \frac{5\sqrt{3}}{2} = 4.33\text{N}$ (3 s.f.)

$P = 5\sin 30° = 2.5\text{N}$

b i $P\cos\theta + 8\sin 40° - 7\cos 35° = 0$

ii $P\sin\theta + 7\sin 35° - 8\cos 40° = 0$

iii $P\cos\theta = 7\cos 35° - 8\sin 40°$
$= 0.5918 \quad (\mathbf{1})$

$P\sin\theta = 8\cos 40° - 7\sin 35°$
$= 2.113 \quad (\mathbf{2})$

Divide equation (**2**) by equation (**1**)

Use $\frac{P\sin\theta}{P\cos\theta} = \tan\theta$ to eliminate P from the equations obtained in **i** and **ii.**

$$\frac{P\sin\theta}{P\cos\theta} = \frac{8\cos 40° - 7\sin 35°}{7\cos 35° - 8\sin 40°}$$

$$\therefore \tan\theta = \frac{2.113}{0.5918}$$

$$= 3.57$$

$$\therefore \theta = 74.4° \quad \text{(3 s.f.)}$$

Substitute θ into equation (**1**)

$$P\cos 74.3569° = 0.5918$$

$$\therefore P = \frac{0.5918}{\cos 74.3569°}$$

$$= 2.19 \text{ (3 s.f.)}$$

c i $9 - P\cos 30° = 0$

ii $Q + P\sin 30° - 8 = 0$

1 c iii Using result from part **i**,

Use part **i** to find P, then substitute into **ii** to find **i**.

$$\begin{aligned}P &= \frac{9}{\cos 30°}\\ &= 9\times\frac{2}{\sqrt{3}}\\ &= \frac{9\times 2}{\sqrt{3}}\\ &= \frac{18}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{18\sqrt{3}}{3}\\ &= 6\sqrt{3}\\ &= 10.4\text{ N}\ \ (3\text{ s.f.})\end{aligned}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

Substitute into result from part **ii**

$$Q + 6\sqrt{3}\sin 30° - 8 = 0$$

$$\begin{aligned}\therefore Q &= 8 - 6\sqrt{3}\times\frac{1}{2}\\ &= 8 - 3\sqrt{3}\\ &= 2.80\text{ N}\ \ (3\text{ s.f.})\end{aligned}$$

$$\sin 30° = \frac{1}{2}$$

d i $Q\cos 60° + 6\cos 45° - P = 0$

Use angles on a straight line to find Q makes an angle of 60° with the x-axis.

ii $Q\sin 60° - 6\sin 45° = 0$

iii Using result from part **ii**,

$$\sin 45° = \frac{1}{\sqrt{2}},\ \cos 60° = \frac{1}{2},\ \sin 60° = \frac{\sqrt{3}}{2}$$

$$\begin{aligned}Q &= \frac{6\sin 45°}{\sin 60°}\\ &= 6\times\frac{1}{\sqrt{2}}\times\frac{2}{\sqrt{3}}\\ &= \frac{12}{\sqrt{6}}\\ &= \frac{12}{\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}\\ &= 2\sqrt{6}\\ &= 4.90\text{ N}\ \ (3\text{ s.f.})\end{aligned}$$

1 d iii Substitute into result from part **i**:

$$2\sqrt{6}\times\frac{1}{2}+6\times\frac{1}{\sqrt{2}}-P=0$$

$$\therefore P=\sqrt{6}+\frac{6}{\sqrt{2}}$$

$$=\sqrt{6}+\frac{6}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

$$=\sqrt{6}+3\sqrt{2}$$

$$=6.69\text{ N (3 s.f.)}$$

e i $6\cos 45°-2\cos 60°-P\sin\theta=0$

ii $6\sin 45°+2\sin 60°-P\cos\theta-4=0$

iii Using result from **i**:

$P\sin\theta=6\cos 45°-2\cos 60°$ **(1)**

Using result from **ii**:

$P\cos\theta=6\sin 45°+2\sin 60°-4$ **(2)**

$(\mathbf{1})\div(\mathbf{2})$:

$$\frac{P\sin\theta}{P\cos\theta}=\frac{6\cos 45°-2\cos 60°}{6\sin 45°+2\sin 60°-4}$$

$$\therefore \tan\theta=\frac{3.24264}{1.97469...}$$

$$=1.642$$

$$\therefore \theta=58.7°\text{ (3 s.f.)}$$

Use $\frac{P\sin\theta}{P\cos\theta}=\tan\theta$ to eliminate P from the equations after resolving

Substitute into (**1**):

$P\sin 58.65°=6\cos 45°-2\cos 60°$

$$\therefore P=\frac{3.24264}{\sin 58.65°}$$

$$P=3.80\text{ N (3 s.f.)}$$

f i $9\cos 40°+3-P\cos\theta-8\sin 20°=0$

ii $P\sin\theta+9\sin 40°-8\cos 20°=0$

1 f iii Using result from **i**:

$$P\cos\theta = 9\cos 40° + 3 - 8\sin 20° \qquad (\mathbf{1})$$

Using result from **ii**:

$$P\sin\theta = 8\cos 20° - 9\sin 40° \qquad (\mathbf{2})$$

$(\mathbf{2}) \div (\mathbf{1})$:

$$\frac{P\sin\theta}{P\cos\theta} = \frac{8\cos 20° - 9\sin 40°}{9\cos 40° + 3 - 8\sin 20°}$$

$$\therefore \tan\theta = \frac{1.732}{7.158}$$

$$= 0.242$$

$$\therefore \theta = 13.6° \text{ (3 s.f.)}$$

Substitute into (**2**):

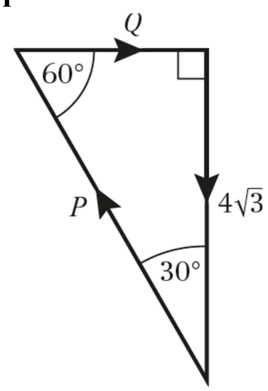
$$P\cos 13.6° = 9\cos 40° + 3 - 8\sin 20°$$

$$= 7.158$$

$$\therefore P = \frac{7.158}{\cos 13.6°}$$

$$= 7.36 \text{ (3 s.f.)}$$

2 a i



ii $R(\rightarrow)$, $\quad Q - P\cos 60° = 0 \qquad (\mathbf{1})$

$R(\uparrow)$, $\quad P\sin 60° - 4\sqrt{3} = 0 \qquad (\mathbf{2})$

From (**2**):

$$P = \frac{4\sqrt{3}}{\sin 60°}$$

$$= 4\sqrt{3} \times \frac{2}{\sqrt{3}}$$

$$= 8\text{ N}$$

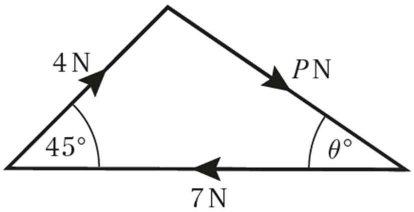
$$\boxed{\sin 60° = \frac{\sqrt{3}}{2}}$$

2 a ii Substitute $P = 8$ N into (**1**):

$$Q = 8\cos 60°$$
$$= 8 \times \frac{1}{2}$$
$$= 4\text{ N}$$

$\boxed{\cos 60° = \frac{1}{2}}$

b i



ii $R(\rightarrow)$, $4\cos 45° + P\cos\theta - 7 = 0$

$\therefore P\cos\theta = 7 - 4\cos 45°$ (**1**)

$R(\uparrow)$, $4\sin 45° - P\sin\theta = 0$

$\therefore P\sin\theta = 4\sin 45°$ (**2**)

(**2**) ÷ (**1**):

$$\frac{P\sin\theta}{P\cos\theta} = \frac{4\sin 45°}{7 - 4\cos 45°}$$
$$\therefore \tan\theta = \frac{2.828}{4.172}$$
$$= 0.678$$
$$\therefore \theta = 34.1° \text{ (3 s.f.)}$$

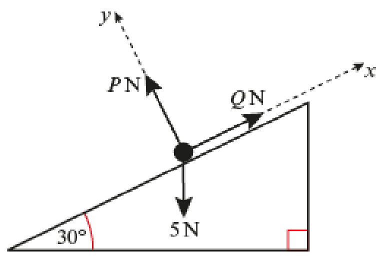
Use $\frac{P\sin\theta}{P\cos\theta} = \tan\theta$ to eliminate P from the equations after resolving

Substitute $\theta = 34.1°$ into equation (**2**):

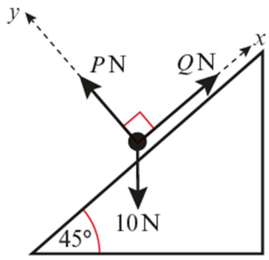
$$P = \frac{2.828}{\sin 34.1°}$$
$$P = 5.04\text{ N (3 s.f.)}$$

3 a $R(\rightarrow)$, $P = 5\cos 30° = 4.33$ N

$R(\uparrow)$, $Q = 5\sin 30° = 2.5$ N



3 b $R(\nearrow)$, $Q - 10\sin 45° = 0$ **(1)**

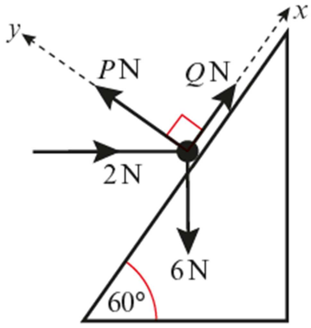


$R(\nwarrow)$, $P - 10\cos 45° = 0$ **(2)**

From **(2)**, $P = 10\cos 45°$
$= 5\sqrt{2}$
$= 7.07\text{ N}$ (3 s.f.)

From **(1)**, $Q = 10\sin 45°$
$= 5\sqrt{2}$
$= 7.07\text{ N}$ (3 s.f.)

c $R(\nearrow)$, $Q + 2\cos 60° - 6\sin 60° = 0$ **(1)**
$R(\nwarrow)$, $P - 2\sin 60° - 6\cos 60° = 0$ **(2)**

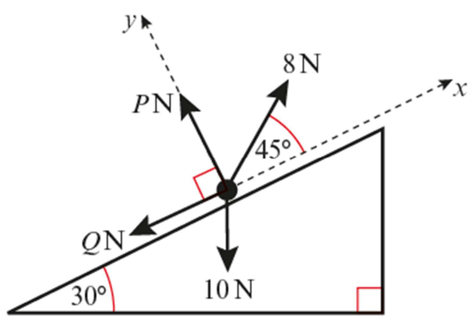


From **(2)**, $P = 2\sin 60° + 6\cos 60°$
$P = 4.73$ (3 s.f.)

From **(1)**, $Q = 6\sin 60° - 2\cos 60°$
$Q = 4.20$ (3 s.f.)

You may give your answers as exact answers using surds as $\cos 30° = \frac{\sqrt{3}}{2}$, $\sin 30° = \frac{1}{2}$, and $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ or you may give answers to 3 significant figures, using a calculator.

d $R(\nearrow)$, $8\cos 45° - 10\sin 30° - Q = 0$ **(1)**



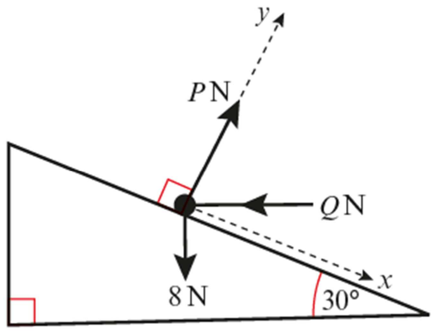
$R(\nwarrow)$, $P + 8\sin 45° - 10\cos 30° = 0$ **(2)**

From **(2)**, $P = 10\cos 30° - 8\sin 45°$
$= 5\sqrt{3} - 4\sqrt{2}$
$= 3.00\text{ N}$ (3 s.f.)

From **(1)**, $Q = 8\cos 45° - 10\sin 30°$
$= 4\sqrt{2} - 5$
$= 0.657\text{ N}$ (3 s.f.)

3 e $R(\searrow)$, $8\sin 30° - Q\cos 30° = 0$ **(1)**

$R(\nearrow)$, $P - Q\sin 30° - 8\cos 30° = 0$ **(2)**



From (**1**), $Q = \dfrac{8\sin 30°}{\cos 30°}$

$= 8\tan 30°$

$= \dfrac{8\sqrt{3}}{3}$

$= 4.62\,\text{N}$ (3 s.f.)

$$\boxed{\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}}$$

Substitute into (**2**):

$P = Q\sin 30° + 8\cos 30°$

$= \dfrac{8\sqrt{3}}{3} \times \dfrac{1}{2} + 8 \times \dfrac{\sqrt{3}}{2}$

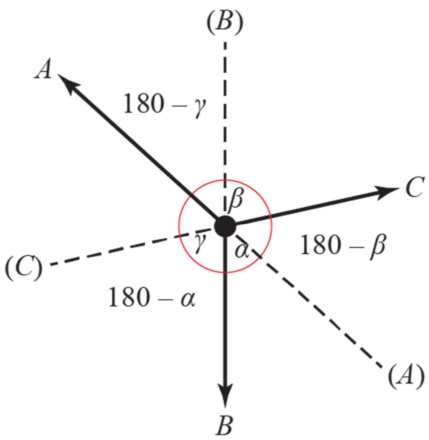
$= \dfrac{4\sqrt{3}}{3} + 4\sqrt{3}$

$= \dfrac{16\sqrt{3}}{3}$

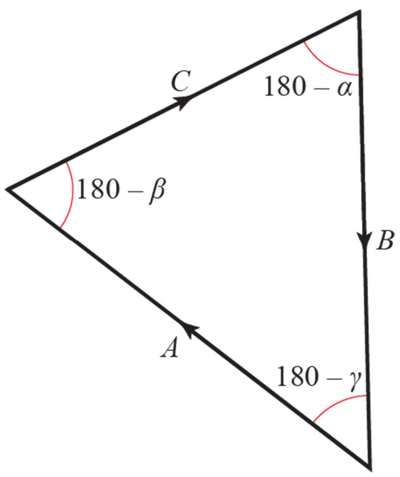
$= 9.24\,\text{N}$ (3 s.f.)

$$\boxed{\sin 30° = \frac{1}{2},\ \cos 30° = \frac{\sqrt{3}}{2}}$$

Challenge
By extending the line of action of each force backwards through the centre, we can find the acute angles between the lines of action of each of the forces.



Since the body is in equilibrium, the forces A, B and C form a closed triangle as shown below:



Using the sine rule:

$$\frac{A}{\sin(180-\alpha)} = \frac{B}{\sin(180-\beta)} = \frac{C}{\sin(180-\gamma)}$$

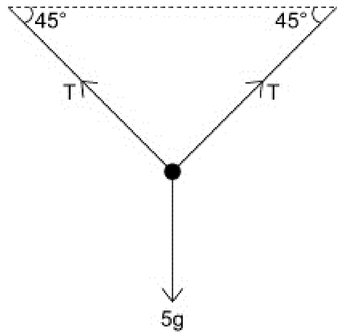
But, for any angle θ, $\sin(180-\theta) = \sin\theta$

Hence,

$$\frac{A}{\sin\alpha} = \frac{B}{\sin\beta} = \frac{C}{\sin\gamma}$$

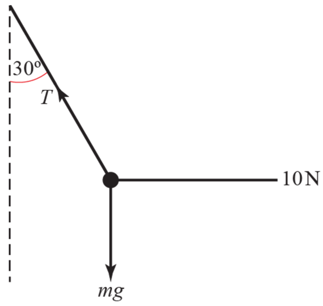
Applications of forces 7B

1 From symmetry the tension in both strings is the same.



$$R(\uparrow)$$
$$T\sin 45° + T\sin 45° - 5g = 0$$
$$\therefore 2T\sin 45° = 5g$$
$$T = \frac{5g}{2\sin 45°}$$
$$= \frac{49\sqrt{2}}{2}$$
$$T = 34.6\text{ N (3 s.f.)}$$

2 **a** Let the tension in the string be T N

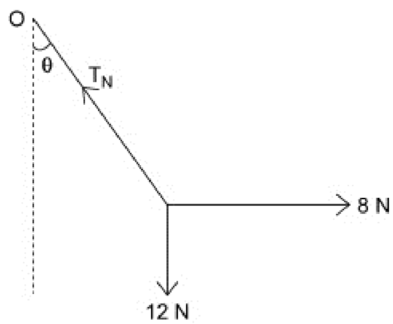


$$R(\leftarrow)$$
$$T\sin 30° - 10 = 0$$
$$\therefore T = \frac{10}{\sin 30°}$$
$$T = 20\text{ N}$$

b $R(\uparrow)$

$$T\cos 30° - mg = 0$$
$$mg = 20\cos 30° \quad (\text{since } T = 20\text{ N})$$
$$\therefore m = \frac{20\cos 30°}{g}$$
$$= \frac{10\sqrt{3}}{g}$$
$$= 1.8\text{ kg (2 s.f.)}$$

3 Let the tension in the string be T N.



$$R(\rightarrow)$$
$$8 - T\sin\theta = 0$$
$$\therefore T\sin\theta = 8 \quad \textbf{(1)}$$

$$R(\uparrow)$$
$$T\cos\theta - 12 = 0$$
$$\therefore T\cos\theta = 12 \quad \textbf{(2)}$$

3 **a** Divide equation (**1**) by equation (**2**) to eliminate the tension T.

$$\frac{T\sin\theta}{T\cos\theta}=\frac{8}{12}$$

$$\therefore \tan\theta=\frac{2}{3}$$

$$\therefore \theta=33.7° \text{ (3 s.f.)}$$

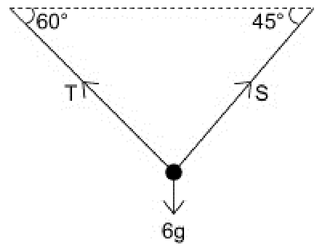
b Substitute into equation (**1**)

$$T\sin 33.7° = 8$$

$$T=\frac{8}{\sin 33.7°}$$

$$=14.4 \text{ (3 s.f.)}$$

4 Let the tension in the strings be T N and S N as shown in the figure.



$R(\leftarrow)$

$$T\cos 60° - S\cos 45° = 0$$

$$\therefore \frac{T}{2}-\frac{S}{\sqrt{2}}=0$$

$$\therefore T = S\sqrt{2} \qquad \textbf{(1)}$$

$R(\uparrow)$

$$T\sin 60° + S\sin 45° - 6g = 0$$

$$\text{T}\frac{\sqrt{3}}{2}+S\frac{1}{\sqrt{2}}=6g \qquad \textbf{(2)}$$

Substitute $T = S\sqrt{2}$ from (**1**) into equation (**2**)

$$S\left(\sqrt{2}\times\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\right)=6g$$

$$S\left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)=6g$$

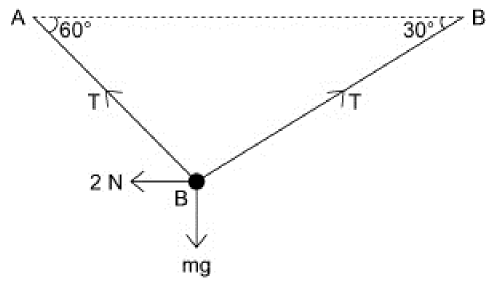
$$S=\frac{6g\sqrt{2}}{\left(\sqrt{3}+1\right)}$$

$$=3g\sqrt{2}\left(\sqrt{3}-1\right)$$

$$=30 \text{ (2 s.f.)}$$

and $T = 6g\left(\sqrt{3}-1\right) = 43$ (2 s.f.)

5 a Let the tension in the string be T and the mass of the bead be m.



Resolve horizontally first to find T:

$R(\rightarrow)$

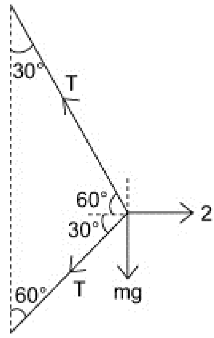
$$T\cos 30° - T\cos 60° - 2 = 0$$
$$T(\cos 30° - \cos 60°) = 2$$
$$\therefore T = \frac{2}{\cos 30° - \cos 60°}$$
$$= \frac{4}{\sqrt{3}-1}$$
$$= \frac{4\left(\sqrt{3}+1\right)}{\left(\sqrt{3}-1\right)\left(\sqrt{3}+1\right)}$$
$$= \frac{4\left(\sqrt{3}+1\right)}{2}$$
$$= 2\left(\sqrt{3}+1\right) = 5.46\text{ N (3 s.f.)}$$

b $R(\uparrow)$

$$T\sin 60° + T\sin 30° - mg = 0$$
$$mg = T(\sin 60° + \sin 30°)$$
$$m = \frac{2}{g}\left(\sqrt{3}+1\right)\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\right) \quad \text{(using } T = 2\left(\sqrt{3}+1\right) \text{ from part } \mathbf{a}\text{)}$$
$$= \frac{4+2\sqrt{3}}{g}$$
$$= 0.76\text{kg (2 s.f.)}$$

c Modelling the bead as smooth assumes there is no friction between it and the string.

6 Let the tension in the string be T and the mass of the bead be m.



a Resolve horizontally first to find T.

$R(\rightarrow)$

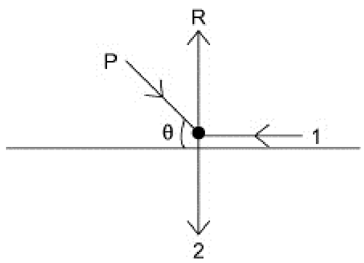
$$2 - T\cos 60° - T\cos 30° = 0$$
$$T(\cos 60° + \cos 30°) = 2$$
$$\therefore T = \frac{2}{\cos 60° + \cos 30°}$$
$$= \frac{4}{1+\sqrt{3}}$$
$$= \frac{4}{1+\sqrt{3}} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} \quad \text{(to rationalise the denomiator)}$$
$$= 2\left(\sqrt{3}-1\right)$$
$$= 1.46 \text{ (3 s.f.)}$$

b $R(\uparrow)$

$$T\sin 60° - T\sin 30° - mg = 0$$
$$mg = T(\sin 60° - \sin 30°)$$
$$= 2\left(\sqrt{3}-1\right)\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \quad \text{(using } T = 2\left(\sqrt{3}-1\right) \text{ from } \mathbf{a}\text{)}$$
$$= \left(\sqrt{3}-1\right)^2$$
$$= 4 - 2\sqrt{3}$$
$$m = \frac{\left(4-2\sqrt{3}\right)}{g}$$
$$= 0.055\,\text{kg} = 55\,\text{g}$$

7 $\tan\theta = \dfrac{12}{5} \Rightarrow \sin\theta = \dfrac{12}{13}$ and $\cos\theta = \dfrac{5}{13}$

Let the normal reaction be R N.



a $R(\rightarrow)$

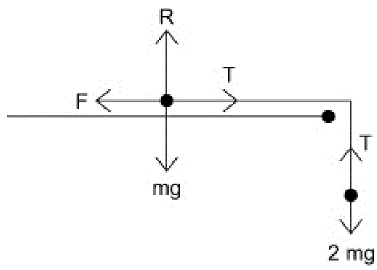
$$P\cos\theta - 1 = 0$$
$$\therefore P = \frac{1}{\cos\theta}$$
$$= \frac{13}{5}$$
$$P = 2.6$$

7 b $R(\uparrow)$

$R - P\sin\theta - 2 = 0$

$$\therefore R = P\sin\theta + 2$$
$$= 2.6 \times \frac{12}{13} + 2$$
$$= 2.4 + 2$$
$$= 4.4$$

8 a Consider the particle of mass $2m$ kg first, as it has only two forces acting on it. This enables you to find the tension.



$R(\uparrow)$

$T - 2mg = 0$

$$\therefore T = 2mg$$

Consider the particle of mass m kg:

$R(\rightarrow)$

$T - F = 0$

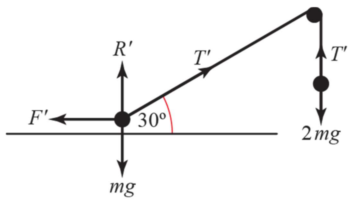
$$\therefore F = T = 2mg$$
$$= 19.6m$$

$R(\uparrow)$

$R - mg = 0$

$$\therefore R = mg$$
$$= 9.8m$$

b Let T' be the new tension in the string.



Consider the particle of mas $2m$ kg:

$R(\uparrow)$: $T' = 2mg$

Consider the particle of mass m kg:

$R(\rightarrow)$

$T'\cos 30° - F' = 0$

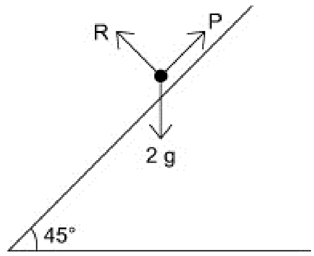
$$\therefore F' = 2mg \times \frac{\sqrt{3}}{2}$$
$$= \sqrt{3}mg$$
$$= 17m \text{ (2 s.f.)}$$

$R(\uparrow)$

$R' + T'\sin 30 - mg = 0$

$$\therefore R' = mg - T'\sin 30$$
$$= mg - 2mg \times \frac{1}{2} \quad \text{(using } T' = 2mg\text{)}$$
$$= 0$$

9 Let the normal reaction be R N.



$R(\nearrow)$:

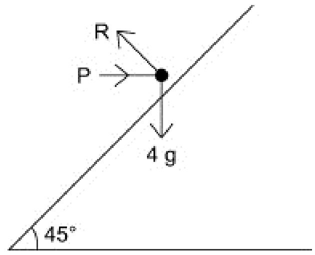
$P - 2g\sin 45° = 0$

$\therefore P = 2g\sin 45°$

$= g\sqrt{2}$

$= 14\text{ N}$ (2 s.f.)

10 Let the normal reaction be R N.



$R(\nearrow)$:

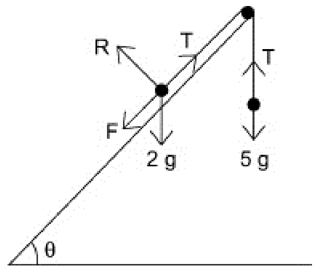
$P\cos 45° - 4g\sin 45° = 0$

$\therefore P = \dfrac{4g\sin 45°}{\cos 45°}$

$= 4g$

$= 39$ (2 s.f.)

11 a Let the normal reaction between the particle P and the plane be R N.
Let the tension in the string be T N.



Consider first the 5 kg mass.

$R(\uparrow)$

$T - 5g = 0$

$\therefore T = 5g$

Consider the 2 kg mass.

$R(\nwarrow)$

$R - 2g\cos\theta = 0$

$R = 2g \times \dfrac{4}{5}$

$= \dfrac{8g}{5}$

$= 16\text{ N}$ (2 s.f.)

b $R(\nearrow)$

$T - F - 2g\sin\theta = 0$

$F = T - 2g\sin\theta$

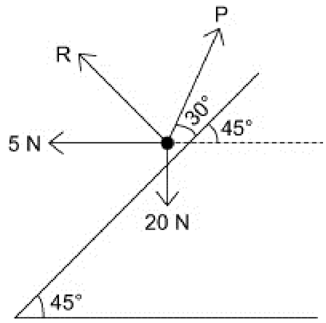
$= 5g - 2g \times \dfrac{3}{5}$ (using $T = 5g$ from above)

$= \dfrac{19g}{5}$

$= 37\text{ N}$ (2 s.f.)

c Assuming the pulley is smooth means there is no friction between it and the string.

12 Let the normal reaction be R N.



First, resolve along the plane to find P as it is the only unknown when resolving in that direction.

$R(\nearrow)$

$P\cos 30° - 5\cos 45° - 20\sin 45° = 0$

$$\therefore P = \frac{5\cos 45° + 20\sin 45°}{\cos 30°}$$
$$= \left(5\times\frac{\sqrt{2}}{2} + 20\times\frac{\sqrt{2}}{2}\right)\times\frac{2}{\sqrt{3}}$$
$$= \frac{25\sqrt{2}}{\sqrt{3}}$$
$$= \frac{25\sqrt{6}}{3}$$
$$= 20.4 \text{ (3 s.f.)}$$

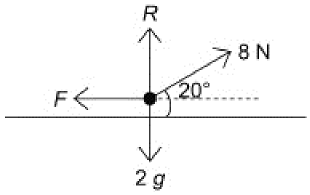
$R(\nwarrow)$

$R + P\sin 30° + 5\sin 45° - 20\cos 45° = 0$

$$R = 20\cos 45° - 5\sin 45° - P\sin 30° \quad (\text{as } P = \frac{25\sqrt{6}}{3})$$
$$R = \frac{15}{\sqrt{2}} - \frac{25\sqrt{6}}{6}$$
$$= \frac{45\sqrt{2} - 25\sqrt{6}}{6}$$
$$= 0.400 \text{ (3 s.f.)}$$

Applications of forces 7C

1 Let the normal reaction be R N, the friction force be F N and the coefficient of friction be μ.



Resolve horizontally to find F, vertically to find R and use $F = \mu R$ to find μ:

$R(\rightarrow)$

$8\cos 20° - F = 0$

$\therefore F = 8\cos 20°$

$R(\uparrow)$

$R + 8\sin 20° - 2g = 0$

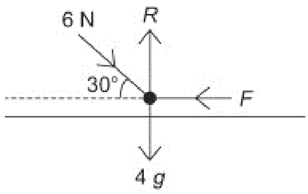
$\therefore R = 2g - 8\sin 20°$

As the book is on the point of slipping the friction is limiting:

$F = \mu R$

$$\begin{aligned}\therefore \mu &= \frac{F}{R}\\ &= \frac{8\cos 20°}{2g - 8\sin 20°}\\ &= \frac{7.518}{16.86}\\ &= 0.446 \text{ (3 s.f.)}\end{aligned}$$

2 Let the normal reaction be R N, the friction force be F N and the coefficient of friction be μ.



$R(\rightarrow)$: $6\cos 30° - F = 0$

$$\begin{aligned}F &= 6\cos 30°\\ &= 3\sqrt{3} = 5.20 \text{ (3 s.f.)}\end{aligned}$$

$R(\uparrow)$: $R - 6\sin 30° - 4g = 0$

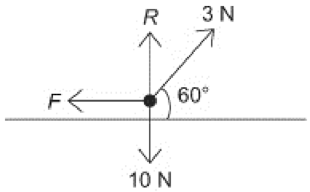
$$\begin{aligned}R &= 6\sin 30° + 4g\\ &= 3 + 4 \times 9.8\\ &= 42.2\end{aligned}$$

As the block is on the point of slipping

$F = \mu R$

$$\begin{aligned}\therefore \mu &= \frac{F}{R}\\ &= 0.123 \text{ (3 s.f.)}\end{aligned}$$

3 Let the normal reaction force be R and the friction force be F.



a Resolve horizontally to find the magnitude of the friction force necessary to maintain equilibrium:

$R(\rightarrow)$

$3\cos 60° - F = 0$

$\therefore F = 3\cos 60°$

$F = 1.5\text{ N}$

b Resolve vertically to calculate R and hence μR :

$R(\uparrow)$

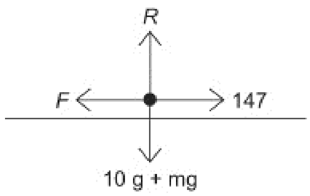
$R + 3\sin 60° - 10 = 0$

$$\begin{aligned}\therefore R &= 10 - 3\sin 60° \\ &= 10 - \frac{3\sqrt{3}}{2} \\ &= 7.40 \text{ (3 s.f.)}\end{aligned}$$

$$\begin{aligned}\therefore \mu R &= 0.3 \times 7.40 \\ &= 2.22 \text{ (3 s.f.)}\end{aligned}$$

Since $F = 1.5\text{ N} < 2.2\text{ N} = \mu R$, the friction required to maintain equilibrium is not limiting friction.

4 a Let the normal reaction be R N and the friction force required to maintain equilibrium be F N.
Let the mass of the books be m kg.



$R(\rightarrow)$

$147 - F = 0$

$\therefore F = 147\text{ N}$

$R(\uparrow)$

$R - 10g - mg = 0$

$\therefore R = 10g + mg$

As the equilibrium is limiting, $F = \mu R$

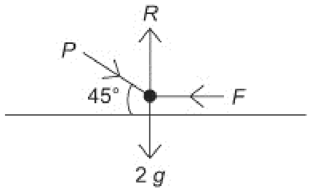
$147 = 0.3(10g + mg)$

$147 = 3g + 0.3mg$

$$\begin{aligned}\therefore m &= \frac{147 - 3g}{0.3g} \\ &= 40 \text{ kg}\end{aligned}$$

b The assumption is that the crate and books may be modelled as a particle.

5 a Let R be the normal reaction and F be the force of friction when P acts downwards.



$R(\rightarrow)$
$P\cos 45° - F = 0$
$\therefore F = P\cos 45°$

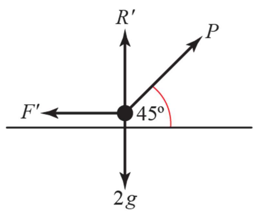
$R(\uparrow)$
$R - P\sin 45° - 2g = 0$
$\therefore R = P\sin 45° + 2g$

Resolve horizontally and vertically to find F and R, then use the condition for limiting friction.

As the equilibrium is limiting, $F = \mu R$

$$\therefore P\cos 45° = 0.3(P\sin 45° + 2g)$$
$$P(\cos 45° - 0.3\sin 45°) = 0.6g$$
$$\therefore P = \frac{0.6g}{\cos 45° - 0.3\sin 45°}$$
$$= \frac{6g\sqrt{2}}{7}$$
$$= 11.9\text{ N (3 s.f.)}$$

b Let R' be the normal reaction and F' be the force of friction when P acts upwards.



$R(\rightarrow)$
$P\cos 45° - F' = 0$
$\therefore F' = P\cos 45°$

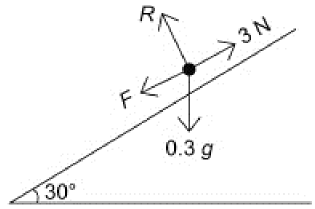
$R(\uparrow)$
$R' + P\sin 45° - 2g = 0$
$\therefore R' = 2g - P\sin 45°$

As the equilibrium is limiting, $F = \mu R$

$$\therefore P\cos 45° = 0.3(2g - P\sin 45°)$$
$$P(\cos 45° + 0.3\sin 45°) = 0.6g$$
$$\therefore P = \frac{6g\sqrt{2}}{13}$$
$$= 6.40\text{ N (3 s.f.)}$$

6 Let R be the normal reaction and F be the force of friction required to maintain equilibrium.

Since the particle is on the point of slipping up the plane, the force of friction acts down the slope.



$R(\nearrow)$

$3 - F - 0.3g\sin 30° = 0$

$\therefore F = 3 - 0.3g\sin 30°$

$= 1.53\,\text{N}$

$R(\nwarrow)$

$R - 0.3g\cos 30° = 0$

$\therefore R = 0.3g\cos 30°$

$= 2.546\,\text{N}$

As the particle is on the point of slipping, $F = \mu R$

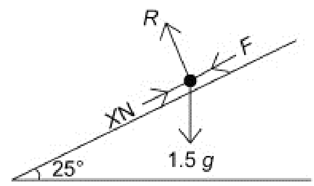
$\therefore 1.53 = \mu \times 2.546$

$\therefore \mu = \dfrac{1.53}{2.546}$

$= 0.601$ (3 s.f.) (accept 0.6)

7 Let R be the normal reaction and F be the force of friction required to maintain equilibrium.

Since the particle is on the point of slipping up the plane, the force of friction acts down the slope.



a $R(\nwarrow)$

$R - 1.5g\cos 25° = 0$

$R - 1.5g\cos 25° = 0$

$\therefore R = 1.5g\cos 25°$

$= 13.3\,\text{N}$ (3 s.f.)

b $R(\nearrow)$

$X - F - 1.5g\sin 25° = 0$

$X = F + 1.5g\sin 25° \qquad \textbf{(1)}$

The particle is in limiting equilibrium, so $F = \mu R$

$\therefore F = 0.25 \times 13.3227$ (using $R = 13.3$ N from **a**)

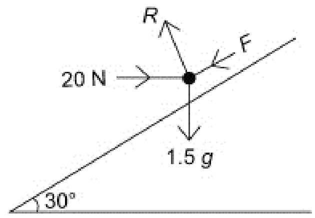
$= 3.3306...$

Sub $F = 3.33$ N into (**1**):

$X = 3.33 + 1.5g\sin 25°$

$= 9.54\,\text{N}$ (3 s.f.)

8 Let the normal reaction be R and the friction force be F acting down the plane.



a $R(\nwarrow)$

$R - 20\sin 30° - 1.5g\cos 30° = 0$

$$\therefore R = 20\sin 30° + 1.5g\cos 30°$$
$$= 22.7 \text{ (3 s.f.)}$$

The normal reaction has magnitude 22.7 N or 23 N (2 s.f.).

b $R(\nearrow)$

$20\cos 30° - F - 1.5g\sin 30° = 0$

$$\therefore F = 20\cos 30° - 1.5g\sin 30°$$
$$= 9.97 \text{ (3 s.f.)}$$

The friction force has magnitude 9.97 N and acts down the plane.

c Minimum possible value of μ occurs when frictional force required to maintain equilibrium is μR :

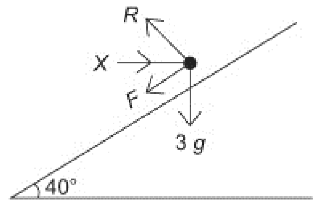
$$F = \mu R$$
$$9.9705... = \mu \times 22.730... \quad \text{(using } F \text{ from } \mathbf{b} \text{ and } R \text{ from } \mathbf{a}\text{)}$$
$$\mu = \frac{22.730...}{9.9705...}$$
$$= 0.43863...$$

The coefficient of friction must be at least 0.439 (3s.f.) to prevent the block sliding.

If you are told the particle is in equilibrium, but not told which way the particle is about to slip, then draw a diagram showing all the forces acting on the particle, with friction acting down the plane.

Resolve forces parallel to the plane. If $F > 0$ then you have chosen the correct direction. If $F < 0$ then you know friction acts up the plane.

9 Let the normal reaction be R and the friction force be F acting down the plane.



a

$R(\nwarrow)$

$R - X\sin 40° - 3g\cos 40° = 0$

$$R = X\sin 40° + 3g\cos 40° \quad \textbf{(1)}$$

$R(\nearrow)$

$X\cos 40° - F - 3g\sin 40° = 0$

$$F = X\cos 40° - 3g\sin 40° \quad \textbf{(2)}$$

As the friction is limiting, $F = \mu R$
Using F from (**2**) and R from (**1**) gives:

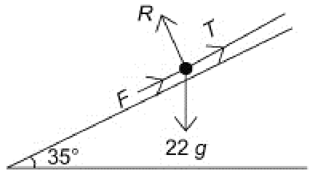
$$X\cos 40° - 3g\sin 40° = 0.3(X\sin 40° + 3g\cos 40°)$$
$$X\cos 40° - 0.3X\sin 40° = 0.9g\cos 40° + 3g\sin 40°$$
$$X(\cos 40° - 0.3\sin 40°) = 0.9g\cos 40° + 3g\sin 40°$$
$$X = \frac{0.9g\cos 40° + 3g\sin 40°}{\cos 40° - 0.3\sin 40°}$$
$$= \frac{25.65}{0.5732}$$
$$X = 44.8 \text{ N (3 s.f.)}$$

b Substituting $X = 44.8$ N into equation (**1**) gives

$$R = 44.8 \times \sin 40° + 3g\cos 40°$$
$$= 51.3 \text{ N (3 s.f.)}$$

10 Let the normal reaction be R and the friction force be F acting up the plane.

The friction acts up the plane, as the sledge is on the point of slipping down the plane.



$R(\nearrow)$

$T + F - 22g\sin 35° = 0$ **(1)**

$R(\nwarrow)$

$R - 22g\cos 35° = 0$

$\therefore R = 22g\cos 35°$

$R = 176.6$ N

As the friction is limiting, $F = \mu R$

$\therefore F = 0.125 \times 176.6$

$= 22.1$ N (3 s.f.)

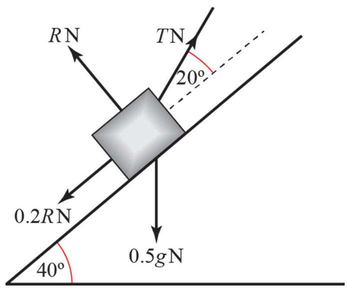
Substituting $T = 22.1$ N into equation (**1**) gives:

$T = 22g\sin 35° - 22.1$

$= 101.6$

$= 102$ N (3 s.f.)

11 $R(\nwarrow)$



$R - 0.5g\cos 40^\circ + T\sin 20^\circ = 0$

$R = 0.5g\cos 40^\circ - T\sin 20^\circ$

T_{MAX} occurs when the particle is on the point of moving up the plane.
At this point, limiting friction $F = \mu R$ acts down the plane:

$R(\nearrow)$

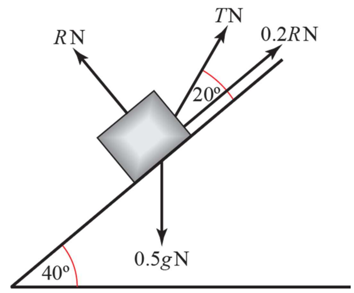
$$T_{\text{MAX}}\cos 20^\circ - 0.5g\sin 40^\circ - F = 0$$

$$T_{\text{MAX}}\cos 20^\circ - 0.5g\sin 40^\circ - \frac{1}{5}\left(0.5g\cos 40^\circ - T\sin 20^\circ\right) = 0$$

$$T_{\text{MAX}}\cos 20^\circ + 0.2T_{\text{MAX}}\sin 20^\circ = 0.5g\sin 40^\circ + 0.1g\cos 40^\circ$$

$$T_{\text{MAX}} = \frac{0.5g\sin 40^\circ + 0.1g\cos 40^\circ}{\cos 20^\circ + 0.2\sin 20^\circ}$$

$$= 3.8690...$$



11 T_{MIN} occurs when the particle is on the point of moving down the plane.
At this point, limiting friction $F = \mu R$ acts up the plane:

$R(\nearrow)$

$$T_{\text{MIN}} \cos 20^\circ - 0.5g \sin 40^\circ - F = 0$$

$$T_{\text{MIN}} \cos 20^\circ - 0.5g \sin 40^\circ + \frac{1}{5}\left(0.5g \cos 40^\circ - T \sin 20^\circ\right) = 0$$

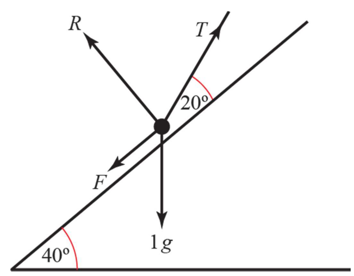
$$T_{\text{MIN}} \cos 20^\circ - 0.2T_{\text{MIN}} \sin 20^\circ = 0.5g \sin 40^\circ - 0.1g \cos 40^\circ$$

$$T_{\text{MAX}} = \frac{0.5g \sin 40^\circ - 0.1g \cos 40^\circ}{\cos 20^\circ - 0.2 \sin 20^\circ}$$

$$= 2.7533...$$

T lies between 2.75 N and 3.87 N (both values to 3s.f.).

12 $R(\nwarrow)$



$$R + 10 \sin 20° - g \cos 40° = 0$$

$$R = g \cos 40° - 10 \sin 20°$$
$$= 4.087$$

$R(\nearrow)$

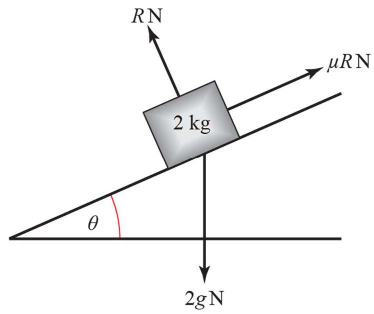
$$10 \cos 20° - F - g \sin 40° = 0$$

$$F = 10 \cos 20° - g \sin 40°$$
$$= 3.0976...$$

As the friction is limiting, $F = \mu R$

$$\therefore \mu = \frac{F}{R}$$
$$= \frac{3.0976}{4.087}$$
$$= 0.758 \quad \text{(3 s.f.)}$$

13 $\tan\theta = \frac{3}{4}$ so $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$



Before P is applied, the particle will be on the point of moving down the slope and the limiting frictional force therefore acts up the slope:

$R(\nwarrow)$

$R = 2g\cos\theta$

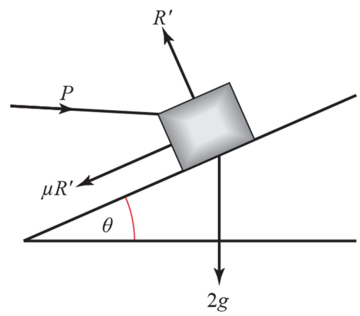
$R(\nearrow)$

$$2g\sin\theta = F_{\text{Max}}$$
$$2g\sin\theta = \mu R$$
$$2g\sin\theta = \mu \times 2g\cos\theta$$
$$\mu = \tan\theta$$
$$\mu = \frac{3}{4}$$

After applying the max value of P that will allow the particle to remain in equilibrium, particle is on the point of moving up the slope. Therefore, the frictional force acts down the slope.

$R(\nwarrow)$

$R' = 2g\cos\theta + P\sin\theta$



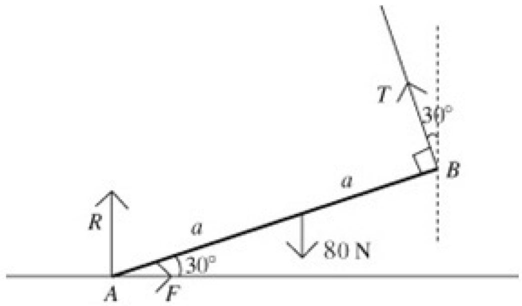
$R(\nearrow)$

$$P\cos\theta = F_{\text{Max}} + 2g\sin\theta$$
$$P\cos\theta = \mu R' + 2g\sin\theta$$
$$P\cos\theta = \mu(2g\cos\theta + P\sin\theta) + 2g\sin\theta$$
$$P(\cos\theta - \mu\sin\theta) = 2g(\mu\cos\theta + \sin\theta)$$
$$P\left(\frac{4}{5} - \frac{3}{4}\times\frac{3}{5}\right) = 2\times 9.8\times\left(\frac{3}{4}\times\frac{4}{5} + \frac{3}{5}\right)$$
$$0.35P = 23.52$$
$$P = 67.2 \text{ N}$$

So max. P is 67.2 N

Applications of forces 7D

1 a Suppose that the rod has length $2a$.



Taking moments about A:

$2aT = 80 \times a\cos 30°$

$2T = 80 \times \frac{\sqrt{3}}{2}$

$T = 20\sqrt{3}$

$= 34.6 \text{ N}$

$R(\rightarrow)$, $\quad F = T\sin 30° = 10\sqrt{3} = 17.3 \text{ N}$

$R(\uparrow)$, $\quad T\cos 30° + R = 80$

$R = 80 - 20\sqrt{3} \times \frac{\sqrt{3}}{2}$

$= 50 \text{ N}$

In order for the rod to remain in equilibrium, we must have $F \leqslant \mu R$:

$10\sqrt{3} \leqslant \mu \times 50$

$\mu \geqslant \frac{10\sqrt{3}}{50}$

$\mu \geqslant \frac{\sqrt{3}}{5}$

$\therefore$ minimum $\mu = 0.35$ (2 s.f.)

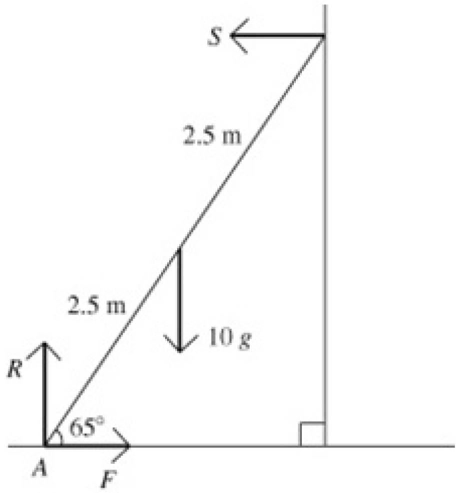
So $T = 34.6$ N, $F = 17.3$ N, $R = 50$ N, minimum $\mu = 0.35$

b Reaction at floor will be resultant of R and F

Magnitude = $\sqrt{50^2 + 17.3^2} = 53$ N (2 s.f.)

Angle above horizontal = $\tan^{-1}\left(\frac{50}{17.3}\right) = 71°$ (2 s.f.)

2 Let A be the end of the ladder on the ground.
Let F be the frictional force at A.



a Taking moments about A:
$10g \times 2.5\cos 65° = S \times 5\sin 65°$

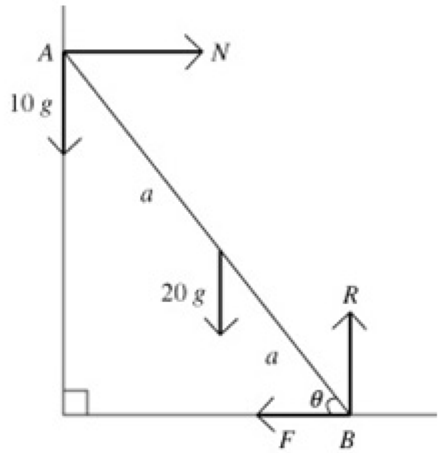
$$S = \frac{25g\cos 65°}{5\sin 65°} = \frac{5g}{\tan 65°} = 22.8\text{ N}$$

b $R(\rightarrow)$, $F = S = 22.8\text{ N}$
$R(\uparrow)$, $R = 10g = 98\text{ N}$

c To ensure ladder remains in equilibrium, we must have
$F \leqslant \mu R$
$22.8 \leqslant \mu \times 98$
$\mu \geqslant 0.233$ (3 s.f.)

d The weight is shown as acting through the midpoint of the ladder because of the assumption that the ladder is uniform.

3 Let the ladder have length $2a$, and be inclined at θ to the horizontal.



a $R(\uparrow)$, $R = 30g$
Taking moments about A:
$20g \times a\cos\theta + F \times 2a\sin\theta = R \times 2a\cos\theta$

$$20g\cos\theta + 2F\sin\theta = 60g\cos\theta \quad (\text{using } R = 30g)$$
$$2F\sin\theta = 40g\cos\theta$$
$$F = \frac{20g}{\tan\theta}$$

The ladder is on the point of slipping, so $F = \mu R$

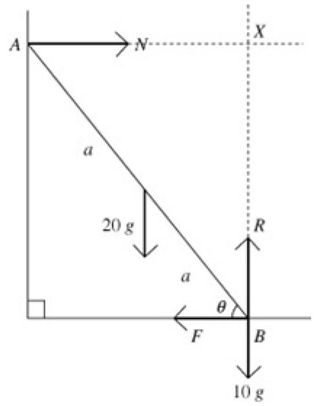
$$\frac{20g}{\tan\theta} = \frac{3}{4} \times 30g$$
$$\therefore \tan\theta = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$$
$$\therefore \theta = 41.6°$$

N is the normal reaction at A,
R is the normal reaction at B,
F is the frictional force at B.

3 b $R(\uparrow)$, $\quad R = 30g$

$R(\rightarrow)$, $\quad N - F = 0$

$$N = F$$



Taking moments about B:

$$20g \times a\cos\theta = N \times 2a\sin\theta$$

$$20g \times a\cos\theta = F \times 2a\sin\theta$$

$$F = \frac{10g\cos\theta}{\sin\theta}$$

$$F = \frac{10g}{\tan\theta}$$

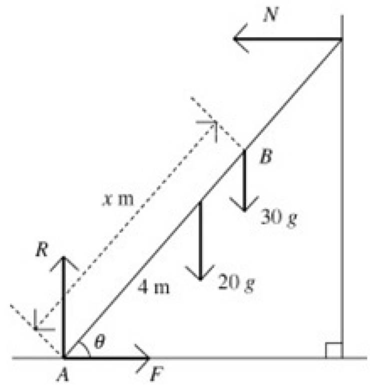
The ladder is on the point of slipping, so $F = \mu R$

$$\frac{10g}{\tan\theta} = \frac{3}{4} \times 30g$$

$$\tan\theta = \frac{4}{9}$$

$$\theta = 24.0°$$

c The assumption that the wall is smooth means there is no friction between the ladder and the wall.



4 a Suppose that the boy reaches the point B, a distance x from A, whilst the end of the ladder is still in contact with the ground.

$R(\rightarrow),\ F = N$

$R(\uparrow),\ R = 50g$

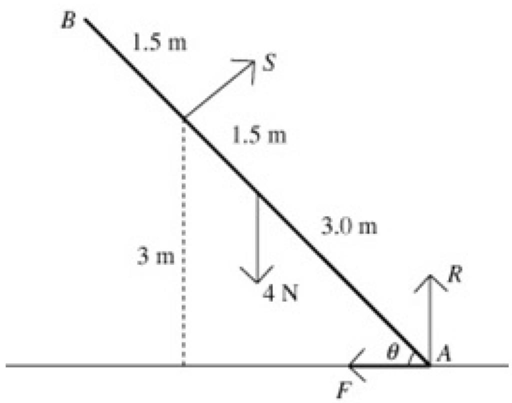
Taking moments about A:

$$20g \times 4\cos\theta + 30g \times x\cos\theta = N \times 8\sin\theta$$
$$80g + 30gx = 8N\tan\theta$$
$$N = \frac{80g + 30gx}{8\tan\theta}$$
$$N = \frac{80g + 30gx}{16} \quad \text{(since } \tan\theta = 2\text{)}$$
$$F = \frac{80g + 30gx}{16} \quad \text{(since } F = N\text{)}$$
$$\mu R = \frac{80g + 30gx}{16} \quad \text{(in limiting equilibrium)}$$
$$0.3 \times 50g = \frac{80g + 30gx}{16}$$
$$240 = 80 + 30x$$
$$x = 5\frac{1}{3} \text{ m}$$

b i The ladder may not be uniform.

ii There would be friction between the ladder and the wall.

5 Let:
S be the normal reaction of the rail on the pole at C,
R be the normal reaction of the ground on the pole at A,
F be the friction between the pole and the ground at A.
θ be the angle between the pole and the ground.



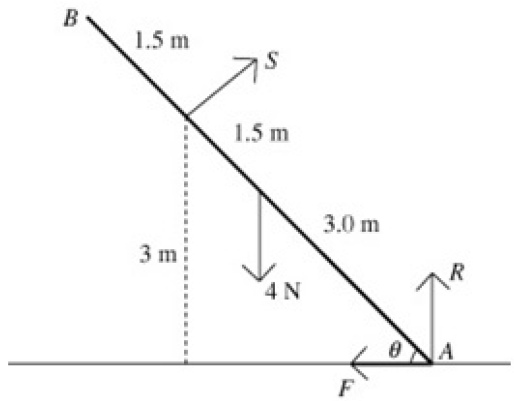
From the diagram,

$$\sin\theta = \frac{3}{4.5} = \frac{2}{3} \text{ and hence } \cos\theta = \frac{\sqrt{9-4}}{3} = \frac{\sqrt{5}}{3}$$

a Taking moments about A:

$$4.5S = 4 \times 3\cos\theta$$
$$= \frac{12\sqrt{5}}{3}$$
$$= 4\sqrt{5}$$
$$S = \frac{8\sqrt{5}}{9} \text{ N}$$

5 b $R(\rightarrow)$



$$F = S\sin\theta$$
$$= \frac{8\sqrt{5}}{9} \times \frac{2}{3}$$
$$= \frac{16\sqrt{5}}{27}$$

$R(\uparrow)$

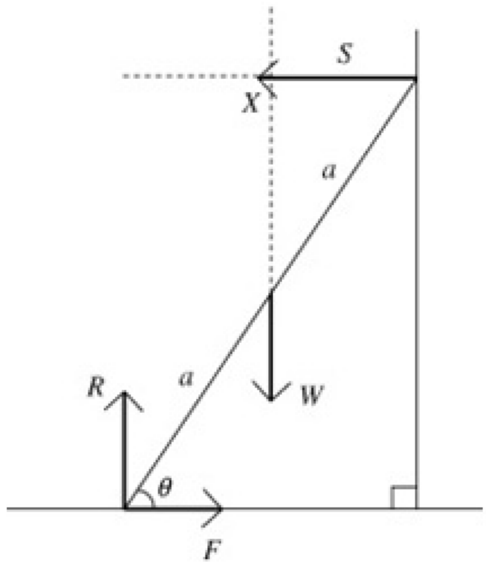
$$R + S\cos\theta = 4$$
$$R = 4 - \frac{8\sqrt{5}}{9} \times \frac{\sqrt{5}}{3}$$
$$= 4 - \frac{40}{27}$$
$$= \frac{68}{27}$$

Pole is in limiting equilibrium, so $F = \mu R$

$$\frac{16\sqrt{5}}{27} = \mu \times \frac{68}{27}$$
$$\therefore \mu = \frac{16\sqrt{5}}{68}$$
$$= \frac{4\sqrt{5}}{17}$$
$$= 0.526 \text{ (3 s.f.)}$$

c The assumption that the rail is smooth means there is no friction between the rail and the pole.

6 Suppose that the ladder has length $2a$ and weight W.
Let:
S be the normal reaction of the wall on the ladder,
R be the normal reaction of the floor on the ladder,
F be the friction between the floor and the ladder.
X be the point where the lines of action of W and S meet.



Taking moments about X:

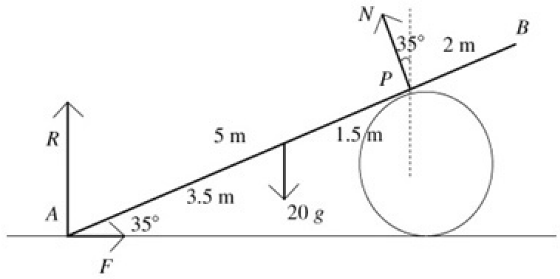
$$2a\sin\theta \times F = R \times a\cos\theta$$
$$2F\sin\theta = R\cos\theta \qquad \textbf{(1)}$$

The ladder is in limiting equilibrium, so $F = \mu R$

Substituting $F = \mu R$ in (**1**):

$$2\mu R\sin\theta = R\cos\theta$$
$$2\mu\sin\theta = \cos\theta$$
$$\frac{2\mu\sin\theta}{\cos\theta} = 1$$
$$2\mu\tan\theta = 1$$

7 Let:
N be the normal reaction of the drum on the ladder at P,
R be the normal reaction of the ground on the ladder at A,
F be the friction between the ground and the ladder at A.



Taking moments about A:

$$20g \times 3.5\cos 35° = 5N$$

$$N = \frac{20g \times 3.5\cos 35°}{5}$$

$$= 14g\cos 35°$$

$R(\uparrow)$

$$N\cos 35° + R = 20g$$

$$R = 20g - 14g\cos 35° \times \cos 35°$$

$$= 103.9\ldots \text{N}$$

$R(\rightarrow)$

$$F = N\sin 35°$$

$$= 14g\cos 35° \times \sin 35°$$

$$= 64.46\ldots \text{N}$$

$F \leqslant \mu R$ to maintain equilibrium:

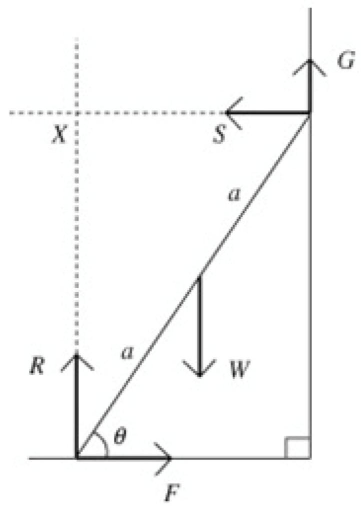
$$14g\cos 35° \sin 35° \leq \mu(20g - 14g\cos^2 35°)$$

$$\mu \geq \frac{14\cos 35° \sin 35°}{20 - 14\cos^2 35°}$$

$$\mu \geq 0.620 \text{ (3 s.f.)}$$

Least possible μ is 0.620 (3 s.f.)

8 Let:
R be the reaction of the ground on the ladder
F be the friction between the ground and the ladder
S be the reaction of the wall on the ladder
G be the friction between the wall and the ladder.
X be the point where the lines of action R and S meet.



Suppose that the ladder has length $2a$ and weight W.

As the ladder rests in limiting equilibrium, $F = \mu_1 R$ and $G = \mu_2 S$.

Taking moments about X:

$$W \times a\cos\theta = F \times 2a\sin\theta + G \times 2a\cos\theta$$

$$W = 2F\tan\theta + 2G \qquad \textbf{(1)}$$

$R(\rightarrow)$, $F = S$

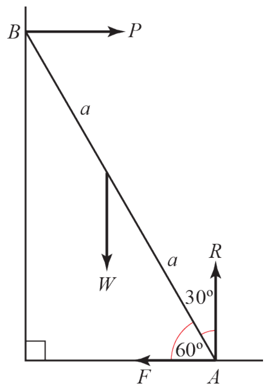
$R(\uparrow)$, $W = R + G$

8 Substituting for W and F in equation (**1**):

$$R + G = 2\mu_1 R\tan\theta + 2G$$
$$R - G = 2\mu_1 R\tan\theta$$
$$R - \mu_1\mu_2 R = 2\mu_1 R\tan\theta \qquad (\text{Since } G = \mu_2 S = \mu_2 F = \mu_2\mu_1 R)$$

Hence $\dfrac{1-\mu_1\mu_2}{2\mu_1} = \tan\theta$

9 Let:
A and B be the ends of the ladder.
P be the normal reaction of the wall on the ladder at B,
R the normal reaction of the ground on the ladder at A
F be the friction at between the ladder and the ground at A



Let the length of the ladder be $2a$.

a Taking moments about A:
$W \times a\cos 60° = P \times 2a\cos 30°$

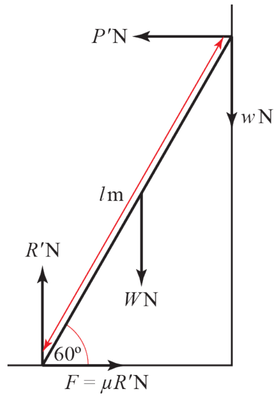
$$P = \frac{Wa\cos 60°}{2a\cos 30°}$$
$$P = \frac{W \times \frac{1}{2}}{2 \times \frac{\sqrt{3}}{2}}$$
$$P = \frac{W}{2\sqrt{3}} \qquad (\mathbf{1})$$

b $R(\uparrow)$, $\quad R = W \qquad (\mathbf{2})$
$R(\rightarrow)$, $\quad F = P \qquad (\mathbf{3})$

Now $F \leqslant \mu R$ since the ladder is in equilibrium (if not, ladder would slide)
Hence, $\quad P \leqslant \mu R \quad$ (by (**3**))

$$\frac{W}{2\sqrt{3}} \leqslant \mu R \quad \text{(by (1))}$$
$$\frac{W}{2\sqrt{3}} \leqslant \mu W \quad \text{(by (2))}$$
$$\mu \geqslant \frac{\sqrt{3}}{6}$$

9 c Let:
R' be the normal reaction of the ground on the ladder at A
P' be the normal reaction of the wall on the ladder at B,
l be the length of the ladder



Since the ladder is in limiting equilibrium, $F' = \mu R'$

$R(\uparrow)$, $R' = W + w$

$R(\rightarrow)$, $\mu R' = P'$

Taking moments about B:

$$\frac{Wl\cos 60^\circ}{2} + \left(F' \times l\sin 60^\circ\right) = \left(R' \times l\cos 60^\circ\right)$$

$$\frac{W}{4} + \left(\mu R' \times \frac{\sqrt{3}}{2}\right) = \frac{R'}{2}$$

$$\frac{W}{4} + \left(\frac{\sqrt{3}}{5}(W + w) \times \frac{\sqrt{3}}{2}\right) = \frac{W + w}{2} \qquad \text{(since } R' = W + w \text{ and } \mu = \frac{\sqrt{3}}{5}\text{)}$$

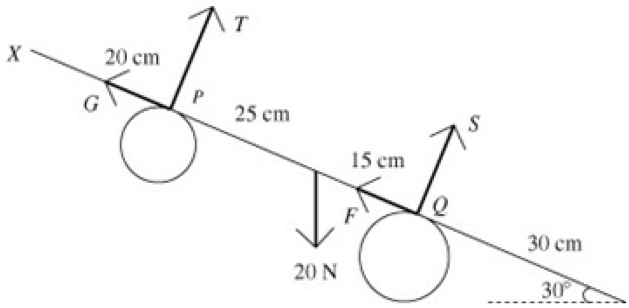
$$W + \frac{6}{5}(W + w) = 2(W + w)$$

$$5W + 6W + 6w = 10W + 10w$$

$$W = 4w$$

$$\Rightarrow w = \frac{W}{4}$$

10 Let:
T be the normal force of the peg on the rod at P,
G be the frictional force at P,
S be the normal force of the peg on the rod at Q,
F be the frictional force at Q.



a Taking moments about P:

$$S \times 40 = 20 \times 25 \times \cos 30^\circ$$

$$S = \frac{20 \times 25 \times \frac{\sqrt{3}}{2}}{40}$$

$$S = \frac{25\sqrt{3}}{4}\text{ N}$$

Taking moments about Q:

$$T \times 40 = 20 \times 15 \times \cos 30^\circ$$

$$T = \frac{20 \times 15 \times \frac{\sqrt{3}}{2}}{40}$$

$$T = \frac{15\sqrt{3}}{4}\text{ N}$$

10 b $R(\searrow)$

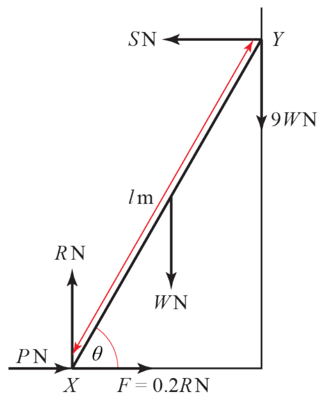
$$G + F = 20\cos 60° = 10 \quad \textbf{(1)}$$

Since the rod is about to slip, friction is limiting and hence $G = \mu T$, $F = \mu S$.
From part **a,**

$$G + F = \mu T + \mu S = \mu \times \frac{40\sqrt{3}}{4} = 10\sqrt{3}\mu \ \textbf{(2)}$$

$$\textbf{(1)} = \textbf{(2)} \Rightarrow \mu = \frac{1}{\sqrt{3}}$$

11 a Let:
S be the normal reaction of the wall on the ladder at Y,
R be the normal reaction of the ground on the ladder at X
F be the friction at between the ladder and the ground at X



$\tan\theta = \sqrt{3}$ so $\sin\theta = \frac{\sqrt{3}}{3}$ and $\cos\theta = \frac{1}{2}$

Ladder is in equilibrium.
Taking moments about X:

$$\frac{Wl\cos\theta}{2} + 9Wl\cos\theta = Sl\sin\theta$$

$$\frac{W}{4} + \frac{9W}{2} = \frac{\sqrt{3}S}{2}$$

$$\sqrt{3}S = \frac{W}{2} + 9W$$

$$\sqrt{3}S = \frac{19W}{2}$$

$$S = \frac{19W}{2\sqrt{3}}$$

b R(↑): $R = W + 9W = 10W$
For the ladder to be in limiting equilibrium,

$$F = \mu R$$

$$F = \frac{1}{5} \times 10W$$

$$F = 2W$$

R(→):
If $P + F > S$, ladder will slide towards and up the wall
If $P < S - F$, ladder will slide away from and down the wall
Therefore $S - F \leq P \leq S + F$

Substituting values for S & F from part **a** and above:

$$\frac{19W}{2\sqrt{3}} - 2W \leq P \leq \frac{19W}{2\sqrt{3}} + 2W$$

$$\left(\frac{19}{2\sqrt{3}} - 2\right)W \leq P \leq \left(\frac{19}{2\sqrt{3}} + 2\right)W$$

11 c Modelling the ladder as uniform allows us to assume the weight acts through the midpoint.

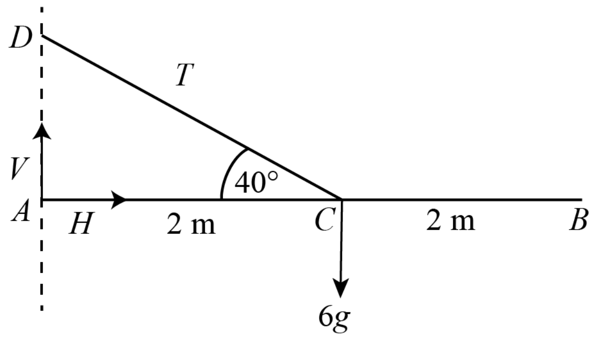
d i The reaction of the wall on the ladder will decrease. To understand why, consider how we took moments about X in part **a**

$$\frac{Wl\cos\theta}{2}+9Wl\cos\theta=Sl\sin\theta$$

The first term in this equation is the turning moment of the weight of the ladder, which acts at a distance $\frac{l}{2}$ from X. If the centre of mass of the ladder is more towards X, say $\frac{l}{a}$ where $a>2$, then this first term would decrease and hence S would also decrease.

ii Ladder remains in equilibrium when $S-F\le P\le S+F$
If S were to decrease, then this range of values for P would also decrease.

12 a



Taking moments about A

$M(A): 6g\times 2=2T\sin 40°$

$T=\dfrac{6g}{\sin 40°}=91.47656...=91.5\text{ N (1 d.p.)}$

b Consider all forces acting on AB

$R(\uparrow)$: $V+T\sin 40°=6g$

$V=6\times 9.8-94.47656...\times\sin(40°)=0\text{ N}$

R(→): $H=T\cos 40°=70.075...=70.1\text{ N (1 d.p.)}$

The force exerted on the rod by the wall is 70.1 N parallel to and towards the rod.

13 a Taking moments about A

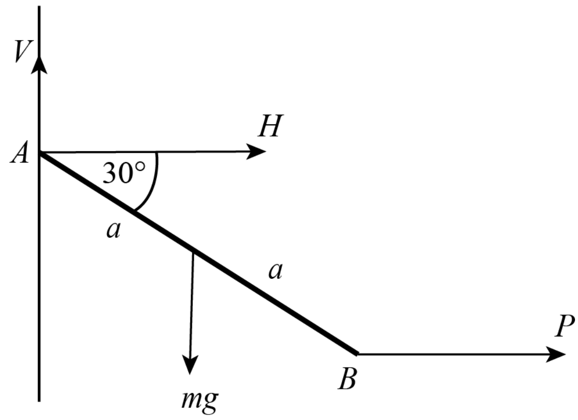
M(A):

$$mg \times a\cos 30° = P \times 2a \times \sin 30°$$

$$P = \frac{mga\cos 30°}{2a\sin 30°}$$

$$= \frac{mga\frac{\sqrt{3}}{2}}{2a\frac{1}{2}}$$

$$= \frac{\sqrt{3}}{2}mg$$



b $R(\uparrow)$: $V = mg$

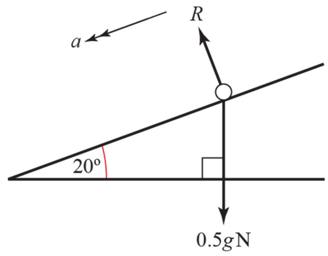
R($\rightarrow$): $H = -\frac{\sqrt{3}}{2}mg$

Magnitude of force at the hinge $= \sqrt{(mg)^2 + \left(-\frac{\sqrt{3}}{2}mg\right)^2} = mg\sqrt{1+\frac{3}{4}} = \frac{\sqrt{7}}{2}mg$

Angle of force at hinge $\theta = \arctan\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} = 49.1°$ above the horizontal away from the rod.

Applications of forces 7E

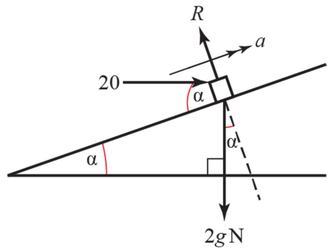
1 $R(\swarrow)$



$$F = ma$$
$$0.5g \sin 20° = 0.5a$$
$$a = 3.35 \text{ (3 s.f.)}$$

2 $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5} \text{ and } \cos\alpha = \frac{4}{5}$

a $R(\nwarrow)$



$$R - 20\sin\alpha - 2g\cos\alpha = 0$$
$$R = 20\sin\alpha + 19.6\cos\alpha$$
$$= 12 + 15.68$$
$$= 27.7 \text{ N}$$

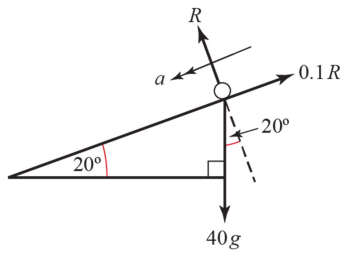
The normal reaction is 27.7 N (3 s.f.).

b $R(\nearrow)$

$$F = ma$$
$$20\cos\alpha - 2g\sin\alpha = 2a$$
$$2a = 20 \times \frac{4}{5} - 2 \times 9.8 \times \frac{3}{5}$$
$$a = 2.12 \text{ m s}^{-2}$$

The acceleration of the box is 2.12 m s^{-2}

3 a $R(\nwarrow)$



$$R - 40g\cos 20° = 0$$
$$R = 368.36$$

$R(\swarrow)$

$$40g\sin 20° - 0.1R = 40a$$
$$392\cos 70° - 36.836 = 40a$$
$$a = 2.43 \text{ (3 s.f.)}$$

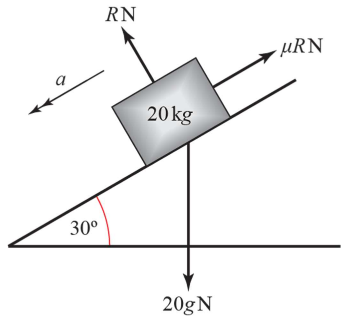
The acceleration of the boy is 2.43 m s^{-2} (3 s.f.).

b $u = 0, \quad a = 2.43, \quad s = 5, \quad v = ?$

$$v^2 = u^2 + 2as$$
$$v^2 = 0^2 + 2 \times 2.43 \times 5 = 24.3$$
$$v = 4.93 \text{ m s}^{-1} \text{ (3 s.f.)}$$

The speed of the boy is 4.93 m s^{-1} (3 s.f.).

4 $u = 0\text{ ms}^{-1}$, $v = 21\text{ ms}^{-1}$, $t = 6$ s, a =?



$$v = u + at$$
$$21 = 0 + 6a$$
$$a = \frac{21}{6} = \frac{7}{2}$$

$R(\nwarrow)$:

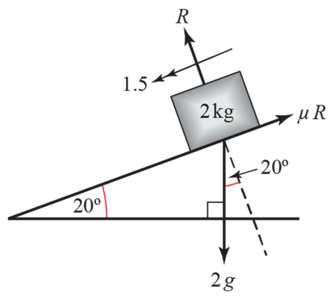
$$R = 20g\cos 30^\circ$$

$R(\swarrow)$

$$F = ma$$
$$20g\sin 30^\circ - \mu R = 20 \times \frac{7}{2}$$
$$20g\sin 30^\circ - (\mu \times 20g\cos 30^\circ) = 20 \times \frac{7}{2}$$
$$\frac{1}{2}g - \frac{\sqrt{3}}{2}\mu g = \frac{7}{2}$$
$$g - \sqrt{3}\mu g = 7$$
$$\mu = \frac{9.8 - 7}{\sqrt{3} \times 9.8}$$
$$= 0.16495...$$

The coefficient of friction is 0.165 (3s.f.).

5 $R(\nwarrow)$



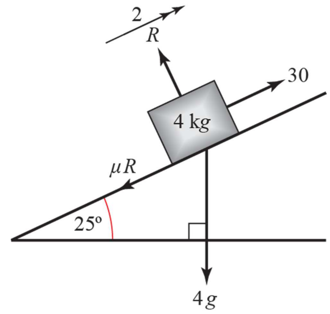
$$R - 2g\cos 20° = 0$$
$$R = 2g\cos 20°$$

$R(\swarrow)$

$$F = ma$$
$$2g\sin 20° - \mu R = 2 \times 1.5$$
$$2g\sin 20° - \mu \times 2g\cos 20° = 3$$
$$\mu = \frac{2g\sin 20° - 3}{2g\cos 20°} = 0.201 \text{ (3 s.f.)}$$

The coefficient of friction is 0.20 (2 s.f.).

6 $R(\nwarrow)$



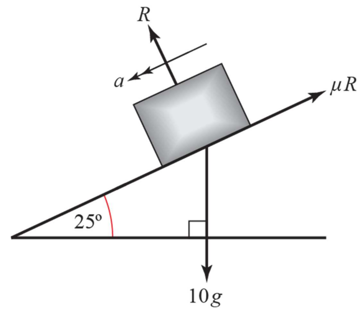
$$R - 4g\cos 25° = 0$$
$$R = 4g\cos 25°$$

$R(\nearrow)$

$$F = ma$$
$$30 - 4g\sin 25° - \mu R = 4 \times 2$$
$$30 - 4g\sin 25° - \mu 4g\cos 25° = 8$$
$$\frac{22 - 4g\sin 25°}{4g\cos 25°} = \mu$$
$$0.15\ (2\text{ s.f.}) = \mu$$

The coefficient of friction is 0.15 (2 s.f.).

7 a $R(\nwarrow)$



$$R - 10g\cos 25° = 0$$
$$R = 98\cos 25°$$
$$= 88.8\,\text{N}\ \ (3\text{ s.f.})$$

The normal reaction is $88.8\,\text{N}$ (3 s.f.).

b $u = 0,\ \ s = 4,\ \ t = 2,\ \ a = ?$

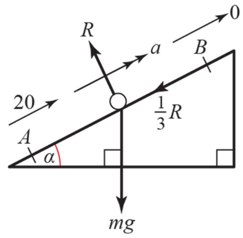
$$s = ut + \frac{1}{2}at^2$$
$$4 = 0 + \frac{1}{2}a \times 2^2$$
$$a = 2\,\text{m s}^{-2}$$

$R(\swarrow)$

$$F = ma$$
$$10g\sin 25° - \mu R = 10 \times 2$$
$$\mu \times 98\cos 25° = 10g\sin 25° - 20$$
$$\mu = \frac{98\sin 25° - 20}{98\cos 25°}$$
$$= 0.241\ (3\text{ s.f.})$$

The coefficient of friction is 0.24 (2 s.f.).

8 a Let mass of particle be m.



$R(\nwarrow)$

$$R - mg\cos\alpha = 0$$

$$R = \frac{4mg}{5}$$

$R(\nearrow)$

$$F = ma$$

$$-mg\sin\alpha - \frac{1}{3}R = ma$$

$$-\frac{3mg}{5} - \frac{1}{3}\times\frac{4mg}{5} = ma$$

$$-\frac{13g}{15} = a$$

The deceleration is $\frac{13g}{15}$.

b $u = 20,\ \ v = 0,\ \ a = -\frac{13g}{15},\ \ s = ?$

$$v^2 = u^2 + 2as$$

$$0 = 20^2 - \frac{26g}{15}s$$

$$s = \frac{6000}{26g} = 23.5\text{m (3 s.f.)}$$

$$AB = 23.5\text{ m (3 s.f.)}$$

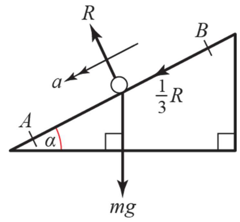
c $u = 20,\ \ v = 0,\ \ a = -\frac{13g}{15},\ \ t = ?$

$$v = u + at$$

$$0 = 20 - \frac{13gt}{15}$$

$$t = \frac{300}{13g} = 2.35\text{s (3 s.f.)}$$

8 d As the particle begins to decelerate downwards from B, friction now acts up the slope.



$R(\nwarrow)$

$R = \frac{4mg}{5}$, as before

$R(\swarrow)$

$$F = ma$$

$$mg\sin\alpha - \frac{1}{3}R = ma$$

$$\frac{3mg}{5} - \frac{1}{3} \times \frac{4mg}{5} = ma$$

$$\frac{g}{3} = a$$

Now use equations of motion for constant acceleration:

$u = 0,\quad a = \frac{g}{3},\quad s = \frac{6000}{26g},\quad v = ?$

$$v^2 = u^2 + 2as$$

$$v^2 = 0 + \frac{2g}{3} \times \frac{6000}{26g}$$

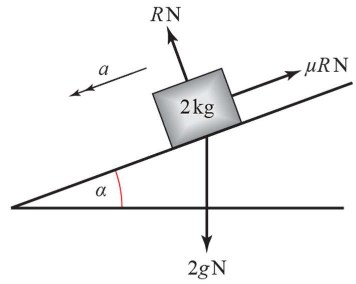
$$= \frac{4000}{26}$$

$v = 12.4\,\text{m s}^{-1}$ (3 s.f.)

The speed of the particle as it passes A on the way down is $12.4\,\text{m s}^{-1}$ (3 s.f.).

9 $\tan\alpha = \frac{2}{5} \Rightarrow \sin\alpha = \frac{2}{\sqrt{29}}$ and $\cos\alpha = \frac{5}{\sqrt{29}}$

$u = 0\text{ ms}^{-1}$, $v = 6\text{ ms}^{-1}$, $t = 3$ s, $a =$?



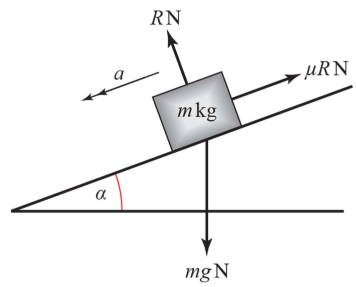
$$v = u + at$$
$$6 = 0 + 3a$$
$$a = \frac{6}{3} = 2\text{ ms}^{-2}$$

$R(\nwarrow)$
$$R = 2g\cos\alpha$$

$R(\swarrow)$
$$F = ma$$
$$2g\sin\alpha - \mu R = 2\times 2$$
$$2g\sin\alpha - (\mu\times 2g\cos\alpha) = 4$$
$$g\sin\alpha - \mu g\cos\alpha = 2$$
$$\mu = \frac{g\sin\alpha - 2}{g\cos\alpha}$$
$$\mu = \frac{\left(9.8\times\frac{2}{\sqrt{29}}\right) - 2}{9.8\times\frac{5}{\sqrt{29}}}$$
$$\mu = \frac{(9.8\times 2) - 2\sqrt{29}}{9.8\times 5}$$
$$= 0.18019...$$

The coefficient of friction is 0.180 (3s.f.).

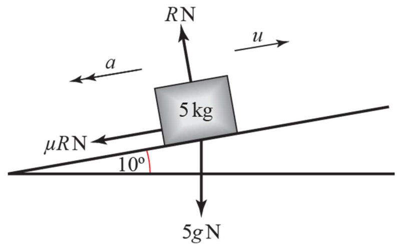
10 $R(\nwarrow)$
$$R = mg\cos\alpha$$



$R(\swarrow)$
$$F = ma$$
$$mg\sin\alpha - \mu R = ma$$
$$mg\sin\alpha - (\mu\times mg\cos\alpha) = ma$$
$$g\sin\alpha - \mu g\cos\alpha = a$$

Since this expression does not contain m, the acceleration is independent of the mass.

11 a $u = 16\text{ ms}^{-1}$, $v = 0\text{ ms}^{-1}$, $t = 5$ s, $a =?$



$$v = u + at$$

$$0 = 16 + 5a$$

$$a = -\frac{16}{5}$$

$R(\nwarrow)$

$$R = 5g\cos 10^\circ$$

$R(\swarrow)$

$$F = ma$$

$$5g\sin 10^\circ + \mu R = ma$$

$$5g\sin 10^\circ + (\mu \times 5g\cos 10^\circ) = 5 \times \frac{16}{5}$$

$$5g\sin 10^\circ + 5\mu g\cos 10^\circ = 16$$

$$\mu = \frac{16 - 5g\sin 10^\circ}{5g\cos 10^\circ}$$

$$= 0.15524...$$

The coefficient of friction is 0.155 (3s.f.).

b The particle will move back down the slope if the component of its weight acting down the slope is greater than the frictional force acting up the slope, i.e. if

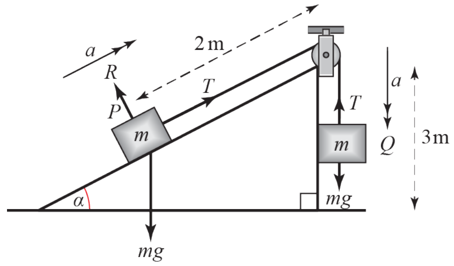
$$5g\sin 10^\circ > 5\mu g\cos 10^\circ$$

$$\sin 10^\circ > 0.155 \times \cos 10^\circ$$

$$0.17364... > 0.15288...$$

Since this inequality is true (i.e. 0.174 > 0.153), the particle will move back down the slope.

Applications of forces, 7F



1 For P: $R(\nearrow)$

$T - mg\sin\alpha = ma$

$$T - \frac{3mg}{5} = ma \quad \textbf{(1)}$$

For Q: $R(\downarrow)$

$mg - T = ma \qquad \textbf{(2)}$

$$\textbf{(1)} + \textbf{(2)}: mg - \frac{3mg}{5} = 2ma$$

$$\frac{g}{5} = a$$

For P:

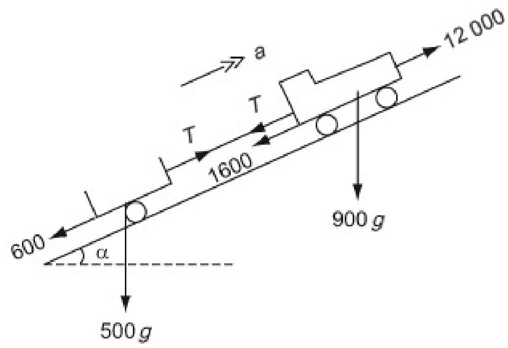
$u = 0,\ a = \frac{g}{5},\ s = 2,\ v = ?$

$v^2 = u^2 + 2as$

$v^2 = 0^2 + \frac{2g}{5} \times 2$

$v = \sqrt{\frac{4g}{5}} = 2.8\,\text{m s}^{-1}$

P hits the pulley with speed $2.8\,\text{m s}^{-1}$.



2 $R(\nearrow)$ For the Van:

$$F = ma$$

$$12\,000 - T - 1600 - 900g\sin\alpha = 900a$$

$$10400 - 900 \times 9.8 \times \frac{3}{5} - T = 900a$$

$$5108 - T = 900a \quad \textbf{(1)}$$

$R(\nearrow)$ For the Trailer:

$$F = ma$$

$$T - 600 - 500g\sin\alpha = 500a$$

$$T - 600 - 500 \times 9.8 \times \frac{3}{5} = 500a$$

$$T - 3540 = 500a \quad \textbf{(2)}$$

a $\textbf{(1)} + \textbf{(2)} \Rightarrow 1568 = 1400a$

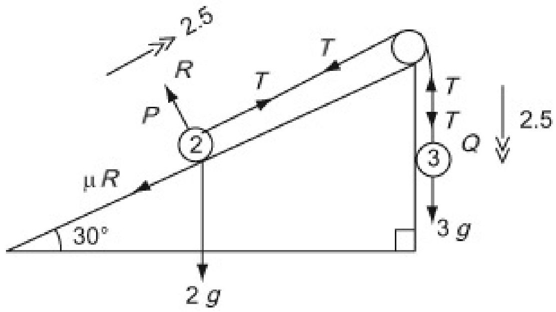
$$a = 1.12\ \text{ms}^{-2}$$

2 b Sub $a = \frac{1568}{1400}$ ms^{-2} in (**2**)

$$T = 3540 + 500 \times \frac{1568}{1400}$$
$$= 4100 \text{ N}$$

c The resistance forces are unlikely to be constant: it is more probable that they will increase as the speed increases.

3 a For P:



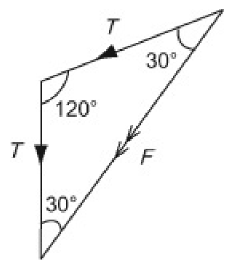
$R(\nwarrow)$
$$R - 2g\cos 30° = 0$$
$$R = g\sqrt{3}$$
$R(\nearrow)$
$$F = ma$$
$$T - \mu R - 2g\cos 60° = 2 \times 2.5$$
$$T - \mu g\sqrt{3} - g = 5 \qquad (\mathbf{1})$$

For Q:

$R(\downarrow)$
$$F = ma$$
$$3g - T = 3 \times 2.5$$
$$3g - T = 7.5 \qquad (\mathbf{2})$$
$$\therefore T = 21.9$$
The tension is 21.9 N.

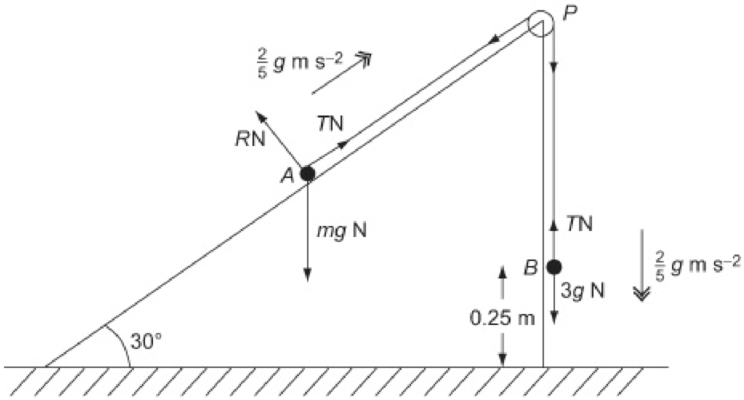
b $(\mathbf{1}) + (\mathbf{2}) \Rightarrow \ 2g - \mu g\sqrt{3} = 12.5$
$$\mu g\sqrt{3} = 7.1$$
$$\mu = \frac{7.1}{g\sqrt{3}}$$
$$= 0.418 \text{ (3 s.f.)}$$
The coefficient of friction is 0.42 (2 s.f.).

c $F = 2T\cos 30°$
$$= 43.8\cos 30°$$
$$= 37.9 \text{ N (3 s.f.)}$$



The force exerted by the string on the pulley is 38N (2 s.f.).

4 a For B:



$R(\downarrow)$

$$F = ma$$

$$3g - T = 3 \times \frac{2}{5}g$$

$$T = 3g - \frac{6}{5}g$$

$$= \frac{9}{5}g$$

$$= 17.64$$

The tension in the string while B is descending is 18 N (2 s.f.).

b For A:

$R(\nearrow)$

$$F = ma$$

$$T - mg\sin 30° = m \times \frac{2}{5}g$$

$$\frac{9}{5}g - \frac{1}{2}mg = \frac{2}{5}mg$$

$$\left(\frac{1}{2} + \frac{2}{5}\right)m = \frac{9}{5}$$

$$\frac{9}{10}m = \frac{9}{5}$$

$$\Rightarrow m = 2$$

4 c Whilst A is still ascending,

$$u = 0,\ a = \frac{2}{5}g,\ s = 0.25,\ v = ?$$

$$v^2 = u^2 + 2as$$

$$v^2 = \frac{4}{5}g \times 0.25$$

$$v = 1.4 \text{ ms}^{-1}$$

After B strikes the ground, there is no tension in the string and the only force acting on A parallel to the plane is the component of its weight acting down the plane.

For A:

$R(\nearrow)$

$$-mg\sin 30° = ma$$

$$a = -\frac{1}{2}g$$

$$u = 1.4,\quad v = 0,\quad a = -\frac{1}{2}g,\quad t = ?$$

$$v = u + at$$

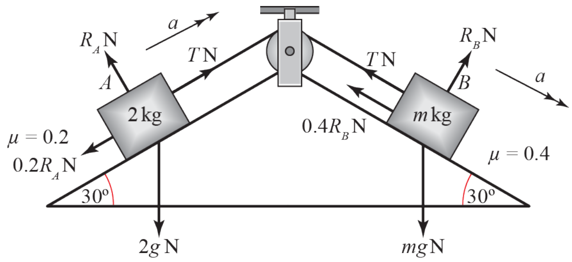
$$0 = 1.4 - \frac{1}{2}gt$$

$$\Rightarrow t = \frac{2.8}{9.8} = \frac{2}{7}$$

The approximate answer, 0.28 s, would also be acceptable.

The time between the instants is $\frac{2}{7}$ s

5 a Let the reaction forces on the blocks be R_A and R_B.
If the system is in limiting equilibrium for the maximum value of m, object B will move down the right-hand slope and object A will move up the left-hand slope.



For A:
$R(\nwarrow)$:
$$R_A = 2g\cos 30^\circ = \sqrt{3}g$$
$R(\nearrow)$
$$T - 2g\sin 30^\circ - 0.2R_A = 0$$
$$T = g + \frac{\sqrt{3}}{5}g$$
$$T = \left(1 + \frac{\sqrt{3}}{5}\right)g \qquad \textbf{(1)}$$

For B
$R(\nearrow)$:
$$R_B = mg\cos 30^\circ = \frac{\sqrt{3}}{2}mg$$
$R(\nwarrow)$:
$$mg\sin 30 - T - 0.4R_B = 0$$
$$T = \frac{1}{2}mg - \frac{4}{10}\times\frac{\sqrt{3}}{2}mg$$
$$T = mg\left(\frac{1}{2} - \frac{\sqrt{3}}{5}\right) \qquad \textbf{(2)}$$

$$\left(\frac{1}{2} - \frac{\sqrt{3}}{5}\right)m = 1 + \frac{\sqrt{3}}{5}$$
$$\left(5 - 2\sqrt{3}\right)m = 10 + 2\sqrt{3}$$
$$m = \frac{10 + 2\sqrt{3}}{5 - 2\sqrt{3}}$$

5 b Since $m = 10$ kg, $R_B = 5\sqrt{3}g$

$R(\nearrow)$ for A, using Newton's second law:

$$2a = T - 2g\sin 30^\circ - 0.2R_A$$

$$2a = T - g - \frac{\sqrt{3}}{5}g \qquad \textbf{(1)}$$

$R(\searrow)$ for B, using Newton's second law:

$$10a = 5g - 2\sqrt{3}g - T \qquad \textbf{(2)}$$

$$5\times\textbf{(1)} = \textbf{(2)} \Rightarrow$$

$$5T - 5g - \sqrt{3}g = 5g - 2\sqrt{3}mg - T$$

$$6T = 10g - \sqrt{3}g$$

$$T = \frac{5}{3}g - \frac{\sqrt{3}}{6}g$$

Substituting this value into (**1**):

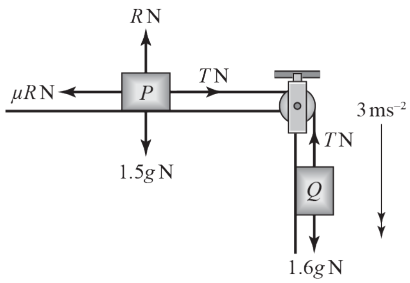
$$2a = \left(\frac{5}{3}g - \frac{\sqrt{3}}{6}g\right) - g - \frac{\sqrt{3}}{5}g$$

$$2a = \frac{2}{3}g - \frac{11\sqrt{3}}{30}g$$

$$a = \left(\frac{1}{3} - \frac{11\sqrt{3}}{60}\right)g = 0.15474...$$

The acceleration is $0.155\ \text{ms}^{-2}$ (3s.f.).

6 a $u = 0\ \text{ms}^{-1}$, $v = 6\ \text{ms}^{-1}$, $t = 2$ s, $a = ?$



$$v = u + at$$

$$6 = 0 + 2a$$

$$a = \frac{6}{3} = 3$$

The acceleration is $3\ \text{ms}^{-2}$

b Considering the box, Q, and using Newton's second law:

$$F = ma$$

$$1.6g - T = 1.6\times 3$$

$$T = 1.6g - 1.6\times 3$$

$$T = 1.6\times(9.8 - 3)$$

$$T = 10.88$$

The tension in the string is 10.88 N.

6 c For P:

$R(\uparrow): R = 1.5g$

$R(\rightarrow)$:

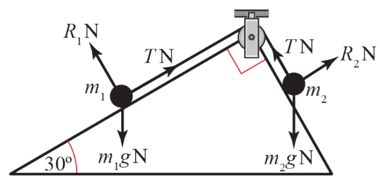
$$F = ma$$
$$T - \mu R = ma$$
$$10.88 - 1.5\mu g = 1.5 \times 3$$
$$1.5\mu g = 10.88 - 4.5$$
$$\mu = \frac{6.38}{1.5 \times 9.8} = 0.43401...$$

To 3 s.f. the coefficient of friction is 0.434, as required.

6 d The tension in the two parts of the string can be assumed to be the same because the string is inextensible.

Challenge

a With wedge smooth, let the reaction forces on the blocks be R_1 and R_2 respectively.
Resolving parallel to the slope on each side:



$T = m_1 g \sin 30^\circ$

$T = m_2 g \cos 30^\circ$

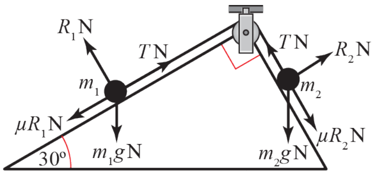
Since the string is inextensible, both values of T are the same, so:

$m_1 g \sin 30^\circ = m_2 g \cos 30^\circ$

$$\frac{m_1}{m_2} = \frac{\cos 30^\circ}{\sin 30^\circ} = \frac{1}{\tan 30^\circ}$$

$\frac{m_1}{m_2} = \sqrt{3}$ as required.

Challenge



b Resolving perpendicular to the slope on each side:

$R_1 = m_1 g \cos 30^\circ$

$R_2 = m_2 g \sin 30^\circ$

<u>Case 1: m_1 is about to move down</u>

Resolving parallel to each slope to find tension in string if m_1 is about to move down:

$T = m_1 g \sin 30^\circ - \mu R_1 = m_1 g \sin 30^\circ - \mu m_1 g \cos 30^\circ$

$T = m_2 g \cos 30^\circ + \mu R_2 = m_2 g \cos 30^\circ + \mu m_2 g \sin 30^\circ$

Since the string is inextensible, both values of T are the same, so:

$$m_1 g\left(\sin 30^\circ - \mu \cos 30^\circ\right) = m_2 g\left(\cos 30^\circ + \mu \sin 30^\circ\right)$$

$$\frac{m_1}{m_2} = \frac{\cos 30^\circ + \mu \sin 30^\circ}{\sin 30^\circ - \mu \cos 30^\circ}$$

$$\frac{m_1}{m_2} = \frac{\sqrt{3} + \mu}{1 - \mu\sqrt{3}}$$

<u>Case 1: m_2 is about to move down</u>

Resolving parallel to each slope to find tension in string if m_2 is about to move down:

$T = m_1 g \sin 30^\circ + \mu R_1 = m_1 g \sin 30^\circ + \mu m_1 g \cos 30^\circ$

$T = m_2 g \cos 30^\circ - \mu R_2 = m_2 g \cos 30^\circ - \mu m_2 g \sin 30^\circ$

Since the string is inextensible, both values of T are the same, so:

$$m_1 g\left(\sin 30^\circ + \mu \cos 30^\circ\right) = m_2 g\left(\cos 30^\circ - \mu \sin 30^\circ\right)$$

$$\frac{m_1}{m_2} = \frac{\cos 30^\circ - \mu \sin 30^\circ}{\sin 30^\circ + \mu \cos 30^\circ}$$

$$\frac{m_1}{m_2} = \frac{\sqrt{3} - \mu}{1 + \mu\sqrt{3}}$$

$\frac{m_1}{m_2}$ must lie between these two values, since they are the values of limiting equilibrium.

So:

$$\frac{\sqrt{3} - \mu}{1 + \mu\sqrt{3}} \le \frac{m_1}{m_2} \le \frac{\sqrt{3} + \mu}{1 - \mu\sqrt{3}} \quad \text{as required.}$$

Applications of forces Mixed exercise 7

1 a Finding the components of P along each axis:

$R(\rightarrow): P_x = 12\cos 70^\circ + 10\sin 75^\circ$

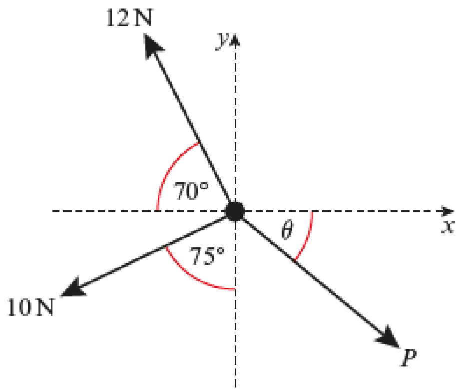
$R(\uparrow): P_y = 12\sin 70^\circ - 10\cos 75^\circ$

$\tan\theta = \dfrac{P_y}{P_x}$

$\tan\theta = \dfrac{12\sin 70^\circ - 10\cos 75^\circ}{12\cos 70^\circ + 10\sin 75^\circ} = 0.63124...$

$\theta = 32.261...$

The angle θ is 32.3° (to 3s.f.).



b Using Pythagoras' theorem:

$P^2 = P_x^{\,2} + P_y^{\,2}$

$P^2 = \left(12\cos 70^\circ + 10\sin 75^\circ\right)^2 + \left(12\sin 70^\circ - 10\cos 75^\circ\right)^2$

$P = \sqrt{264.91...} = 16.276...$

P has a magnitude of 16.3 N (3s.f.).

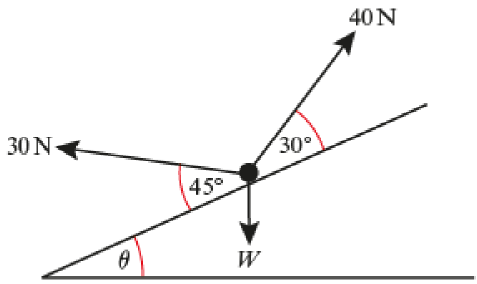
2 a $R(\nwarrow)$:

$W\cos\theta = 40\sin 30^\circ + 30\sin 45^\circ$

$R(\swarrow)$:

$30\cos 45^\circ + W\sin\theta = 40\cos 30^\circ$

$W\sin\theta = 40\cos 30^\circ - 30\cos 45^\circ$



$\dfrac{W\sin\theta}{W\cos\theta} = \dfrac{40\cos 30^\circ - 30\cos 45^\circ}{40\sin 30^\circ + 30\sin 45^\circ}$

$\tan\theta = \dfrac{20\sqrt{3} - 15\sqrt{2}}{20 + 15\sqrt{2}} = 0.35281...$

$\theta = 18.046...$

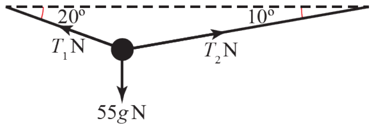
The angle θ is 18.0° (to 3s.f.).

b Using Pythagoras' theorem, $|W|^2$ is the sum of the squares of the two components.

$W^2 = \left(20\sqrt{3} - 15\sqrt{2}\right)^2 + \left(20 + 15\sqrt{2}\right)^2$

$W = \sqrt{1878.8...} = 43.345...$

The weight of the particle is 43.3 N (3s.f.).



3 Resolving horizontally:

$$T_1 \cos 20^\circ = T_2 \cos 10^\circ$$

$$T_2 = \frac{T_1 \cos 20^\circ}{\cos 10^\circ} \qquad \textbf{(1)}$$

Resolving vertically:

$$T_1 \sin 20^\circ + T_2 \sin 10^\circ = 55g \qquad \textbf{(2)}$$

Substituting $T_2 = \dfrac{T_1 \cos 20^\circ}{\cos 10^\circ}$ from (**1**) into (**2**):

$$T_1 \sin 20^\circ + \frac{T_1 \cos 20^\circ \times \sin 10^\circ}{\cos 10^\circ} = 55g$$

$$T_1(\sin 20^\circ + \cos 20^\circ \tan 10^\circ) = 55g$$

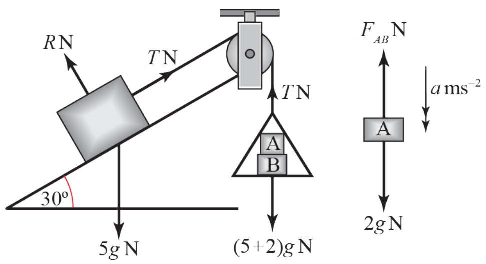
$$T_1 = \frac{55 \times 9.8}{\sin 20^\circ + \cos 20^\circ \tan 10^\circ} = 1061.6...$$

Subsituting this value of T_1 into (**1**):

$$T_2 = \frac{\cos 20^\circ \times 1061.6...}{\cos 10^\circ} = 1012.9...$$

The tensions in the two parts of the rope are 1062 N and 1013 N (nearest whole number).

4 Let the acceleration of the system be a ms^{-2}



For the block, $R(\nearrow)$:

$$F = ma$$

$$T - 5g \sin 30^\circ = 5a$$

$$T = 5a + \frac{5g}{2} \qquad \textbf{(1)}$$

For the pan and masses, $R(\downarrow)$:

$$F = ma$$

$$(2+5)g - T = (2+5)a$$

$$T = 7g - 7a \qquad \textbf{(2)}$$

Since the string is inextensible, T is constant and hence (**1**) = (**2**):

$$5a + \frac{5g}{2} = 7g - 7a$$

$$12a = \frac{9g}{2}$$

$$a = \frac{3g}{8}$$

Hence, the scale-pan accelerates downward, away from the pulley with magnitude $a = \dfrac{3g}{8}$ ms^{-2}

Applying Newton's second law to mass A, and denoting the force exerted by B on A as F_{AB}:

$$2g - F_{AB} = 2a$$

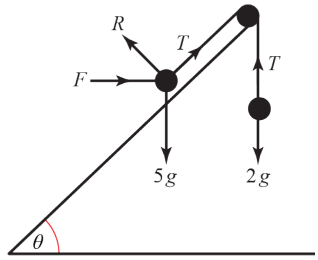
$$F_{AB} = 2g - \frac{2 \times 3g}{8}$$

$$F_{AB} = \frac{10}{8}g$$

$$F_{AB} = 1.25 \times 9.8 = 12.25$$

4 However, Newton's third law of motion gives
|force exerted by B on A| = |force exerted by A on B|.
Therefore the force exerted by A on B is 12.25 N.

5 a For the 2 kg particle



$R(\uparrow)$

$T - 2g = 0$

$\therefore T = 2g$

For the 5 kg particle:

$R(\nearrow)$

$T + F\cos\theta - 5g\sin\theta = 0$

$\therefore F\cos\theta = 5g\sin\theta - T$

As $T = 2g$, $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$

$F \times \frac{4}{5} = 5g \times \frac{3}{5} - 2g$

$\therefore F = g \times \frac{5}{4}$

$= \frac{5g}{4}$

$= 12$ (2 s.f.)

b $R(\nwarrow)$

$R - F\sin\theta - 5g\cos\theta = 0$

$\therefore R = F\sin\theta + 5g\cos\theta$

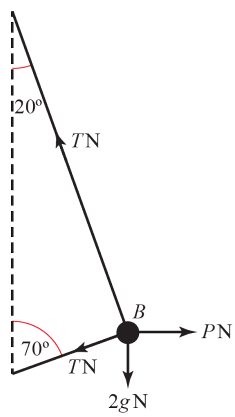
$= \left(\frac{5}{4}g \times \frac{3}{5}\right) + \left(5g \times \frac{4}{5}\right)$

$= \frac{19}{4}g$

$= 47$ (2 s.f.)

c F will be smaller

6 a Resolving vertically:



$T\cos 20^\circ = T\cos 70^\circ + 2g$

$T(\cos 20^\circ - \cos 70^\circ) = 2g$

$T = \frac{2 \times 9.8}{\cos 20^\circ - \cos 70^\circ} = 32.793...$

The tension in the string, to two significant figures, is 33 N.

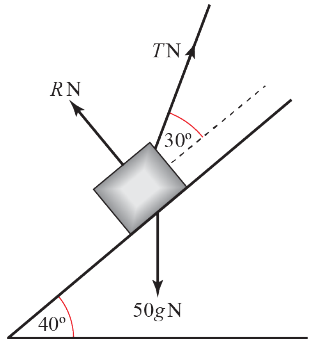
6 b Resolving horizontally:

$P = T\sin 20^\circ + T\sin 70^\circ$

$P = (\sin 20^\circ + \sin 70^\circ) \times 32.793...$

$P = 42.032...$

The value of P is 42 N (2 s.f.).



7 a $R(\nearrow)$:

$T\cos 30^\circ = 50g\sin 40^\circ$

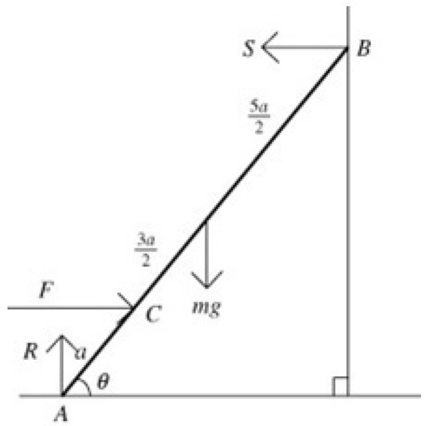
$$T = \frac{50 \times 9.8\sin 40^\circ}{\cos 30^\circ}$$

$$= 363.69...$$

The tension in the string is 364 N (3s.f.).

b Even when the hill is covered in snow, there is likely to be some friction between the runners of the sled and the slope, so modelling the hill as a smooth slope is unrealistic.

8 Let S be the reaction of the wall on the ladder at B.
Let R be the reaction of the ground on the ladder at A.
(Both surfaces are smooth, so no friction.)



$\boldsymbol{R(\rightarrow)}$: $\boldsymbol{F = S}$

Taking moments about A:

$$mg \times \frac{5}{2}a \times \cos\theta + F \times a \times \sin\theta = S \times 5a \times \sin\theta$$

$$\frac{5mg}{2} + F\tan\theta = 5S\tan\theta \qquad \text{(dividing by } a\cos\theta\text{)}$$

$$\frac{5mg}{2} + F\tan\theta = 5F\tan\theta \qquad \text{(Since } F = S\text{)}$$

$$\frac{5mg}{2} = 4F\tan\theta$$

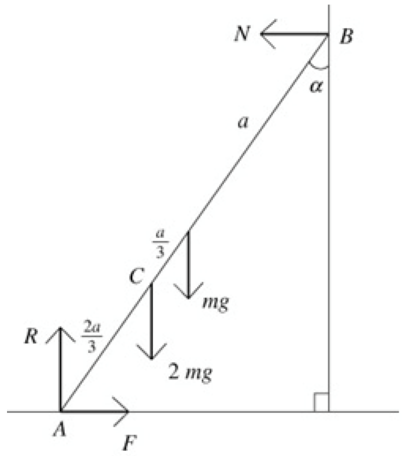
$$= 4 \times \frac{9}{5}F \qquad \text{(Since } \tan\theta = \frac{9}{5}\text{)}$$

$$= 7.2F$$

$$F = \frac{5mg}{2 \times 7.2}$$

$$= \frac{25mg}{72} \text{ as required.}$$

9 Let N be the reaction of the wall on the ladder at B.
Let R be the reaction of the ground on the ladder at A,
Let F the friction between the ladder and the ground at A.



$\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

$R(\uparrow)$: $R = mg + 2mg = 3mg$

Taking moments about B:

$$mg \times a\sin\alpha + 2mg \times \frac{4}{3}a\sin\alpha + F \times 2a\cos\alpha = R \times 2a\sin\alpha$$

$$mga \times \frac{3}{5} + \frac{8mga}{3} \times \frac{3}{5} + F \times 2a \times \frac{4}{5} = 6mga \times \frac{3}{5}$$

$$F \times \frac{8a}{5} = \frac{18mga}{5} - \frac{8mga}{5} - \frac{3mga}{5}$$

$$F \times \frac{8a}{5} = \frac{7mga}{5}$$

$$F = \frac{7mga}{5} \times \frac{5}{8a}$$

$$= \frac{7mg}{8}$$

The ladder and the child are in equilibrium, so

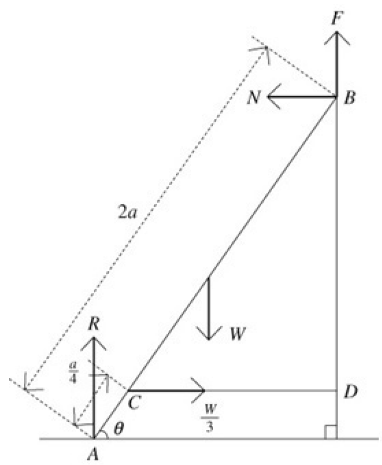
$$F \le \mu R$$

$$\frac{7mg}{8} \le \mu \times 3mg$$

$$\mu \ge \frac{7}{24}$$

The least possible value for μ is $\frac{7}{24}$

10 Let R be the reaction of the ground on the ladder at A,
Let N be the reaction of the wall on the ladder at B
Let F be the friction between the wall and the ladder at B.



a Since you do not know the magnitude of F, you cannot resolve vertically to find R.
Therefore, take moments about B (since this eliminates F):

$$\frac{W}{3} \times \frac{7a}{4}\sin\theta + W \times a\cos\theta = R \times 2a\cos\theta$$

$$\frac{7W}{12} \times \tan\theta + W = 2R \qquad \text{(dividing through by } a\cos\theta\text{)}$$

$$\frac{7}{12} \times \frac{4}{3}W + W = 2R \qquad \text{(since } \tan\theta = \frac{4}{3}\text{)}$$

$$\frac{16W}{9} = 2R$$

$$R = \frac{8W}{9}$$

10 b $R(\rightarrow)$: $N = \frac{W}{3}$

$R(\uparrow)$:

$R + F = W$

$F = W - R$

$= W - \frac{8}{9}W$

$= \frac{W}{9}$

For the ladder to remain in equilibrium,

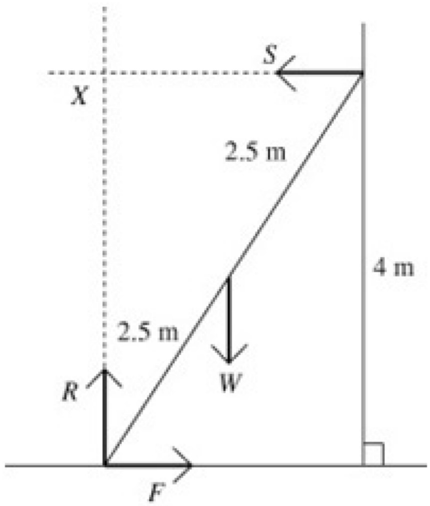
$F \le \mu N$

$\frac{W}{9} \le \mu \frac{W}{3}$

$\mu \ge \frac{1}{3}$

c The ladder had negligible thickness / the ladder does not bend.

11 a Let S be the reaction of the wall on the ladder
Let R be the reaction of the ground on the ladder
Let F the friction between the ladder and the ground
Let X be the point where the lines of action of R and S intersect, as shown in the diagram.



By Pythagoras's Theorem, distance from base of ladder to wall is 3 m.

$R(\rightarrow)$: $F = S$

$R(\uparrow)$: $R = W$

Taking moments about X:

$1.5W = 4F$

Suppose the ladder can rest in equilibrium in this position. Then

$F \le \mu R$

$\frac{1.5W}{4} \le 0.3 \times W$

$\frac{3W}{8} \le \frac{3W}{10}$

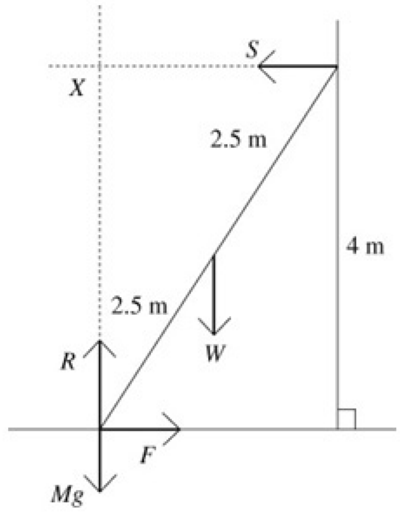
$30 \le 24$

which is false, therefore the assumption that $F \le \mu R$ must be false – the ladder cannot be resting in equilibrium.

11 b With the brick in place, take moments about X:
$1.5W = 4F$ so

$$F = \frac{1.5W}{4} = \frac{3W}{8}$$

which is independent of M, the mass of the brick.



c $R(\uparrow)\quad R = W + Mg$
$R(\rightarrow)\quad F = S$

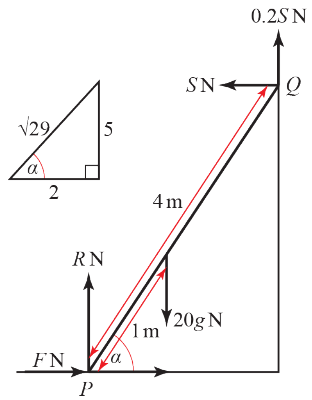
$$F \le \mu R \Rightarrow \frac{3W}{8} \le 0.3(W + Mg) = \frac{3(W + Mg)}{10}$$
$$\Rightarrow 10W \le 8W + 8Mg$$
$$8Mg \ge 2W,\ M \ge \frac{W}{4g}$$

So the smallest value for M is $\frac{W}{4g}$

12 Let S be the reaction of the wall on the ladder at Q
Let R be the reaction of the ground on the ladder at P

$\tan\alpha = \frac{5}{2} \Rightarrow \sin\alpha = \frac{5}{\sqrt{29}}$ and $\cos\alpha = \frac{2}{\sqrt{29}}$

Since the ladder is in limiting equilibrium,
frictional force at the wall = $\mu S = 0.2S$.



Taking moments about P:
$20g \times 1\cos\alpha = S \times 4\sin\alpha + 0.2S \times 4 \times \cos\alpha$

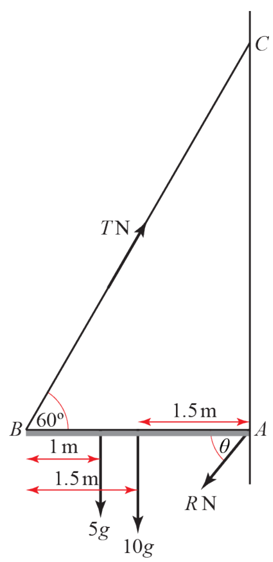
$$\frac{20\times 2}{\sqrt{29}}g = \left(\frac{4\times 5}{\sqrt{29}} + \frac{0.8\times 2}{\sqrt{29}}\right)S$$
$$40g = 21.6S$$
$$S = \frac{392}{21.6} = 18.148...$$

R($\rightarrow$): $F = S$
The force F required to hold the ladder still is 18 N (2 s.f.).

13 Since the rod is uniform, the weight acts from the midpoint of AB.

a Take moments about A:

$$(10g \times 1.5) + (5g \times 2) = T \times 3\sin 60^\circ$$
$$T = \frac{25g}{3\sin 60^\circ}$$
$$T = \frac{25\times 9.8\times 2}{3\sqrt{3}} = 94.300...$$



The tension in the string is 94.3 N (3s.f.).

b Let the horizontal and vertical components of the reaction R at the hinge be R_x and R_y respectively.

Resolving horizontally:

$$R_x = T\cos 60^\circ$$
$$R_x = \frac{25g}{3\sin 60^\circ}\cos 60^\circ$$
$$R_x = \frac{25g}{3\tan 60^\circ} = \frac{25g}{3\sqrt{3}}$$

Resolving vertically upwards:

$$R_y = T\sin 60^\circ - 10g - 5g$$
$$R_y = \frac{25g}{3\sin 60^\circ}\sin 60^\circ - 15g$$
$$R_y = \left(\frac{25}{3} - 15\right)g = -\frac{20g}{3}$$

The reaction at the hinge is given by:

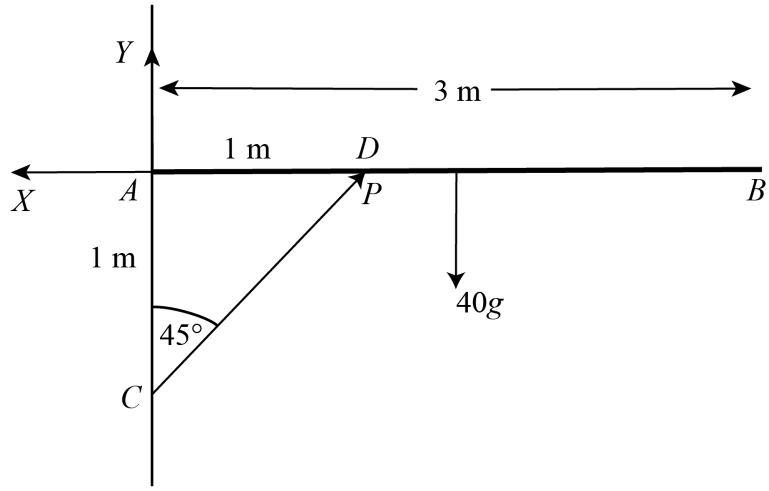
$$R^2 = R_x{}^2 + R_y{}^2$$
$$R^2 = \left(\frac{25g}{3\sqrt{3}}\right)^2 + \left(-\frac{20g}{3}\right)^2$$
$$R^2 = \left(\frac{625}{27} + \frac{400}{9}\right)g^2$$
$$R = \sqrt{\frac{1825}{27}} \times 9.8 = 80.570...$$
$$\tan\theta = \frac{R_y}{R_x}$$
$$\tan\theta = \frac{20g}{3} \times \frac{3\sqrt{3}}{25g} = \frac{4\sqrt{3}}{5}$$
$$\theta = 54.182...$$

The reaction at the hinge is 80.6 N acting at 54.2° below the horizontal (both values to 3 s.f.).

14 Let the horizontal and vertical components or the force at A be X and Y respectively. Let the thrust in the rod be P.

a $\text{M}(A)\text{: } 1 \times P \times \cos 45° = 40g \times \frac{3}{2}$

$P = \frac{60g}{\cos 45°} = 60\sqrt{2}g = 830 \text{ N (2 s.f.)}$



b $\text{R}(\rightarrow): X = P\cos 45° = 60g$

$\text{R}(\uparrow): Y + P\cos 45° = 40g$

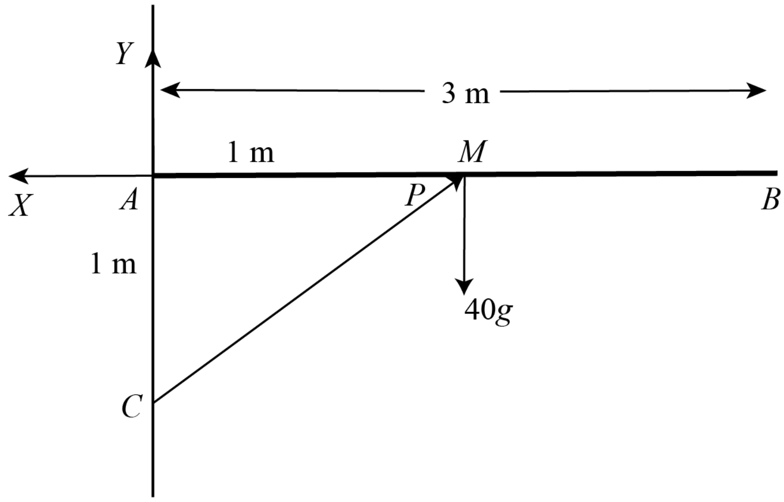
$Y = 40g - 60g = -20g$

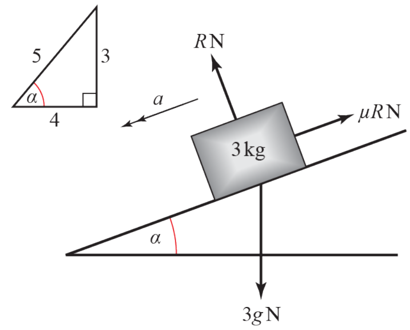
$\text{resultant} = \sqrt{X^2 + Y^2}$

$= 10g\sqrt{4^2 + 2^2} = 10g\sqrt{40}$

$= 620 \text{ N (2 s.f.)}$

c The lines of action of P and the weight meet at M, hence the line of action of the resultant of X and Y must also pass through M (3 forces acting on a body in equilibrium). Therefore the reaction must act horizontally (i.e. no vertical component).





15 $\tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

$u = 0$ ms^{-1}, $s = 6$ m, $t = 1.5$ s, a =?

$$s = ut + \frac{1}{2}at^2$$

$$6 = 0 + \frac{1}{2}a \times 1.5^2 = \frac{9a}{8}$$

$$a = 6 \times \frac{9}{8} = \frac{16}{3}$$

$R(\nwarrow)$: $R = 3g\cos\alpha$

$R(\swarrow)$:

$$F = ma$$

$$3g\sin\alpha - \mu R = 3 \times \frac{16}{3}$$

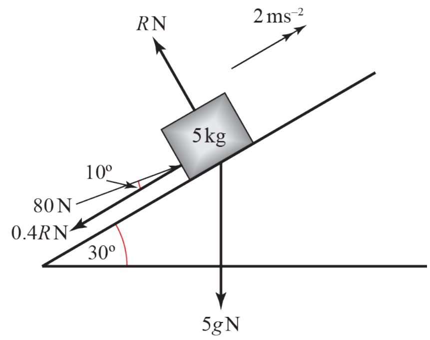
$$3g\sin\alpha - (\mu \times 3g\cos\alpha) = 16$$

$$\frac{3}{5}g - \frac{4}{5}\mu g = \frac{16}{3}$$

$$9g - 12\mu g = 80$$

$$\mu = \frac{(9 \times 9.8) - 80}{12 \times 9.8} = 0.06972...$$

The coefficient of friction is 0.070 (3s.f.).



16 $R(\nwarrow)$: $R = 5g\cos 30^\circ + 80\sin 10^\circ$

$R(\nearrow)$:

$ma = F$

$$5a = 80\cos 10^\circ - 0.4R - 5g\sin 30^\circ$$

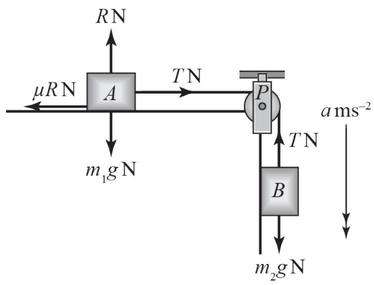
$$5a = 80\cos 10^\circ - 0.4(5g\cos 30^\circ + 80\sin 10^\circ) - 5g\sin 30^\circ$$

$$a = 16\cos 10^\circ - 0.4(g\cos 30^\circ + 16\sin 10^\circ) - g\sin 30^\circ$$

$$a = 6.3507...$$

The acceleration of the particle is 6.53 ms^{-2} (3s.f.).

17 Since $m_2 > \mu m_1$, when system is released from rest then B moves downwards and A moves towards the pulley P



For particle A:

$R(\uparrow)$: $R = \mu m_1 g$

$R(\rightarrow)$:

$$F = m_1 a$$
$$T - \mu R = m_1 a$$
$$T - \mu m_1 g = m_1 a$$
$$T = m_1 a + \mu m_1 g \quad \textbf{(1)}$$

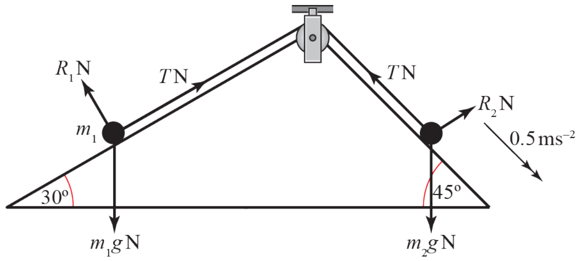
For particle B:

$R(\uparrow)$:

$$F = m_2 a$$
$$m_2 g - T = m_2 a$$
$$T = m_2 g - m_2 a \quad \textbf{(2)}$$

Since string is inextensible, the values of T are identical, so (**1**) = (**2**):

$$m_1 a + \mu m_1 g = m_2 g - m_2 a$$
$$m_1 a + m_2 a = m_2 g - \mu m_1 g$$
$$a = \frac{g(m_2 - \mu m_1)}{m_1 + m_2} \quad \text{as required.}$$



18 For particle with mass m_1:

$R(\nearrow)$

$$F = ma$$

$$T - m_1 g \sin 30^\circ = \frac{m_1}{2}$$

$$T = \left(\frac{1}{2} + g\sin 30^\circ\right)m_1 \qquad \textbf{(1)}$$

For particle with mass m_2:

$R(\nearrow)$

$$F = ma$$

$$m_2 g \sin 45^\circ - T = \frac{m_2}{2}$$

$$\left(g\sin 45^\circ - \frac{1}{2}\right)m_2 = T \qquad \textbf{(2)}$$

Since string is inextensible, T is constant throughout and hence (**1**) = (**2**):

$$\left(\frac{1}{2} + g\sin 30^\circ\right)m_1 = \left(g\sin 45^\circ - \frac{1}{2}\right)m_2$$

$$\left(\frac{1}{2} + \frac{g}{2}\right)m_1 = \left(\frac{g\sqrt{2}}{2} - \frac{1}{2}\right)m_2$$

$$\frac{m_1}{m_2} = \frac{g\sqrt{2}-1}{1+g} \quad \text{as required}$$

Further kinematics 8A

1 **a** Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, $\mathbf{r} = (2\mathbf{i}) + (\mathbf{i} + 3\mathbf{j}) \times 4 = 2\mathbf{i} + 4\mathbf{i} + 12\mathbf{j} = 6\mathbf{i} + 12\mathbf{j}$

b Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, $\mathbf{r} = (3\mathbf{i} - \mathbf{j}) + (-2\mathbf{i} + \mathbf{j}) \times 5 = 3\mathbf{i} - \mathbf{j} - 10\mathbf{i} + 5\mathbf{j} = -7\mathbf{i} + 4\mathbf{j}$

c Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, $(4\mathbf{i} + 3\mathbf{j}) = \mathbf{r}_0 + (2\mathbf{i} - \mathbf{j}) \times 3$, $\mathbf{r}_0 = (4\mathbf{i} + 3\mathbf{j}) - (6\mathbf{i} - 3\mathbf{j}) = 4\mathbf{i} + 3\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} = -2\mathbf{i} + 6\mathbf{j}$

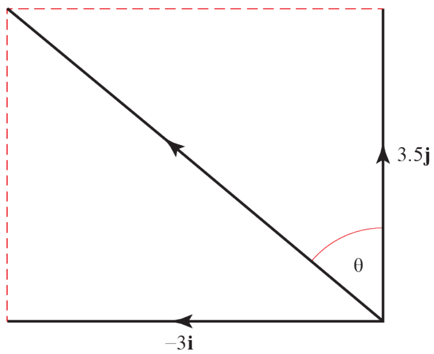
d Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, $(-2\mathbf{i} + 5\mathbf{j}) = \mathbf{r}_0 + (-2\mathbf{i} + 3\mathbf{j}) \times 6$, $\mathbf{r}_0 = (-2\mathbf{i} + 5\mathbf{j}) - (-12\mathbf{i} + 18\mathbf{j})$
$= -2\mathbf{i} + 5\mathbf{j} + 12\mathbf{i} - 18\mathbf{j} = 10\mathbf{i} - 13\mathbf{j}$

e Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, $(8\mathbf{i} - 7\mathbf{j}) = (2\mathbf{i} + 2\mathbf{j}) + \mathbf{v} \times 3$, $3\mathbf{v} = (8\mathbf{i} - 7\mathbf{j}) - (2\mathbf{i} + 2\mathbf{j}) = 6\mathbf{i} - 9\mathbf{j}$
$\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$

f Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$, $(12\mathbf{i} - 11\mathbf{j}) = (4\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - 3\mathbf{j}) \times t$
$12 = 4 + 2t \Rightarrow t = 4$ s

2 $\mathbf{r}_0 = (10\mathbf{i} - 5\mathbf{j})$ m, $\mathbf{r} = (-2\mathbf{i} + 9\mathbf{j})$ m, $t = 4$ s, $\mathbf{v} = ?$

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$$
$$-2\mathbf{i} + 9\mathbf{j} = (10\mathbf{i} - 5\mathbf{j}) + 4\mathbf{v}$$
$$4\mathbf{v} = -2\mathbf{i} + 9\mathbf{j} - (10\mathbf{i} - 5\mathbf{j})$$
$$\mathbf{v} = -3\mathbf{i} + \frac{7}{2}\mathbf{j}$$



$$\text{Speed} = \sqrt{3^2 + \left(\frac{7}{2}\right)^2} = \frac{\sqrt{85}}{2}$$

$$\text{Bearing} = 360^\circ - \theta \text{ where } \tan\theta = \frac{3}{3.5}$$
$$\theta = 40.601...^\circ$$

The boat travels at a speed of $\frac{\sqrt{85}}{2}$ ms^{-1} at a bearing of 319° (3s.f.).

3 $\mathbf{r}_0 = (-2\mathbf{i} + 3\mathbf{j})$ m, $\mathbf{r} = (6\mathbf{i} - 3\mathbf{j})$ m, $t = ?$, $v = 4$ ms^{-1}
Change in position $= (6\mathbf{i} - 3\mathbf{j}) - (-2\mathbf{i} + 3\mathbf{j}) = (8\mathbf{i} - 6\mathbf{j})$

Distance travelled $= \sqrt{8^2 + 6^2} = 10$

$$v = \frac{s}{t}$$
$$4 = \frac{10}{t}$$
$$t = 2.5$$

The mouse takes 2.5 s to travel to the new position.

4 a $\mathbf{r_0} = \begin{pmatrix} 120 \\ -10 \end{pmatrix}$m, $\mathbf{v} = \begin{pmatrix} -30 \\ 40 \end{pmatrix}$ms^{-1}, $t = t$, $\mathbf{r} = ?$

$\mathbf{r} = \mathbf{r_0} + \mathbf{v}t$

$\mathbf{r} = \begin{pmatrix} 120 \\ -10 \end{pmatrix} + \begin{pmatrix} -30 \\ 40 \end{pmatrix}t$

$\mathbf{r} = \begin{pmatrix} 120 - 30t \\ -10 + 40t \end{pmatrix}$

b When the helicopter is due north of the origin, the **i** component of its position vector is 0.

$120 - 30t = 0$

$t = \frac{120}{30} = 4$

The helicopter is due north of the origin after 4 s.

5 Using $\mathbf{r} = \mathbf{r_0} + \mathbf{v}t$ for P

$\mathbf{r} = (4\mathbf{i}) + (\mathbf{i} + \mathbf{j}) \times 8$

$= 4\mathbf{i} + 8\mathbf{i} + 8\mathbf{j}$

$= 12\mathbf{i} + 8\mathbf{j}$

Using $\mathbf{r} = \mathbf{r_0} + \mathbf{v}t$ for Q

$\mathbf{r} = (-3\mathbf{j}) + \mathbf{v} \times 8$

At $t = 8$ s, position vectors of P and Q are equal:

$12\mathbf{i} + 8\mathbf{j} = (-3\mathbf{j}) + \mathbf{v} \times 8$

$8\mathbf{v} = 12\mathbf{i} + 8\mathbf{j} + 3\mathbf{j}$

$= 12\mathbf{i} + 11\mathbf{j}$

$\mathbf{v} = \frac{1}{8}(12\mathbf{i} + 11\mathbf{j})$

$= 1.5\mathbf{i} + 1.375\mathbf{j}$

speed $= |\mathbf{v}|$

$= \sqrt{1.5^2 + 1.375^2}$

$= \sqrt{2.25 + 1.890625}$

$\approx 2.03\,\text{m s}^{-1}$

6 a Using $\mathbf{r} = \mathbf{r_0} + \mathbf{v}t$ for F

$\mathbf{r} = 400\mathbf{j} + (7\mathbf{i} + 7\mathbf{j}) \times t$

$= 400\mathbf{j} + 7t\,\mathbf{i} + 7t\,\mathbf{j}$

$= 7t\,\mathbf{i} + (400 + 7t)\mathbf{j}$

Using $\mathbf{r} = \mathbf{r_0} + \mathbf{v}t$ for S

$\mathbf{r} = 500\mathbf{i} + (-3\mathbf{i} + 15\mathbf{j}) \times t$

$= 500\mathbf{i} - 3t\,\mathbf{i} + 15t\,\mathbf{j}$

$= (500 - 3t)\mathbf{i} + 15t\,\mathbf{j}$

6 b For F and S to collide, $7t\mathbf{i}+(400+7t)\mathbf{j}=(500-3t)\mathbf{i}+15t\mathbf{j}$

$$\begin{aligned}\textbf{i components equal: } 7t&=500-3t\\ 10t&=500\\ t&=50\end{aligned}$$

$$\begin{aligned}\textbf{j components equal: } 400+7t&=15t\\ 400&=8t\\ t&=50\end{aligned}$$

Both conditions give the same value of t, so the two position vectors are equal when $t=50$, i.e. F and S collide at $\mathbf{r}=7\times50\mathbf{i}+(400+7\times50)\mathbf{j}=350\mathbf{i}+750\mathbf{j}$.

7 a $\mathbf{u}=\begin{pmatrix}0\\0\end{pmatrix}\text{ms}^{-1}$, $\mathbf{v}=\begin{pmatrix}3\\4\end{pmatrix}\text{ms}^{-1}$, $t=5\,\text{s}$, $\mathbf{a}=?$

Using $\mathbf{v}=\mathbf{u}+\mathbf{a}t$

$$\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}+5\mathbf{a}$$

$$\mathbf{a}=\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}$$

The acceleration of the particle is $\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}\text{ms}^{-2}$

b Using

$$\mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s}=\begin{pmatrix}0\\0\end{pmatrix}+\frac{1}{2}\begin{pmatrix}\frac{3}{5}\\\frac{4}{5}\end{pmatrix}5^2$$

$$\mathbf{s}=\frac{1}{2}\begin{pmatrix}15\\20\end{pmatrix}$$

After 5 s, the displacement vector of the particle is $\begin{pmatrix}\frac{15}{2}\\10\end{pmatrix}$ m.

8 a $\mathbf{u}=(15\mathbf{i}+4\mathbf{j})\text{ ms}^{-1}$, $\mathbf{v}=(5\mathbf{i}-3\mathbf{j})\text{ ms}^{-1}$, $t=4$ s, $\mathbf{a}=?$

Using $\mathbf{v}=\mathbf{u}+\mathbf{a}t$

$$5\mathbf{i}-3\mathbf{j}=15\mathbf{i}+4\mathbf{j}+4\mathbf{a}$$

$$5\mathbf{i}-3\mathbf{j}-(15\mathbf{i}+4\mathbf{j})=4\mathbf{a}$$

$$\mathbf{a}=-\frac{5}{2}\mathbf{i}-\frac{7}{4}\mathbf{j}$$

The acceleration of the particle is $\left(-\frac{5}{2}\mathbf{i}-\frac{7}{4}\mathbf{j}\right)\text{ ms}^{-2}$

8 b $\mathbf{r} = \mathbf{r_o} + \mathbf{s}$ where $\mathbf{r_o} = (10\mathbf{i} - 8\mathbf{j})$ m

Using

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = (10\mathbf{i} - 8\mathbf{j}) + (15\mathbf{i} + 4\mathbf{j})t + \frac{1}{2}\left(-\frac{5}{2}\mathbf{i} - \frac{7}{4}\mathbf{j}\right)t^2$$

$$\mathbf{r} = \left(10 + 15t - \frac{5}{4}t^2\right)\mathbf{i} + \left(-8 + 4t - \frac{7}{8}t^2\right)\mathbf{j}$$

The position vector of the particle after t s is $\left(10 + 15t - \frac{5}{4}t^2\right)\mathbf{i} + \left(-8 + 4t - \frac{7}{8}t^2\right)\mathbf{j}$ m.

9 a $\mathbf{a} = \begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$ ms^{-2}, $\mathbf{u} = \begin{pmatrix} 70 \\ -30 \end{pmatrix}$ ms^{-1}, $\mathbf{v} = ?$, $t = 10$ s,

Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v} = \begin{pmatrix} 70 \\ -30 \end{pmatrix} + 10\begin{pmatrix} -1 \\ 1.5 \end{pmatrix} = \begin{pmatrix} 60 \\ -15 \end{pmatrix}$$

After 10s, the velocity of the plane is $\begin{pmatrix} 60 \\ -15 \end{pmatrix}$ ms^{-1}

b Using

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = 10\begin{pmatrix} 70 \\ -30 \end{pmatrix} + \frac{10^2}{2}\begin{pmatrix} -1 \\ 1.5 \end{pmatrix}$$

$$\mathbf{s} = \begin{pmatrix} 650 \\ -225 \end{pmatrix}$$

Distance travelled $= \sqrt{650^2 + 225^2} = 687.84...$

The plane is 688 m (3s.f.) from its starting point after 10 s.

10 $\mathbf{v} = (4\mathbf{i} + 3\mathbf{j})$ ms^{-1}, $t = 20$ s, $\mathbf{a} = (0.2\mathbf{i} + 0.6\mathbf{j})$ ms^{-2}, $\mathbf{s} = ?$

Using

$$\mathbf{s} = \mathbf{v}t - \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = 20 \times (4\mathbf{i} + 3\mathbf{j}) - \frac{20^2}{2}(0.2\mathbf{i} + 0.6\mathbf{j})$$

$$\mathbf{s} = 80\mathbf{i} + 60\mathbf{j} - 40\mathbf{i} - 120\mathbf{j}$$

$$\mathbf{s} = 40\mathbf{i} - 60\mathbf{j}$$

After 20 s, the displacement vector of the boat from its starting position is $(40\mathbf{i} - 60\mathbf{j})$ m.

11 a Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$ and $t = 3$

For A:

$$\mathbf{v} = (-\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - 4\mathbf{j}) \times 3$$
$$= (-1 + 6)\mathbf{i} + (1 - 12)\mathbf{j}$$
$$= 5\mathbf{i} - 11\mathbf{j}$$

$$\text{Speed} = \sqrt{5^2 + 11^2} = \sqrt{25 + 121}$$
$$= \sqrt{146} = 12.1\,\text{m s}^{-1} \text{ (3 s.f.)}$$

For B:

$$v = \mathbf{i} + 2\mathbf{j} \times 3$$
$$= \mathbf{i} + 6\mathbf{j}$$

$$\text{Speed} = \sqrt{1^2 + 6^2} = \sqrt{1 + 36}$$
$$= \sqrt{37} = 6.08\,\text{m s}^{-1} \text{ (3 s.f.)}$$

b Using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ for A,

$$\mathbf{s} = (-\mathbf{i} + \mathbf{j}) \times 3 + \frac{1}{2} \times (2\mathbf{i} - 4\mathbf{j}) \times 9$$
$$= -3\mathbf{i} + 3\mathbf{j} + 9\mathbf{i} - 18\mathbf{j}$$
$$= 6\mathbf{i} - 15\mathbf{j}$$

So at the instant of the collision, A is at the point with position vector

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{s}$$
$$\mathbf{r} = (12\mathbf{i} + 12\mathbf{j}) + (6\mathbf{i} - 15\mathbf{j})$$
$$= 18\mathbf{i} - 3\mathbf{j}$$

c First find the displacement through which B travels during the motion:

Using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ for B,

$$\mathbf{s} = (\mathbf{i}) \times 3 + \frac{1}{2} \times (2\mathbf{j}) \times 9$$
$$= 3\mathbf{i} + 9\mathbf{j}$$

So B's starting point is given by:

$$\mathbf{r}_0 = (\text{final position}) - (\text{displacement through which B travels})$$
$$\mathbf{r}_0 = \mathbf{r} - \mathbf{s}$$
$$\mathbf{r}_0 = (18\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 9\mathbf{j}) = 15\mathbf{i} - 12\mathbf{j}$$

12 a $\mathbf{u} = (-4\mathbf{i} + 8\mathbf{j})$ kmh^{-1}, $\mathbf{v} = (-2\mathbf{i} - 6\mathbf{j})$ kmh^{-1}, $t = 2$ h, $\mathbf{a} = ?$
Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$-2\mathbf{i} - 6\mathbf{j} = -4\mathbf{i} + 8\mathbf{j} + 2\mathbf{a}$$
$$2\mathbf{a} = 2\mathbf{i} - 14\mathbf{j}$$
$$\mathbf{a} = \mathbf{i} - 7\mathbf{j}$$

The acceleration of the ship is $(\mathbf{i} - 7\mathbf{j})$ kmh^{-2}

b Using

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s} = (-4\mathbf{i} + 8\mathbf{j})t + \frac{1}{2}(\mathbf{i} - 7\mathbf{j})t^2$$
$$\mathbf{s} = (-4t + 0.5t^2)\mathbf{i} + (8t - 3.5t^2)\mathbf{j}$$

After t h, the ship's displacement vector from O is $(-4t + 0.5t^2)\mathbf{i} + (8t - 3.5t^2)\mathbf{j}$ km.

c When the ship is SW of O, then the coefficients of **i** and **j** are equal (and negative) so:

$$-4t + 0.5t^2 = 8t - 3.5t^2$$
$$4t^2 = 12t$$
$$t = 3$$

(The solution t = 0 can be ignored as at this time both coefficients are zero, ship is at O.)

The ship is SW of O 3 h after 12:00, i.e. at 15:00.

d When the two ships meet $\mathbf{r} = \mathbf{s}$. Since **r** has no **i** component, the **i** component of **s** must also be 0.

$$-4t + 0.5t^2 = 0$$
$$0.5t^2 = 4t$$
$$t = 8 \qquad \text{(solution } t = 0 \text{ can again be ignored)}$$
$$\mathbf{r} = (40 - 25t)\mathbf{j}$$
$$\mathbf{r} = (40 - 25 \times 8)\mathbf{j}$$
$$\mathbf{r} = -160\mathbf{j}$$

The two ships meet at position vector $-160\mathbf{j}$ km (i.e. 160 km S of O).

13 a For the particle to be NE of O, the coefficients of **i** and **j** are equal so:

$$2t^2 - 3 = 7 - 4t$$
$$2t^2 + 4t - 10 = 0$$
$$t^2 + 2t - 5 = 0 \qquad \text{as required.}$$

b Using formula for the roots of a quadratic equation:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$t = \frac{-2 \pm \sqrt{4 + 20}}{2}$$
$$t = \sqrt{6} - 1$$

Negative root can be ignored as equation only applies for t ≥ 0.

13 b Since the two coefficients are equal, we need only calculate one of them:

$$7-4t = 7-4\times\left(\sqrt{6}-1\right)$$
$$= 11-4\sqrt{6}$$

$$\text{Distance} = \sqrt{\left(11-4\sqrt{6}\right)^2+\left(11-4\sqrt{6}\right)^2}$$
$$= 1.6999...$$

The particle is 1.70 m from O when it is NE of O.

c $\mathbf{u} = (5\mathbf{i} + 6\mathbf{j})\ \text{ms}^{-1}$, $\mathbf{v} = (b\mathbf{i} + 2b\mathbf{j})\ \text{ms}^{-1}$, $t = 2$ s, $\mathbf{a} = (3a\mathbf{i} - 2a\mathbf{j})$

Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\begin{pmatrix} b \\ 2b \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} + 2\begin{pmatrix} 3a \\ -2a \end{pmatrix}$$

Considering coefficients of **i**:

$$b = 5+6a \qquad \textbf{(1)}$$

Considering coefficients of **j**:

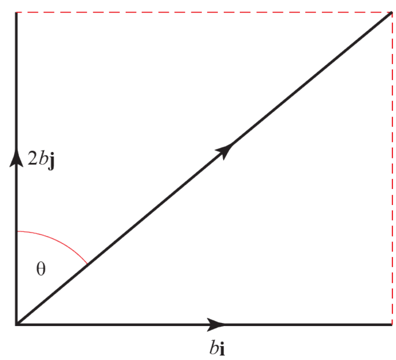
$$2b = 6-4a \qquad \textbf{(2)}$$

Substituting $b = 5+6a$ from (**1**) into (**2**):

$$2(5+6a) = 6-4a$$
$$5+6a = 3-2a$$
$$8a = -2$$
$$a = -0.25$$

Substituting $a = -0.25$ into (**1**):

$$b = 5-1.5 = 3.5$$



Therefore at $t = 2$ s, $\mathbf{v} = (3.5\mathbf{i} + 7\mathbf{j})\ \text{ms}^{-1}$

$$\text{Speed} = \sqrt{3.5^2+7^2} = 7.8262...$$

Bearing = θ where

$$\tan\theta = \frac{b}{2b} = 0.5$$
$$\theta = 26.565...$$

The particle is travelling at speed of 7.83 ms^{-1} at a bearing of 026.6° (both to 3s.f.).

13 d At $t = 2$ s, for the first particle:

$\mathbf{r}_A = (2\times 4 - 3)\mathbf{i} + (7 - 4\times 2)\mathbf{j}$

$\mathbf{r}_A = 5\mathbf{i} - \mathbf{j}$

For the second particle, the displacement since $t = 0$ is given by:

$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$

$\mathbf{s} = 2\begin{pmatrix}5\\6\end{pmatrix} + \frac{2^2}{2}\begin{pmatrix}-3\times 0.25\\2\times 0.25\end{pmatrix}$

$\mathbf{s} = \begin{pmatrix}10-1.5\\12+1\end{pmatrix} = 8.5\mathbf{i} + 13\mathbf{j}$

Displacement of second particle from O,

$\mathbf{r}_B = \mathbf{r}_\mathbf{o} + \mathbf{s}$ where $\mathbf{r}_\mathbf{o} = 5\mathbf{j}$

$\mathbf{r}_B = 5\mathbf{j} + 8.5\mathbf{i} + 13\mathbf{j} = 8.5\mathbf{i} + 18\mathbf{j}$

Relative displacement of the two particles:

$\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix}8.5\\18\end{pmatrix} - \begin{pmatrix}5\\-1\end{pmatrix}$

$= \begin{pmatrix}3.5\\19\end{pmatrix}$

Distance $= |\mathbf{r}_B - \mathbf{r}_A|$

$= \sqrt{3.5^2 + 19^2}$

$= 19.319...$

The distance between the two particles is 19.3 m (3s.f.).

Challenge

The planes cross at **r** relative to the control tower after a time T after the first plane passes it.
For the first plane:

$$\mathbf{u} = \begin{pmatrix} 20 \\ -100 \end{pmatrix} \text{ms}^{-1},\ \mathbf{a} = \begin{pmatrix} 0 \\ 6 \end{pmatrix} \text{ms}^{-2},\ \mathbf{s} = \mathbf{r},\ t = T$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = \begin{pmatrix} 20 \\ -100 \end{pmatrix} T + \frac{T^2}{2}\begin{pmatrix} 0 \\ 6 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 20T \\ -100T + 3T^2 \end{pmatrix}$$

For the second plane:

$$\mathbf{u} = \begin{pmatrix} 70 \\ 40 \end{pmatrix} \text{ms}^{-1},\ \mathbf{a} = \begin{pmatrix} 0 \\ -8 \end{pmatrix} \text{ms}^{-2},\ \mathbf{s} = \mathbf{r},\ t = (T - t)$$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = \begin{pmatrix} 70 \\ 40 \end{pmatrix}(T - t) + \frac{(T - t)^2}{2}\begin{pmatrix} 0 \\ -8 \end{pmatrix}$$

$$\mathbf{r} = \begin{pmatrix} 70T - 70t \\ 40T - 40t - 4T^2 - 4t^2 + 8Tt \end{pmatrix}$$

Equating **i** components:

$$\begin{aligned} 20T &= 70T - 70t \\ 70T - 20T &= 70t \\ T &= \frac{7}{5}t \end{aligned}$$

Equating **j** components and substituting in this value of T:

$$\begin{aligned} -100T + 3T^2 &= 40T - 40t - 4T^2 - 4t^2 + 8Tt \\ 7T^2 + 4t^2 - 8Tt &= 140T - 40t \\ \frac{7 \times 7^2}{5^2}t^2 + 4t^2 - \frac{8 \times 7}{5}t^2 &= \frac{140 \times 7}{5}t - 40t \\ \left(\frac{343}{25} + 4 - \frac{56}{5}\right)t^2 &= (196 - 40)t \\ \frac{163}{25}t^2 &= 156t \\ t &= 23.926... \end{aligned}$$

The second plane passes over the control tower 24 s after the first plane (2s.f.).

Further kinematics 8B

1 a $\mathbf{u} = (12\mathbf{i} + 24\mathbf{j})\ \text{ms}^{-1}$, $t = 3$ s, $\mathbf{a} = -9.8\mathbf{j}\ \text{ms}^{-2}$, $\mathbf{s} = ?$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s} = (12\mathbf{i} + 24\mathbf{j}) \times 3 + \frac{1}{2}(-9.8\mathbf{j}) \times 9$$
$$\mathbf{s} = 36\mathbf{i} + 27.9\mathbf{j}$$

The position vector of P after 3 s is $(36\mathbf{i} + 27.9\mathbf{j})\text{m}$.

b $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v} = (12\mathbf{i} + 24\mathbf{j}) - 3 \times 9.8\mathbf{j}$$
$$\mathbf{v} = 12\mathbf{i} + (24 - 29.4)\mathbf{j}$$
$$\mathbf{v} = 12\mathbf{i} - 5.4\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{12^2 + (-5.4)^2}$$
$$= \sqrt{173.16}$$
$$|\mathbf{v}| = 13.159...$$

The speed of P after 3 s is 13 m s^{-1} (2 s.f.).

2 a $\mathbf{u} = (4\mathbf{i} + 5\mathbf{j})\ \text{ms}^{-1}$, $t = t$ s, $\mathbf{a} = -10\mathbf{j}\ \text{ms}^{-2}$, $\mathbf{s} = ?$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$
$$\mathbf{s} = (4\mathbf{i} + 5\mathbf{j})t + \frac{1}{2}(-10\mathbf{j})t^2$$
$$\mathbf{s} = 4t\mathbf{i} + 5(t - t^2)\mathbf{j}$$

The position vector of the particle after t s is $4\mathbf{i} + 5(t - t^2)\mathbf{j}$ m.

b When particle reaches greatest height, $\mathbf{j}$ component of velocity = 0 (the $\mathbf{i}$ component remains unchanged throughout).
$\mathbf{u} = (4\mathbf{i} + 5\mathbf{j})\ \text{ms}^{-1}$, $\mathbf{v} = (4\mathbf{i})\ \text{ms}^{-1}$, $\mathbf{a} = -10\mathbf{j}\ \text{ms}^{-2}$, $t = ?$

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$
$$4\mathbf{i} = 4\mathbf{i} + 5\mathbf{j} - 10t\mathbf{j}$$
$$10t\mathbf{j} = 5\mathbf{j}$$
$$t = 0.5 \text{ s}$$

Using this value of t to determine the coefficient of $\mathbf{j}$ in the equation derived in part **a**:

$$h = 5(0.5 - 0.5^2)$$
$$h = 5 \times 0.25 = 1.25$$

The greatest height of the particle is 1.25 m.

3 a Both assumptions are made in order to facilitate the calculation. Either could be better or worse than the other. Possible answers include:
The sea is likely to be horizontal and relatively flat, whereas the ball is subject to air resistance, so the assumption that sea is a horizontal plane is most reasonable.
Although the sea is horizontal it is unlikely to be flat because of waves, so the assumption that the ball is a particle is most reasonable.

b $\mathbf{u} = (3p\mathbf{i} + p\mathbf{j})\text{ ms}^{-1}$, $\mathbf{a} = -9.8\mathbf{j}\text{ ms}^{-2}$, $t = 2$ s, $\mathbf{v} = ?$
Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$

$$\mathbf{v} = \begin{pmatrix} 3p \\ p \end{pmatrix} + 2\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} 3p \\ p - 19.6 \end{pmatrix} \qquad \mathbf{(1)}$$

We also know that $\mathbf{r_0} = 25\mathbf{j}$ m, $\mathbf{r} = (q\mathbf{i} + 10\mathbf{j})$ m.
The change in displacement of the ball is:

$$\mathbf{s} = \mathbf{r} - \mathbf{r_o}$$
$$= q\mathbf{i} + 10\mathbf{j} - 25\mathbf{j}$$
$$\mathbf{s} = q\mathbf{i} - 15\mathbf{j}$$

Using:

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\begin{pmatrix} q \\ -15 \end{pmatrix} = 2\begin{pmatrix} 3p \\ p \end{pmatrix} + \frac{2^2}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

Comparing $\mathbf{j}$ components:

$$-15 = 2p - 19.6$$
$$2p = 19.6 - 15$$
$$p = \frac{4.6}{2} = 2.3$$

Substitute $p = 2.3$ in $\mathbf{(1)}$:

$$\mathbf{v} = \begin{pmatrix} 3 \times 2.3 \\ 2.3 - 19.6 \end{pmatrix} = \begin{pmatrix} 6.9 \\ -17.3 \end{pmatrix}$$

The velocity of the ball at B is $(6.9\mathbf{i} - 17\mathbf{j})\text{ ms}^{-1}$ (both coefficients to 2s.f.).

c In order to determine the acceleration on the boat, we first need to find the time at which the ball reaches the sea.
The displacement of the ball relative to A is given by:

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s} = \begin{pmatrix} 3 \times 2.3 \\ 2.3 \end{pmatrix}t + \frac{1}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}t^2$$

$$\mathbf{s} = \begin{pmatrix} 6.9t \\ 2.3t - 4.9t^2 \end{pmatrix}$$

3 c When the ball lands at C, $\mathbf{s} = x\mathbf{i} - 25\mathbf{j}$, where $x = OC$.

$$\begin{pmatrix} x \\ -25 \end{pmatrix} = \begin{pmatrix} 6.9t \\ 2.3t - 4.9t^2 \end{pmatrix}$$

Considering **j** components only:

$$-25 = 2.3t - 4.9t^2$$

$$4.9t^2 - 2.3t - 25 = 0$$

Finding the positive root of this quadratic equation (negative solution can be ignored as before ball thrown or ship sets out):

$$t = \frac{2.3 \pm \sqrt{2.3^2 + (4 \times 4.9 \times 25)}}{2 \times 4.9}$$

$$t = 2.5056...$$

Considering **i** components only:

$$x = 6.9t = 6.9 \times 2.506 = 17.289 \text{ m}$$

For the boat:

$s = 17.289$ m, $t = 2.506$ s, $u = 0$ ms^{-1}, $a = ?$

$$s = ut + \frac{1}{2}at^2$$

$$17.289 = 0 + \frac{1}{2}a(2.506)^2$$

$$a = \frac{34.578}{6.28} = 5.50...$$

(solution of $t = 0$ ignored – shows that both boat and ball start at same place)

The acceleration of the boat is 5.5 ms^{-2} (2s.f.).

4 a $\mathbf{u} = (3u\mathbf{i} + 4u\mathbf{j}),\ \mathbf{a} = -9.8\mathbf{j},\ \mathbf{s} = 750\mathbf{i},\ t = t$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$750\mathbf{i} = \left(3u\mathbf{i} + 4u\mathbf{j}\right)t + \frac{1}{2}\left(-9.8\mathbf{j}\right)t^2$$

$$750\mathbf{i} = 3ut\mathbf{i} + \left(4ut - 4.9t^2\right)\mathbf{j}$$

Comparing **i** coefficients:

$$750 = 3ut$$

$$\therefore t = \frac{250}{u}$$

Comparing **j** coefficients:

$$0 = 4ut - 4.9t^2$$

$$0 = \frac{4u \times 250}{u} - 4.9\left(\frac{250}{u}\right)^2 \quad \text{(substituting } t = \frac{250}{u} \text{ from above)}$$

$$= 1000 - \frac{306\,250}{u^2}$$

$$u^2 = \frac{306\,250}{1000}$$

$$= 306.25$$

$$u = \sqrt{306.25} = 17.5, \text{ as required.}$$

b Greatest height when **j** component of velocity is zero.
Considering **j** components:

$$u_y = 4u = 4 \times 17.5 = 70,\ a = -9.8,\ v_y = 0,\ s = ?$$

$$v^2 = u^2 + 2as$$

$$0^2 = 70^2 - 2 \times 9.8 \times s$$

$$s = \frac{70^2}{2 \times 9.8}$$

$$= 250$$

P reaches a max height of 250 m above the ground.

c Find the **i** and **j** components of the velocity when $t = 5$, and then find the angle between them.

$$\mathbf{u} = (52.5\mathbf{i} + 70\mathbf{j}),\ \mathbf{a} = -9.8\mathbf{j},\ t = 5,\ \mathbf{v} = ?$$

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{v} = (52.5\mathbf{i} + 70\mathbf{j}) - 5 \times 9.8\mathbf{j}$$

$$\mathbf{v} = 52.5\mathbf{i} - 21\mathbf{j}$$

$$\tan\theta = \frac{v_y}{u_x} = \frac{21}{52.5} = 0.4$$

$$\Rightarrow \theta = 21.8°$$

The angle the direction of motion of P makes with **i** when $t = 5$ is $22°$ (to the nearest degree).

5 Let the point S be $x\mathbf{i}+y\mathbf{j}$
$\mathbf{u}=(8\mathbf{i}+10\mathbf{j}),\ \mathbf{a}=-9.8\mathbf{j},\ t=6,\ \mathbf{s}=x\mathbf{i}+y\mathbf{j}$

a Considering **i** components,

$$\begin{aligned} x &= u_x \times t \\ &= 8\times 6 \\ &= 48 \end{aligned}$$

The horizontal distance between O and S is 48 m.

b Considering **j** components,

$$\begin{aligned} y &= ut+\frac{1}{2}at^2 \\ &= 10\times 6-4.9\times 6^2 \\ &= -116.4 \end{aligned}$$

The vertical distance between O and S is 120m (2 s.f.).

c $\mathbf{u}=(8\mathbf{i}+10\mathbf{j}),\ \mathbf{a}=-9.8\mathbf{j},\ t=T,\ \mathbf{v}=8\mathbf{i}-14.5\mathbf{j}$

$$\begin{aligned} \mathbf{v} &= \mathbf{u}+\mathbf{a}t \\ 8\mathbf{i}-14.5\mathbf{j} &= (8\mathbf{i}+10\mathbf{j})-T\times 9.8\mathbf{j} \end{aligned}$$

Considering **j** components,

$$\begin{aligned} -14.5 &= 10-9.8T \\ T &= \frac{24.5}{9.8}=\frac{5}{2} \\ &= 2\frac{1}{2} \end{aligned}$$

At $T=\frac{5}{2}$ s,

$$\mathbf{s}=\mathbf{u}t+\frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{s}=\frac{5}{2}(8\mathbf{i}+10\mathbf{j})+\frac{1}{2}(-9.8\mathbf{j})\left(\frac{5}{2}\right)^2$$

$$\mathbf{s}=\left(8\times\frac{5}{2}\right)\mathbf{i}+\left(10\times\frac{5}{2}-4.9\times\left(\frac{5}{2}\right)^2\right)\mathbf{j}$$

$$\mathbf{s}=20\mathbf{i}-\frac{45}{8}\mathbf{j}$$

The position vector of the particle after $2\frac{1}{2}$ seconds is $\left(20\mathbf{i}-\frac{45}{8}\mathbf{j}\right)$ m.

6 a $\mathbf{u} = (a\mathbf{i} + b\mathbf{j})\text{ ms}^{-1}$, $t = t$ s, $\mathbf{a} = -10\mathbf{j}\text{ ms}^{-2}$, $\mathbf{s} = (x\mathbf{i} + y\mathbf{j})$ m

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$x\mathbf{i} + y\mathbf{j} = (a\mathbf{i} + b\mathbf{j})t + \frac{1}{2}(-10\mathbf{j})t^2$$

Considering coefficients of **i**:

$x = at$

$t = \dfrac{x}{a}$ **(1)**

Considering coefficients of **j**:

$y = bt - 5t^2$ **(2)**

Substituting $t = \dfrac{x}{a}$ from (**1**) into (**2**):

$y = \dfrac{bx}{a} - \dfrac{5x^2}{a^2}$ as required.

b i X is the value of x when y = 0:

$$0 = \frac{bX}{8} - \frac{5X^2}{64}$$

$$\frac{5X^2}{64} = \frac{bX}{8}$$

$5X^2 = 8bX$ disregarding $X = 0$

$$X = \frac{8b}{5}$$

X is $1.6b$

ii Y is the value of y when $x = \dfrac{X}{2} = \dfrac{4b}{5}$:

$$Y = \frac{b \times 4b}{8 \times 5} - \frac{5 \times (4b)^2}{64 \times 5^2}$$

$$Y = \frac{b^2}{10} - \frac{b^2}{4 \times 5}$$

$$Y = \frac{b^2}{20}$$

Y is $0.05b^2$

Further kinematics 8C

1 a $a = 1 - \sin \pi t \text{ ms}^{-2}$, $t = 0$ s, $v = 0 \text{ ms}^{-1}$, $s = 0$ m

$$v = \int a \, dt$$

$$v = \int (1 - \sin \pi t) \, dt$$

$$v = t + \frac{\cos \pi t}{\pi} + c$$

Substituting $v = 0$ when $t = 0$ gives:

$$0 = 0 + \frac{\cos 0}{\pi} + c$$

$$c = -\frac{1}{\pi}$$

$$\Rightarrow v = t + \frac{\cos \pi t}{\pi} - \frac{1}{\pi}$$

b Using expression for v, above:

$$s = \int v \, dt$$

$$s = \int \left(t + \frac{\cos \pi t}{\pi} - \frac{1}{\pi} \right) dt$$

$$s = \frac{t^2}{2} + \frac{\sin \pi t}{\pi^2} - \frac{t}{\pi} + c$$

Substituting $s = 0$ when $t = 0$ gives:

$$0 = 0 + 0 - 0 + c$$

$$c = 0$$

$$\Rightarrow s = \frac{t^2}{2} + \frac{\sin \pi t}{\pi^2} - \frac{t}{\pi}$$

2 a $a = \sin 3\pi t \text{ ms}^{-2}$, $t = 0$ s, $v = \frac{1}{3\pi} \text{ ms}^{-1}$, $s = 1$ m

$$v = \int a \, dt$$

$$v = \int \sin 3\pi t \, dt$$

$$v = -\frac{\cos 3\pi t}{3\pi} + c$$

Using values given:

$$\frac{1}{3\pi} = -\frac{\cos 0}{3\pi} + c$$

$$c = \frac{1}{3\pi} + \frac{1}{3\pi}$$

$$\Rightarrow v = \frac{2}{3\pi} - \frac{\cos 3\pi t}{3\pi}$$

2 b The maximum value of v occurs when $\cos 3\pi t$ has its minimum value, i.e. −1.

$$v_{\max} = \frac{2}{3\pi} - \frac{-1}{3\pi} = \frac{1}{\pi}$$

The maximum value of v is $\frac{1}{\pi}$ ms^{-1}

c Using expression for v, above:

$$s = \int v \, dt$$

$$s = \int \left(\frac{2}{3\pi} - \frac{\cos 3\pi t}{3\pi} \right) dt$$

$$s = \frac{2t}{3\pi} - \frac{\sin 3\pi t}{9\pi^2} + c$$

Using values given:

$$1 = 0 - 0 + c$$

$$c = 1$$

$$\Rightarrow s = \frac{2t}{3\pi} - \frac{\sin 3\pi t}{9\pi^2} + 1$$

3 a $a = -\cos 4\pi t$ ms^{-2}, $t = 0$ s, $v = 0$ ms^{-1}, $s = 0$ m

$$v = \int a \, dt$$

$$v = \int -\cos 4\pi t \, dt$$

$$v = -\frac{\sin 4\pi t}{4\pi} + c$$

Using values given:

$$0 = -0 + c$$

$$c = 0$$

$$\Rightarrow v = -\frac{\sin 4\pi t}{4\pi}$$

b The maximum value of v occurs when $\sin 4\pi t$ has its minimum value, i.e. −1.

$$v_{\max} = -\frac{-1}{4\pi}$$

The maximum value of v is $\frac{1}{4\pi}$ ms^{-1}

c Using expression for v, above:

$$s = \int v \, dt$$

$$s = \int -\frac{\sin 4\pi t}{4\pi} dt$$

$$s = \frac{\cos 4\pi t}{16\pi^2} + c$$

3 c Using values given:

$$0 = \frac{1}{16\pi^2} + c$$

$$c = -\frac{1}{16\pi^2}$$

$$\Rightarrow s = \frac{\cos 4\pi t}{16\pi^2} - \frac{1}{16\pi^2}$$

d The maximum value of s occurs when $\cos 4\pi t$ is -1.

$$s_{\max} = -\frac{1}{16\pi^2} - \frac{1}{16\pi^2}$$

The maximum distance from O is $\frac{1}{8\pi^2}$ ms^{-1}

e The particle changes direction when $4\pi t = n\pi$ where n is a whole number.

It therefore changes direction whenever $t = \frac{n\pi}{4\pi}$ s i.e. every 0.25 s.

Between 0 and 4 s it therefore changes direction $\frac{4}{0.25} - 1 = 15$ times (it is stationary at 0 and 4 s).

4 a $v = \frac{\mathrm{d}s}{\mathrm{d}t}$

$$v = \frac{2}{3}3t^{-\frac{1}{3}} + (-3 \times 2\mathrm{e}^{-3t})$$

$$v = 2t^{-\frac{1}{3}} - 6\mathrm{e}^{-3t}$$

At $t = 0.5$ s:

$$v = 2\left(0.5^{-\frac{1}{3}}\right) - 6\mathrm{e}^{-1.5}$$

$$v = 1.1810...$$

At $t = 0.5$ s, the velocity of M is 1.18 ms^{-1} (3s.f.).

b $a = \frac{\mathrm{d}v}{\mathrm{d}t}$

$$a = \left(-\frac{1}{3}\right)2t^{-\frac{4}{3}} - (-3 \times 6\mathrm{e}^{-3t})$$

$$a = -\frac{2}{3}t^{-\frac{4}{3}} + 18\mathrm{e}^{-3t}$$

At $t = 3$ s:

$$a = -\frac{2}{3}\left(3^{-\frac{4}{3}}\right) + 18\mathrm{e}^{-9}$$

$$a = -0.15185...$$

At $t = 3$ s, the acceleration of M is -0.152 ms^{-2} (3s.f.).

c Using Newton's second law of motion when $t = 3$ s

$$\begin{aligned} F &= ma \\ &= 5 \times (-0.152) \\ &= -0.759 \text{ N (3s.f.)}. \end{aligned}$$

So F acts in opposition to the direction of motion.

5 a At $t = 4$ s, the relevant equation is $s = \frac{t}{2}$. Since $v = \frac{ds}{dt}$

$v = \frac{1}{2}$

At $t = 4$ s, the velocity of P is 0.5 ms^{-1}

b At $t = 22$ s, the relevant equation is $s = \sqrt{t+3}$. Since $v = \frac{ds}{dt}$

$v = \frac{1}{2}(t+3)^{-\frac{1}{2}}$

$v = \frac{1}{2} \times \frac{1}{\sqrt{25}} = \frac{1}{10}$

At $t = 4$ s, the velocity of P is 0.1 ms^{-1}

6 a $t = 2$ s, $s = 3^t + 3t$ m

$v = \frac{ds}{dt}$

$v = 3^t \ln 3 + 3$

$v(2) = 9\ln 3 + 3$

$v(2) = 12.887...$

At $t = 2$ s, the velocity of P is 12.9 ms^{-1} (3s.f.).

b $t = 10$ s, $s = -252 + 96t - 6t^2$ m

$v = \frac{ds}{dt}$

$v = 96 - 12t$

$v(10) = 96 - 120$

$v(10) = -24$

At $t = 10$ s, the velocity of P is -24 ms^{-1}

c For $0 \le t \le 3$, the displacement is $s = \left(3^t + 3t\right)$ m, which is always positive and increasing for $0 \le t \le 3$, so maximum displacement does not occur then.

For $3 < t \le 6$, the displacement is $s = \left(24t - 36\right)$ m, which is also always positive and increasing for $3 < t \le 6$, so maximum displacement does not occur then.

Therefore maximum displacement must occur when $t \ge 6$ s.

6 c For $t > 6$, max displacement occurs when

$$0 = \frac{ds}{dt}$$
$$0 = 96 - 12t$$
$$12t = 96$$
$$t = 8$$

Note that $\frac{d^2s}{dt^2} = -12 < 0 \Rightarrow t = 8$ is a max.

$$s(8) = -252 + (96 \times 8) - (6 \times 8^2)$$
$$= -252 + 768 - 384$$
$$= 132$$

The maximum displacement of P is 132 m.

d Check to see if there is a value of t for $0 \le t \le 3$ for which $\frac{ds}{dt}$ = 18 ms^{-1}:

$$18 = 3^t \ln 3 + 3$$
$$3^t = \frac{18-3}{\ln 3}$$
$$t \ln 3 = \ln 15 - \ln(\ln 3)$$
$$t = \frac{\ln 15 - \ln(\ln 3)}{\ln 3} = 2.3793...$$

At $t = 2.739$ s,

$$s(2.379) = 3^{2.379} + (3 \times 2.379)$$
$$= 20.791...$$

For $3 < t \le 6$, $24t - 36 = \pm 18 \Rightarrow t = 19.5$ s or $t = 0.75$ s, both of which do not lie in the interval $3 < t \le 6$, so no values of t in this interval for qwhich speed is 18 ms^{-1}

For $t > 6$,

$$\pm 18 = 96 - 12t$$
$$12t = 96 \pm 18$$
$$t = 6.5 \text{ and } t = 9.5$$

$$s(6.5) = -252 + (96 \times 6.5) - (6 \times 6.5^2)$$
$$= -252 + 624 - 253.5$$
$$= 118.5$$

$$s(9.5) = -252 + (96 \times 9.5) - (6 \times 9.5^2)$$
$$= -252 + 912 - 541.5$$
$$= 118.5$$

P has a speed of 18 ms^{-1} at displacements of 20.8 m (3s.f.) and 118.5 m (twice).

7 We will integrate twice to find an expression for the displacement, then find how long it takes to travel 16 m.

$a = 3\sqrt{t}$ ms^{-2}, $t = 1$ s, $v = 2$ ms^{-1}

$$v = \int a \, \mathrm{d}t$$

$$v = \int 3\sqrt{t} \, \mathrm{d}t$$

$$v = 3 \times \frac{2}{3} t^{\frac{3}{2}} + c$$

Using values given:

$$2 = \left(2 \times 1^{\frac{3}{2}}\right) + c$$

$$c = 2 - 2 = 0$$

$$\Rightarrow v = 2t^{\frac{3}{2}}$$

$$s = \int v \, \mathrm{d}t$$

$$s = \int 2t^{\frac{3}{2}} \, \mathrm{d}t$$

$$s = \frac{2}{5} \times 2t^{\frac{5}{2}} + c$$

Since we are interested in a time interval, we do not need to find c.

$$16 = \frac{4}{5} t^{\frac{5}{2}}$$

$$20 = t^{\frac{5}{2}}$$

$$t = 20^{\frac{2}{5}} = 3.3144...$$

The particle takes 3.31 s to travel 16 m.

8 **a** $s = k\sqrt{t}$ m; when $s = 200$ m, $t = 25$ s

$T = 25$ s because the runner completes the race in 25 s.

Also,

$$200 = k\sqrt{25}$$

$$k = \frac{200}{5}$$

$$= 40$$

The values of k and T are 40 and 25 s, respectively.

b $v = \dfrac{\mathrm{d}s}{\mathrm{d}t}$

$$v = \frac{1}{2} \times 40t^{-\frac{1}{2}}$$

$$= \frac{20}{\sqrt{t}}$$

Runner finishes the race in 25 s:

$$v(25) = \frac{20}{\sqrt{25}} = 4$$

The speed of the runner when she crosses the finish line is 4 ms^{-1}.

8 c For small values of t, v is unrealistically large: For example, at $t = 0.01$s $v = 20 \times 0.01^{-\frac{1}{2}} = 200\,\text{ms}^{-1}$ and no human could run this fast!

9 a $v = 2 + 8\sin kt$

$a = \dfrac{dv}{dt} = 8k\cos kt$

For any constant k, $\dfrac{d}{dt}(\sin kt) = k\cos kt$

When $t = 0$, $a = 4$

$4 = 8k \Rightarrow k = \frac{1}{2}$

The initial condition, that the acceleration is $4\,\text{m s}^{-1}$, gives an equation in k which you solve.

b $a = 8 \times \frac{1}{2}\cos\frac{1}{2}t = 4\cos\frac{1}{2}t$

When $a = 0$

$\cos\frac{1}{2}t = 0$

$\Rightarrow \frac{1}{2}t = \frac{\pi}{2},\ \frac{3\pi}{2}$

$\Rightarrow t = \pi,\ 3\pi$

In all differentiation and integration of trigonometric functions, it is assumed that angles are measured in radians. $\cos\theta = 0$ when θ is an odd multiple of $\frac{\pi}{2}$.

c $64 - (v-2)^2 = 64 - \left(8\sin\frac{1}{2}t\right)^2$

$= 64 - 64\sin^2\frac{1}{2}t$

$= 64\left(1 - \sin^2\frac{1}{2}t\right)$

$= 64\cos^2\frac{1}{2}t$

$= 4\left(4\cos\frac{1}{2}t\right)^2$

$= 4a^2$, as required.

Using the identity $\sin^2\theta + \cos^2\theta = 1$

d At maximum velocity, $a = 0\ \text{ms}^{-2}$

From part **b**, this occurs when $t = \pi$ s and $t = 3\pi$ s.

In both cases, $\sin kt = 1$, so $v = 2 + 8 = 10$.

The maximum value of a occurs when $\dfrac{da}{dt} = 0$

Since a is a multiple of $\cos kt$, $\dfrac{da}{dt}$ is a multiple of $\sin kt$,

so $\dfrac{da}{dt} = 0$ when $\sin kt = 0$ and hence $v = 2 + 0 = 2$ (from **a**)

By the result from **c**, at $v = 2$ we have

$4a^2 = 64 - (2-2)^2$

$a = \sqrt{\dfrac{64}{4}} = 4$

The maximum values of velocity and acceleration are $10\ \text{ms}^{-1}$ and $4\ \text{ms}^{-2}$ respectively.

10 a Find acceleration:

$$a = \frac{dv}{dt} = \frac{d\left(10t - 2t^{\frac{3}{2}}\right)}{dt}$$

$$a = 10 - 3t^{\frac{1}{2}}$$

For $0 \leq t \leq 4$, a ≥ 0, so v always increasing and hence maximum value of v occurs at $t = 4$ s.

$$v(4) = (10 \times 4) - \left(2 \times 4^{\frac{3}{2}}\right)$$

$$= 40 - 16$$

$$= 24$$

The maximum velocity for $0 \leq t \leq 4$ is 24 ms^{-1}

b For first 4 s:

$$s = \int v \, dt$$

$$s = \int \left(10t - 2t^{\frac{3}{2}}\right) dt$$

$$s = 5t^2 - \frac{4}{5}t^{\frac{5}{2}} + c$$

At $t = 0$, $s = 0$ so $c = 0$

Hence,

$$s(4) = \left(5 \times 4^2\right) - \left(\frac{4}{5} \times 4^{\frac{5}{2}}\right)$$

$$= 80 - \frac{128}{5}$$

$$= 54.4$$

When $t = 4$ s, P is 54.4 m from O.

c When P is at rest, $v = 0$

$$0 = 24 - \left(\frac{t-4}{2}\right)^4$$

$$\frac{t-4}{2} = \sqrt[4]{24}$$

$$t = \left(2 \times \sqrt[4]{24}\right) + 4 = 8.4267...$$

P is at rest after 8.43 s (3s.f.).

10 d In first 4 s, P travels 54.4 m (see part **b**).
For remaining time:

$$s = \int_4^{10} v \, \mathrm{d}t$$

$$s = \int_4^{8.43} 24 - \left(\frac{t-4}{2}\right)^4 \mathrm{d}t + \left|\int_{8.43}^{10} 24 - \left(\frac{t-4}{2}\right)^2 \mathrm{d}t\right|$$

$$s = \left[24t - \frac{(t-4)^5}{5 \times 2^4}\right]_4^{8.43} + \left|\left[24t - \frac{(t-4)^5}{5 \times 2^4}\right]_{8.43}^{10}\right|$$

$$s = \left(24 \times 8.43 - \frac{4.43^5}{80}\right) - (96 - 0) + \left|\left(240 - \frac{6^5}{80}\right) - \left(24 \times 8.43 - \frac{4.43^5}{80}\right)\right|$$

$$= 85 + |-38.2| - 85 + 38.2 - 123.2$$

P travels a total distance of 54.4 + 123.2 = 177.6 m.

Further kinematics 8D

1 a $\mathbf{v} = \frac{\mathrm{d}r}{\mathrm{d}t} = 3\mathbf{i} + (3t^2 - 4)\mathbf{j}$

When $t = 3$,

$\mathbf{v} = 3\mathbf{i} + 23\mathbf{j}$

The velocity of P when $t = 3$ is $(3\mathbf{i} + 23\mathbf{j})\,\mathrm{m\,s^{-1}}$

b $\mathbf{a} = \dot{\mathbf{v}} = 6t\,\mathbf{j}$

When $t = 3$,

$\mathbf{a} = 18\,\mathbf{j}$

The acceleration of P when $t = 3$ is $18\,\mathbf{j}\,\mathrm{m\,s^{-2}}$

2 $m = 3\text{ g} = 0.003\text{ kg}$, $\mathbf{v} = (t^2\mathbf{i} + (2t - 3)\mathbf{j})\,\mathrm{ms^{-1}}$, $t = 4$ s, $\mathbf{F} = ?$

$\mathbf{a} = \dot{\mathbf{v}}$

$\mathbf{a} = 2t\mathbf{i} + 2\mathbf{j}$

When $t = 4$ s, $\mathbf{a} = 8\mathbf{i} + 2\mathbf{j}$

$\mathbf{F} = m\mathbf{a}$

$\mathbf{F} = 0.003 \times (8\mathbf{i} + 2\mathbf{j})$

$= 0.024\mathbf{i} + 0.006\mathbf{j}$

The force $\mathbf{F}$ is $(0.024\mathbf{i} + 0.006\mathbf{j})$ N.

3 $\mathbf{r} = 5\mathrm{e}^{-3t}\mathbf{i} + 2\mathbf{j}$ m

a When P is directly NE of O, coefficients of **i** and **j** are identical.

$5\mathrm{e}^{-3t} = 2$

$\mathrm{e}^{-3t} = 0.4$

$-3t = \ln 0.4$

$t = \frac{\ln 0.4}{-3} = 0.30543...$

P is directly NE of O at $t = 0.305$ s (3s.f.).

b $\mathbf{v} = \dot{\mathbf{r}}$

$\mathbf{v} = -15\mathrm{e}^{-3t}\mathbf{i}$

However, when particle is north east of O, by part **a** we see that $\mathrm{e}^{-3t} = 0.4$

Hence

$\mathbf{v} = -(15 \times 0.4)\mathbf{i} = 6\mathbf{i}$

The speed at this time is 6 ms^{-1}

c The velocity vector has a single component in the direction of **i** and the coefficient is always negative (since e^{-3t} is always positive) so P is always moving west.

4 a $\mathbf{v} = \dot{\mathbf{r}} = 8t\,\mathbf{i} + (24 - 6t)\,\mathbf{j}$
When $t = 2$,
$\mathbf{v} = (16\mathbf{i} + 12\mathbf{j})$
$|\mathbf{v}|^2 = 16^2 + 12^2 = 400$
$\Rightarrow |\mathbf{v}| = \sqrt{400} = 20$

The speed of P when $t = 2$ is $20\,\text{m s}^{-1}$

b $\mathbf{a} = \dot{\mathbf{v}} = 8\,\mathbf{i} - 6\,\mathbf{j}$
Neither component is dependent on t, hence the acceleration is a constant.
$|\mathbf{a}|^2 = 8^2 + (-6)^2 = 100$
$\Rightarrow |\mathbf{a}| = \sqrt{100} = 10$

The magnitude of the acceleration is $10\,\text{m s}^{-1}$

5 a $\mathbf{v} = \dot{\mathbf{r}} = (3t^2 - 12)\,\mathbf{i} + (8t - 6)\,\mathbf{j}$
When $t = 0$,
$\mathbf{v} = -12\,\mathbf{i} - 6\,\mathbf{j}$
$|\mathbf{v}|^2 = (-12)^2 + (-6)^2 = 180$
$\Rightarrow |\mathbf{v}| = \sqrt{180} = 6\sqrt{5}$

The speed of projection is $6\sqrt{5}\,\text{m s}^{-1}$

b When P is moving parallel to **j** the velocity has no **i** component.
$3t^2 - 12 = 0$
$\Rightarrow t^2 = 4$
$\Rightarrow t = 2 \ (t \geq 0)$

c When $t = 2$
$\mathbf{r} = (2^3 - 12 \times 2)\,\mathbf{i} + (4 \times 2^2 - 6 \times 2)\,\mathbf{j} = -16\,\mathbf{i} + 4\,\mathbf{j}$
The position vector of P at the instant when P is moving parallel to **j** is $(-16\,\mathbf{i} + 4\,\mathbf{j})\,\text{m}$.

d $\mathbf{r} = \left(t^3 - 12t\right)\mathbf{i} + \left(4t^2 - 6t\right)\mathbf{j}$ m, $t = 5$ s, $m = 0.5$ kg, $\mathbf{F} = ?$
$\mathbf{v} = \dot{\mathbf{r}} = \left(3t^2 - 12\right)\mathbf{i} + \left(8t - 6\right)\mathbf{j}$
$\mathbf{a} = \dot{\mathbf{v}} = 6t\mathbf{i} + 8\mathbf{j}$
When $t = 5$ s, $\mathbf{a} = 30\mathbf{i} + 8\mathbf{j}$
Hence, $\mathbf{F} = m\mathbf{a}$
$= 0.5\left(30\mathbf{i} + 8\mathbf{j}\right)$
$\mathbf{F} = 15\mathbf{i} + 4\mathbf{j}$
$\left|\mathbf{F}\right| = \sqrt{15^2 + 4^2}$
$= 15.524...$
The magnitude of the force acting on P at $t = 5$ s is 15.5 N (3s.f.).

6 a $\mathbf{v} = \dot{\mathbf{r}} = (6t-6)\mathbf{i} + (3t^2 + 2kt)\mathbf{j}$

When $t = 3$,

$\mathbf{v} = 12\mathbf{i} + (27 + 6k)\mathbf{j}$

$\left(12\sqrt{5}\right)^2 = |v|^2$

$720 = 12^2 + \left(27 + k\right)^2$

$720 = 144 + 729 + 324k + 36k^2$

$0 = 36k^2 + 324k + 153$

$0 = (2k+1)(2k+17)$

$k = -0.5,\ -8.5$

b If $k = -0.5$

$\mathbf{v} = (6t-6)\mathbf{i} + (3t^2 - t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (6t-1)\mathbf{j}$

When $t = 1.5$,

$\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$

$|\mathbf{a}|^2 = 6^2 + 8^2 = 100$

$\Rightarrow |\mathbf{a}| = 10$

If $k = -8.5$

$\mathbf{v} = (6t-6)\mathbf{i} + (3t^2 - 17t)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\mathbf{i} + (6t-17)\mathbf{j}$

When $t = 1.5$,

$\mathbf{a} = 6\mathbf{i} - 8\mathbf{j}$

$|\mathbf{a}|^2 = 6^2 + (-8)^2 = 100$

$\Rightarrow |\mathbf{a}| = 10$

For both of the values of k the magnitude of the acceleration of P when $t = 1.5$ is $10\,\mathrm{m\,s^{-2}}$

7 a $\mathbf{v} = \dot{\mathbf{r}} = 12t\mathbf{i} + \frac{5}{2}t^{\frac{3}{2}}\mathbf{j}$

When $t = 4$,

$\mathbf{v} = 48\mathbf{i} + \frac{5}{2} \times 4^{\frac{3}{2}}\mathbf{j}$

$= 48\mathbf{i} + 20\mathbf{j}$

$|\mathbf{v}|^2 = 48^2 + 20^2 = 2704^2$

$\Rightarrow |\mathbf{v}| = \sqrt{2704} = 52$

$\boxed{4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8}$

The speed of P when $t = 4$ is $52\,\mathrm{m\,s^{-1}}$

b $\mathbf{a} = \dot{\mathbf{v}} = 12\mathbf{i} + \dfrac{5}{2} \times \dfrac{3}{2}t^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \dfrac{15}{4}t^{\frac{1}{2}}\mathbf{j}$

You need to know that $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$

When $t = 4$

$\mathbf{a} = 12\mathbf{i} - \dfrac{15}{4} \times 4^{\frac{1}{2}}\mathbf{j} = 12\mathbf{i} + \dfrac{15}{2}\mathbf{j}$

The acceleration of P when $t = 4$ is $\left(12\mathbf{i} + \dfrac{15}{2}\mathbf{j}\right)\mathrm{m\,s^{-2}}$

8 a $\mathbf{v} = \dot{\mathbf{r}} = \left(18 - 12t^2\right)\mathbf{i} + 2ct\,\mathbf{j}$

When $t = 1.5$,

$\mathbf{v} = \left(18 - 12 \times 1.5^2\right)\mathbf{i} + 3c\,\mathbf{j}$

$= -9\,\mathbf{i} + 3c\,\mathbf{j}$

$15^2 = |v|^2$

$15^2 = (-9)^2 + (3c)^2$

$9c^2 = 15^2 - 9^2$

$9c^2 = 144$

$\Rightarrow c^2 = \dfrac{144}{9} = 16$

As c is positive, $c = 4$

b $\mathbf{a} = \dot{\mathbf{v}} = -24t\,\mathbf{i} + 2c\,\mathbf{j}$

Using $c = 4$ and $t = 1.5$

$\mathbf{a} = -36\,\mathbf{i} + 8\,\mathbf{j}$

The acceleration of P when $t = 1.5$ is $\left(-36\,\mathbf{i} + 8\,\mathbf{j}\right)\mathrm{m\,s^{-2}}$

Acceleration is a vector and the answer should be given in vector form.

9 $\mathbf{r} = \left(2t^2 - 3t\right)\mathbf{i} + \left(5t + t^2\right)\mathbf{j}$ m

$\mathbf{v} = \dot{\mathbf{r}} = \left(4t - 3\right)\mathbf{i} + \left(5 + 2t\right)\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 4\mathbf{i} + 2\mathbf{j}$

$|\mathbf{a}| = \sqrt{4^2 + 2^2} = 2\sqrt{5}$

The acceleration is constant because the expression for it does not contain t, and it has a magnitude of $2\sqrt{5}$ ms^{-2}

10 a $\mathbf{r} = \left(20t - 2t^3\right)\mathbf{i} + kt^2\mathbf{j}$ m, $t = 2$ s, $|\mathbf{v}| = 16$ ms^{-1}

$\mathbf{v} = \dot{\mathbf{r}} = \left(20 - 6t^2\right)\mathbf{i} + 2kt\mathbf{j}$

$\mathbf{v}(2) = (20 - 24)\mathbf{i} + 4k\mathbf{j}$

$= -4\mathbf{i} + 4k\mathbf{j}$

$16^2 = |\mathbf{v}(2)|^2 = (-4)^2 + \left(4k\right)^2$

$256 = 16 + 16k^2$

$k^2 = \dfrac{256 - 16}{16} = 15$

$k = \sqrt{15}$

The value of k is $\sqrt{15}$.

10 b When P is moving parallel to $\mathbf{j}$, the coefficient of the $\mathbf{i}$ component of velocity is zero.
From part **a**, since $\mathbf{v} = \left(20 - 6t^2\right)\mathbf{i} + 2kt\mathbf{j}$, P is moving parallel to $\mathbf{j}$ when:

$$20 - 6t^2 = 0$$

$$t^2 = \frac{20}{6}$$

$$t = \sqrt{\frac{10}{3}}$$

Now $\mathbf{a} = \dot{\mathbf{v}} = -12t\mathbf{i} + 2\sqrt{15}\mathbf{j}$

At $t = \sqrt{\frac{10}{3}}$ s, the acceleration is given by:

$$\mathbf{a} = -12\sqrt{\frac{10}{3}}\mathbf{i} + 2\sqrt{15}\mathbf{j}$$

$$\mathbf{a} = -4\sqrt{30}\mathbf{i} + 2\sqrt{15}\mathbf{j}$$

When P is moving parallel to $\mathbf{j}$ its acceleration is $\left(-4\sqrt{30}\mathbf{i} + 2\sqrt{15}\mathbf{j}\right)\text{ms}^{-2}$

Further kinematics 8E

1 a $\mathbf{v} = \int \mathbf{a}\,dt = \int (6t^2\,\mathbf{i} + (8 - 4t^3)\,\mathbf{j})dt$

$= 2t^3\,\mathbf{i} + (8t - t^4)\mathbf{j} + C$

When $t = 0$, $\mathbf{v} = 0\,\mathbf{i} + 0\,\mathbf{j}$

$0\,\mathbf{i} + 0\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + C \Rightarrow C = 0\,\mathbf{i} + 0\,\mathbf{j}$

Hence

$\mathbf{v} = 2t^3\,\mathbf{i} + (8t - t^4)\,\mathbf{j}$

When $t = 2$

$\mathbf{v} = 16\,\mathbf{i} + (8 \times 2 - 2^4)\,\mathbf{j} = 16\,\mathbf{i}$

The velocity of P when $t = 2$ is $16\,\mathbf{i}$ m s^{-1}

b $\mathbf{r} = \int \mathbf{v}dt = \int (2t^3\,\mathbf{i} + (8t - t^4)\,\mathbf{j})dt$

$= \frac{1}{2}t^4\,\mathbf{i} + \left(4t^2 - \frac{1}{5}t^5\right)\mathbf{j} + D$

When $t = 0$, $\mathbf{r} = 0\,\mathbf{i} + 0\,\mathbf{j}$

$0\,\mathbf{i} + 0\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + D \Rightarrow D = 0\,\mathbf{i} + 0\,\mathbf{j}$

Hence

$\mathbf{r} = \frac{t^4}{2}\mathbf{i} + \left(4t^2 - \frac{t^5}{5}\right)\mathbf{j}$

When $t = 4$

$\mathbf{r} = \frac{4^4}{2}\mathbf{i} + \left(4 \times 4^2 - \frac{4^5}{5}\right)\mathbf{j} = 128\,\mathbf{i} - 140.8\,\mathbf{j}$

The position vector of P when $t = 4$ is $(128\,\mathbf{i} - 140.8\,\mathbf{j})$m

2 a $\mathbf{r} = \int \mathbf{v}\,dt = \int ((3t^2 + 2)\,\mathbf{i} + (6t - 4)\,\mathbf{j})\,dt$

$= (t^3 + 2t)\,\mathbf{i} + \left(3t^2 - 4t\right)\mathbf{j} + A$

When $t = 2$, $\mathbf{v} = 9\,\mathbf{j}$

$9\,\mathbf{j} = 12\,\mathbf{i} + 4\,\mathbf{j} + A \Rightarrow A = -12\,\mathbf{i} + 5\,\mathbf{j}$

Hence

$\mathbf{r} = (t^3 + 2t - 12)\,\mathbf{i} + \left(3t^2 - 4t + 5\right)\mathbf{j}$

When $t = 0$

$\mathbf{r} = -12\,\mathbf{i} + 5\,\mathbf{j}$

$|\,\mathbf{r}\,|^2 = (-12)^2 + 5^2 = 169 \Rightarrow |\,\mathbf{r}\,| = \sqrt{169} = 13$

The distance of P from O when $t = 0$ is 13 m.

2 b When P is moving parallel to $\mathbf{i}$, $\mathbf{v}$ has no $\mathbf{j}$ component.

$\Rightarrow 6t - 4 = 0$

$\Rightarrow t = \frac{2}{3}$

$\mathbf{a} = \dot{\mathbf{v}} = 6t\,\mathbf{i} + 6\,\mathbf{j}$

When $t = \frac{2}{3}$ s, $\mathbf{a} = 4\,\mathbf{i} + 6\,\mathbf{j}$

The acceleration of P at the instant when it is moving parallel to the vector $\mathbf{i}$ is $(4\,\mathbf{i} + 6\,\mathbf{j})\,\text{m s}^{-2}$

3 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \left((2t-4)\mathbf{i} + 6\sin t\mathbf{j}\right)\,\mathrm{d}t$

$= \left(t^2 - 4t\right)\mathbf{i} - 6\cos t\mathbf{j} + c$

When $t = \frac{\pi}{2}$ s, $\mathbf{v} = 0\ \text{ms}^{-1}$, so

$0 = \left(\frac{\pi^2}{4} - \frac{4\pi}{2}\right)\mathbf{i} - 0\mathbf{j} + c$

$c = \left(2\pi - \frac{\pi^2}{4}\right)\mathbf{i}$

The velocity of the particle is given by $\left[\left(t^2 - 4t + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 6\cos t\mathbf{j}\right]\ \text{ms}^{-1}$

b When $t = \frac{3\pi}{2}$ s,

$\mathbf{v} = \left(\frac{9\pi^2}{4} - \frac{12\pi}{2} + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 0\mathbf{j}$

$\mathbf{v} = \left(\frac{8\pi^2}{4} - 6\pi + 2\pi\right)\mathbf{i}$

$\mathbf{v} = \left(2\pi^2 - 4\pi\right)\mathbf{i}$

Since the velocity only has an $\mathbf{i}$ component when $t = \frac{3\pi}{2}$ s, this is also the speed.

The speed of P at $\frac{3\pi}{2}$ s is $\left(2\pi^2 - 4\pi\right)\ \text{ms}^{-1}$

4 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int ((5t-3)\,\mathbf{i} + (8-t)\,\mathbf{j})\,\mathrm{d}t$

$= \left(\frac{5}{2}t^2 - 3t\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2\right)\mathbf{j} + C$

When $t = 0$, $\mathbf{v} = 2\,\mathbf{i} - 5\,\mathbf{j}$

$2\,\mathbf{i} - 5\,\mathbf{j} = 0\,\mathbf{i} + 0\,\mathbf{j} + C \Rightarrow C = 2\,\mathbf{i} - 5\,\mathbf{j}$

Hence

$\mathbf{v} = \left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}$

The velocity of P after t seconds is $\left(\left(\frac{5}{2}t^2 - 3t + 2\right)\mathbf{i} + \left(8t - \frac{1}{2}t^2 - 5\right)\mathbf{j}\right)\ \text{m s}^{-1}$

4 b P is moving parallel to $\mathbf{i}-\mathbf{j}$ when, in the expression giving the velocity of P

$(\text{coefficient of } \mathbf{i} \text{ component}) = -1\times(\text{coefficient of j component})$

$$\left(\frac{5}{2}t^2-3t+2\right)=-\left(8t-\frac{1}{2}t^2-5\right)$$

$$\frac{5}{2}t^2-3t+2=-8t+\frac{1}{2}t^2+5$$

$$2t^2+5t-3=0$$

$$(2t-1)(t+3)=0$$

Hence,

$$t=\frac{1}{2},\ -3$$

As $t\geq 0$, $t=\frac{1}{2}$

c When $t=\frac{1}{2}$

$$\mathbf{v}=\left(\frac{5}{8}-\frac{3}{2}+2\right)\mathbf{i}+\left(4-\frac{1}{8}-5\right)\mathbf{j}$$

$$=\frac{9}{8}\mathbf{i}-\frac{9}{8}\mathbf{j}$$

$$|\mathbf{v}|^2=\left(\frac{9}{8}\right)^2+\left(\frac{9}{8}\right)^2=2\times\left(\frac{9}{8}\right)^2$$

$$\Rightarrow|\mathbf{v}|=\frac{9\sqrt{2}}{8}$$

The speed of P when it is moving parallel to $\mathbf{i}-\mathbf{j}$ is $\frac{9\sqrt{2}}{8}\ \mathrm{m\,s^{-1}}$

5 a $\mathbf{v}=\int\mathbf{a}\,\mathrm{d}t=\int(2\mathbf{i}-2t\,\mathbf{j})\,\mathrm{d}t$

$$=2t\,\mathbf{i}-t^2\mathbf{j}+A$$

When $t=0$, $\mathbf{v}=2\,\mathbf{j}$

$$2\,\mathbf{j}=0\,\mathbf{i}+0\,\mathbf{j}+A\Rightarrow A=2\,\mathbf{j}$$

Hence

$$\mathbf{v}=2t\,\mathbf{i}+(2-t^2)\,\mathbf{j}$$

Let the position vector of P at time t seconds be $\mathbf{p}$ m.

$$\mathbf{p}=\int\mathbf{v}\,\mathrm{d}t=\int 2t\,\mathbf{i}+(2-t^2)\,\mathbf{j}$$

$$=t^2\,\mathbf{i}+\left(2t-\frac{1}{3}t^3\right)\mathbf{j}+B$$

When $t=0$, $\mathbf{v}=6\,\mathbf{i}$

$$6\,\mathbf{i}=0\,\mathbf{i}+0\,\mathbf{j}+B\Rightarrow B=6\,\mathbf{i}$$

Hence

$$\mathbf{p}=(t^2+6)\,\mathbf{i}+\left(2t-\frac{1}{3}t^3\right)\mathbf{j}$$

The position vector of P at time t seconds is $\left((t^2+6)\,\mathbf{i}+\left(2t-\frac{1}{3}t^3\right)\mathbf{j}\right)$ m

5 b Let the position vector of Q at time t seconds be $\mathbf{q}$ m.

$$\mathbf{q} = \int \mathbf{v}\,dt = \int ((3t^2 - 4)\mathbf{i} - 2t\,\mathbf{j})dt$$
$$= (t^3 - 4t)\mathbf{i} - t^2\,\mathbf{j} + C$$

From part **a**, when $t = 3$

$$\mathbf{p} = (3^2 + 6)\mathbf{i} + \left(2 \times 3 - \frac{3^3}{3}\right)\mathbf{j} = 15\mathbf{i} - 3\mathbf{j}$$

As the particles collide when $t = 3$, $\mathbf{q}(3) = \mathbf{p}(3)$

$$\mathbf{p}(3) = \mathbf{q}(3)$$
$$15\mathbf{i} - 3\mathbf{j} = (3^3 - 4 \times 3)\mathbf{i} - 3^2\,\mathbf{j} + C$$
$$15\mathbf{i} - 3\mathbf{j} = 15\mathbf{i} - 9\mathbf{j} + C$$
$$C = 6\mathbf{j}$$

Hence,

$$\mathbf{q} = (t^3 - 4t)\mathbf{i} + (6 - t^2)\mathbf{j}$$

When $t = 0$, $\mathbf{q} = 6\mathbf{j}$

The position vector of Q at time $t = 0$ is $6\mathbf{j}$ m

6 a

$$\mathbf{v} = \int \mathbf{a}\,dt = \int ((4t - 3)\mathbf{i} - 6t^2t\,\mathbf{j})dt$$
$$= (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j} + A$$

When $t = 0$, $\mathbf{v} = 0$

$$0 = 0\mathbf{i} + 0\mathbf{j} + A \Rightarrow A = 0$$
$$\mathbf{v} = (2t^2 - 3t)\mathbf{i} - 2t^3\mathbf{j}$$

When $t = \frac{1}{2}$

$$\mathbf{v} = \left(2\left(\frac{1}{2}\right)^2 - 3 \times \frac{1}{2}\right)\mathbf{i} - 2\left(\frac{1}{2}\right)^3\mathbf{j}$$
$$= -\mathbf{i} - \frac{1}{4}\mathbf{j}$$

The velocity of P when $t = \frac{1}{2}$ is $\left(-\mathbf{i} - \frac{1}{4}\mathbf{j}\right)\text{m s}^{-1}$

b

$$\mathbf{r} = \int \mathbf{v}\,dt = \int ((2t^2 - 3t)\mathbf{i} - 2t^3\,\mathbf{j})dt$$
$$= \left(\frac{2}{3}t^3 - \frac{3}{2}t^2\right)\mathbf{i} - \frac{1}{2}t^4\,\mathbf{j} + B$$

When $t = 0$, $\mathbf{r} = 4\mathbf{i} - 6\mathbf{j}$

$$4\mathbf{i} - 6\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + B \Rightarrow B = 4\mathbf{i} - 6\mathbf{j}$$
$$r = \left(\frac{2}{3}t^3 - \frac{3}{2}t^2 + 4\right)\mathbf{i} - \left(\frac{1}{2}t^4 + 6\right)\mathbf{j}$$

When $t = 6$

$$\mathbf{r} = (144 - 54 + 4)\mathbf{i} - (648 + 6)\mathbf{j} = 94\mathbf{i} - 654\mathbf{j}$$

The position vector of P when $t = 6$ is $(94\mathbf{i} - 654\mathbf{j})$ m

7 a $\mathbf{v} = \int \mathbf{a}\,dt = \int ((8t^3 - 6t)\mathbf{i} + (8t-3)\mathbf{j})dt$

$= (2t^4 - 3t^2)\mathbf{i} + (4t^2 - 3t)\mathbf{j} + C$

When $t = 2$, $\mathbf{v} = 16\mathbf{i} + 3\mathbf{j}$

$16\mathbf{i} + 3\mathbf{j} = 20\mathbf{i} + 10\mathbf{j} + C \Rightarrow C = -4\mathbf{i} - 7\mathbf{j}$

$\mathbf{v} = (2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}$

The velocity of P after t seconds is $\left((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j}\right)$ m s^{-1}

b When P is moving parallel to **i**, the **j** component of the velocity is zero.

$4t^2 - 3t - 7 = 0$

$(t+1)(4t-7) = 0$

$t \geq 0 \Rightarrow t = \frac{7}{4}$ s

8 a $\mathbf{r}_P = \int \mathbf{v}_P\,dt = \int \left((4t-3)\mathbf{i} + 4\mathbf{j}\right)dt$

$= \left(2t^2 - 3t\right)\mathbf{i} + 4t\mathbf{j} + c$

When $t = 0$ s, $\mathbf{r}_P = \left(\mathbf{i} + 2\mathbf{j}\right)$ m

$\mathbf{i} + 2\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = \mathbf{i} + 2\mathbf{j}$

The position of P at time t is given by $\left(\left(2t^2 - 3t + 1\right)\mathbf{i} + \left(4t + 2\right)\mathbf{j}\right)$m.

b i $\mathbf{r}_Q = \int \mathbf{v}_Q\,dt = \int 5\mathbf{i} + k\mathbf{j}\,dt$

$= 5t\mathbf{i} + kt\mathbf{j} + c$

When $t = 0$ s, $\mathbf{r} = \left(11\mathbf{i} + 5\mathbf{j}\right)$ m

$11\mathbf{i} + 5\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c$

$c = 11\mathbf{i} + 5\mathbf{j}$

$\mathbf{r}_Q = \left(5t + 11\right)\mathbf{i} + \left(kt + 5\right)\mathbf{j}$

When the particles collide, their position vectors are identical, so:

$$\mathbf{r}_P = \mathbf{r}_Q$$

$\left(2t^2 - 3t + 1\right)\mathbf{i} + \left(4t + 2\right)\mathbf{j} = \left(5t + 11\right)\mathbf{i} + \left(kt + 5\right)\mathbf{j}$

Considering the coefficients of **i**:

$2t^2 - 3t + 1 = 5t + 11$

$2t^2 - 8t - 10 = 0$

$t^2 - 4t - 5 = 0$

$\left(t - 5\right)\left(t + 1\right) = 0$

The negative root can be ignored, so the particles collide when $t = 5$ s

Equating the coefficients of **j** when $t = 5$ s:

$20 + 2 = 5k + 5$

$k = \frac{22 - 5}{5} = 3.4$

The value of k is 3.4

8 b ii Substituting $k = 3.4$ and $t = 5$ into equation for $\mathbf{r}_Q$:

$$\mathbf{r}_Q = (25+11)\mathbf{i} + ((5\times3.4)+5)\mathbf{j}$$
$$= 36\mathbf{i} + 22\mathbf{j}$$

The position vector of the points where the particles meet is $(36\mathbf{i}+22\mathbf{j})$ m.

Challenge

$\mathbf{v} = (3t\cos t\mathbf{i} + 5t\mathbf{j})$ ms^{-1}, $\mathbf{r_o} = (4\mathbf{i}+\mathbf{j})$m, $t = 0$ s

$$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int (3t\cos t\mathbf{i} + 5t\mathbf{j})\,\mathrm{d}t$$

To evaluate $\int t\cos t\,\mathrm{d}t$, let $u = t$ and $\frac{\mathrm{d}v}{\mathrm{d}t} = \cos t$

Then $\frac{\mathrm{d}u}{\mathrm{d}t} = 1$ and $v = \sin t$

Using integration by parts, $\int t\cos t\,\mathrm{d}t = t\sin t - \int \sin t\,\mathrm{d}t$

$$= t\sin t + \cos t \qquad \textbf{(1)}$$

$$\mathbf{r} = \int (3t\cos t\mathbf{i} + 5t\mathbf{j})\,\mathrm{d}t$$
$$= \left(3\int t\cos t\,\mathrm{d}t\right)\mathbf{i} + \left(5\int t\,\mathrm{d}t\right)\mathbf{j}$$
$$= 3(t\sin t + \cos t)\mathbf{i} + \frac{5t^2}{2}\mathbf{j} + c \qquad \text{(using } \textbf{(1)}\text{)}$$

When $t = 0$ s, $\mathbf{r} = (4\mathbf{i}+\mathbf{j})$m

$$4\mathbf{i}+\mathbf{j} = 3(0+1)\mathbf{i} + 0\mathbf{j} + c$$
$$c = \mathbf{i}+\mathbf{j}$$

Hence, $\mathbf{r} = (3(t\sin t + \cos t)+1)\mathbf{i} + \left(\frac{5t^2}{2}+1\right)\mathbf{j}$

When $t = \frac{\pi}{2}$,

$$\mathbf{r} = \left(3\left(\frac{\pi}{2}\sin\frac{\pi}{2}+0\right)+1\right)\mathbf{i} + \left(\frac{5\pi^2}{2\times4}+1\right)\mathbf{j}$$
$$\mathbf{r} = \left(\frac{3\pi}{2}+1\right)\mathbf{i} + \left(\frac{5\pi^2}{8}+1\right)\mathbf{j}$$

The position of P at time $t = \frac{\pi}{2}$ s is $\left(\left(\frac{3\pi}{2}+1\right)\mathbf{i} + \left(\frac{5\pi^2}{8}+1\right)\mathbf{j}\right)$ m relative to O.

Further kinematics Mixed exercise 8

1 $\mathbf{u} = 0,\ t = 5,\ \mathbf{v} = 6\mathbf{i} - 8\mathbf{j},\ \mathbf{a} = ?$

Using $\mathbf{v} = \mathbf{u} + \mathbf{a}t$,

$$(6\mathbf{i} - 8) = \mathbf{a} \times 5,$$

$$\mathbf{a} = \frac{1}{5}(6\mathbf{i} - 8\mathbf{j})$$

Using $\mathbf{F} = m\mathbf{a}$,

$$\mathbf{F} = 4 \times \frac{1}{5}(6\mathbf{i} - 8\mathbf{j})$$

$$= 4.8\mathbf{i} - 6.4\mathbf{j}$$

2 Using $\mathbf{F} = m\mathbf{a}$,

$$(2\mathbf{i} - \mathbf{j}) = 2\mathbf{a},$$

$$\mathbf{a} = \mathbf{i} - \frac{1}{2}\mathbf{j}$$

$\mathbf{u} = \mathbf{i} + 3\mathbf{j},\ t = 3,\ \mathbf{a} = \mathbf{i} - \frac{1}{2}\mathbf{j},\ \mathbf{s} = ?$

Using $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$,

$$\mathbf{s} = (\mathbf{i} + 3\mathbf{j}) \times 3 + \frac{1}{2}\left(\mathbf{i} - \frac{1}{2}\mathbf{j}\right) \times 3^2$$

$$= 3\mathbf{i} + 9\mathbf{j} + \frac{9}{2}\mathbf{i} - \frac{9}{4}\mathbf{j}$$

$$= \frac{15}{2}\mathbf{i} + \frac{27}{4}\mathbf{j}$$

$$\text{distance} = \sqrt{\left(\frac{15}{2}\right)^2 + \left(\frac{27}{4}\right)^2}$$

$$= \sqrt{56.25 + 45.5625}$$

$$= \sqrt{101.8125}$$

$$= 10.1\text{ m (3 s.f.)}$$

3 a Using $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$,

$$\mathbf{r} = -500\mathbf{j} + (2\mathbf{i} + 3\mathbf{j}) \times t$$

$$= -500\mathbf{j} + 2t\mathbf{i} + 3t\mathbf{j}$$

$$= 2t\mathbf{i} + (-500 + 3t)\mathbf{j}$$

3 b 5 minutes $= 5 \times 60$ seconds

$= 300$ seconds

At 2.05 pm, the dinghy has position:

$$\mathbf{r} = 2 \times 300\,\mathbf{i} + (-500 + 3 \times 300)\,\mathbf{j}$$
$$= 600\,\mathbf{i} + 400\,\mathbf{j}$$

$$\text{distance} = \sqrt{600^2 + 400^2}$$
$$= \sqrt{360\,000 + 160\,000}$$
$$= \sqrt{520\,000}$$
$$= 721\text{m} \quad (3\text{ s.f.})$$

4 a Using $\mathbf{r}_A = \mathbf{r}_{A_0} + \mathbf{v}_A t$ for **A**,

$$\mathbf{r}_A = (\mathbf{i} + 3\,\mathbf{j}) + (2\mathbf{i} - \mathbf{j}) \times t$$
$$= (1 + 2t)\mathbf{i} + (3 - t)\mathbf{j}$$

Using $\mathbf{r}_B = \mathbf{r}_{B_0} + \mathbf{v}_B t$ for **B**,

$$\mathbf{r}_B = (5\mathbf{i} - 2\,\mathbf{j}) + (-\mathbf{i} + 4\mathbf{j}) \times t$$
$$= (5 - t)\mathbf{i} + (-2 + 4t)\mathbf{j}$$

b $$\mathbf{r}_{\overline{AB}} = \mathbf{r}_B - \mathbf{r}_A$$
$$= \big((5-t)\mathbf{i} + (-2+4t)\mathbf{j}\big) - \big((1+2t)\mathbf{i} + (3-t)\mathbf{j}\big)$$
$$= (5 - t - 1 - 2t)\mathbf{i} + (-2 + 4t - 3 + t)\mathbf{j}$$
$$= (4 - 3t)\mathbf{i} + (-5 + 5t)\mathbf{j}$$

c If A and B collide, the vector $\boldsymbol{AB}$ would be zero, so $4 - 3t = 0$ and $-5 + 5t = 0$, but these two equations are not consistent $(t = 1 \text{ and } t \neq 1)$, so vector $\boldsymbol{AB}$ can never be zero and A and B will not collide.

d At 10 am, $t = 2$:

$$\mathbf{r}_{\overline{AB}} = (4 - 3 \times 2)\,\mathbf{i} + (-5 + 5 \times 2)\,\mathbf{j}$$
$$= -2\,\mathbf{i} + 5\,\mathbf{j}$$

$$\text{Distance} = \sqrt{(-2)^2 + 5^2}$$
$$= \sqrt{29}$$
$$= 5.39\,\text{km}$$

5 Let x be the horizontal distance between O and S, and y be the vertical distance between O and S.
$\mathbf{u} = 8\mathbf{i} + 10\mathbf{j},\ \mathbf{a} = -9.8\mathbf{j},\ t = 6,\ \mathbf{s} = x\mathbf{i} + y\mathbf{j}$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$x\mathbf{i} + y\mathbf{j} = (8\mathbf{i} + 10\mathbf{j}) \times 6 + \frac{1}{2}(-9.8\mathbf{j}) \times 36$$

a Equating **i** components:
$x = 48$
The horizontal distance between O and S is 48 m.

b Equating **j** components:
$y = 60 - 4.9 \times 36$
$= -116$
The vertical distance between O and S is 116 m (3 s.f.).

6 $\mathbf{u} = (p\mathbf{i} + q\mathbf{j})\ \mathrm{ms}^{-1},\ \mathbf{r}_0 = 0.8\mathbf{j}\ \mathrm{m},\ \mathbf{a} = -9.8\mathbf{j}\ \mathrm{ms}^{-2};\ t = 4\ \mathrm{s},\ \mathbf{r} = 64\mathbf{i}\ \mathrm{m}$

a Combining $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ and $\mathbf{s} = \mathbf{r} - \mathbf{r}_0$ gives

$$\mathbf{r} - \mathbf{r}_0 = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

Using vector notation:

$$\begin{pmatrix} 64 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0.8 \end{pmatrix} = 4\begin{pmatrix} p \\ q \end{pmatrix} + \frac{4^2}{2}\begin{pmatrix} 0 \\ -9.8 \end{pmatrix}$$

$$\begin{pmatrix} 64 \\ -0.8 \end{pmatrix} = \begin{pmatrix} 4p \\ 4q - 78.4 \end{pmatrix}$$

Considering **i** components:
$64 = 4p$
$p = 16$
Considering **j** components:
$-0.8 = 4q - 78.4$
$4q = 78.4 - 0.8$
$q = 19.6 - 0.2 = 19.4$

The values of p and q are 16 and 19.4 respectively. $\mathbf{s} = \begin{pmatrix} 16 \\ 19.4 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4.9 \end{pmatrix} t^2$

b $|\mathbf{u}| = \sqrt{16^2 + 19.4^2} = 25.146...$
The initial speed of the ball is 25.1 ms^{-1}.

c $\tan\alpha = \dfrac{q}{p} = \dfrac{19.4}{16}$

The exact value of $\tan\alpha$ is $\dfrac{97}{80}$

6 d Use $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$ and $\mathbf{s} = \mathbf{r} - \mathbf{r}_0$ to find values of t for which $\mathbf{r} = x\mathbf{i}+5\mathbf{j}$:

$$\begin{pmatrix} x \\ 5 \end{pmatrix} - \begin{pmatrix} x \\ 0.8 \end{pmatrix} = \begin{pmatrix} 16 \\ 19.4 \end{pmatrix} t + \begin{pmatrix} 0 \\ -4.9 \end{pmatrix} t^2$$

Considering $\mathbf{j}$ components:

$$5 - 0.8 = 19.4t - 4.9t^2$$

$$4.9t^2 - 19.4t + 4.2 = 0$$

Using the equation for the roots of a quadratic equation:

$$t = \frac{19.4 \pm \sqrt{19.4^2 - (4 \times 5 \times 4.2)}}{2 \times 4.9}$$

$t = 3.7243...$ or $t = 0.2348...$

The ball is above 5 m between these two times, i.e. for 3.7243… − 0.2348…= 3.50 s (3s.f.).

e To make the model more realistic, one should consider factors such as air resistance and how it is affected by the shape (especially the seam) and the spin of the ball.

7 a $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int t\left(2t^2 + 14\right)^{\frac{1}{2}}\mathrm{d}t$

$$= \frac{2}{3 \times 2 \times 2}\left(2t^2 + 14\right)^{\frac{3}{2}} + c$$

$$= \frac{1}{6}\left(2t^2 + 14\right)^{\frac{3}{2}} + c$$

$\mathbf{r} = 0$ when $t = 0$, hence

$$0 = \frac{1}{6}(0 + 14)^{\frac{3}{2}} + c$$

$c = -8.73$

$$\Rightarrow \mathbf{r} = \frac{1}{6}\left(2t^2 + 14\right)^{\frac{3}{2}} - 8.73$$

At $t = 5$ s, $\mathbf{r} = \frac{1}{6}(50 + 14)^{\frac{3}{2}} + 8.73 = 76.6$

At $t = 5$ s, the displacement of P from O is 76.6 m (3s.f.).

b $\mathbf{v} = \frac{1000}{t^2}$ ms^{-1}, $t = 5$ s, $\mathbf{r} = 76.6$ m; $t = 6$ s, $\mathbf{r} = ?$

$\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int 1000t^{-2}\mathrm{d}t$

$$= -\frac{1000}{t} + c$$

Using that fact that at $t = 5$ s the position of the particle will be as given in part **a**:

$$76.6 = \frac{-1000}{5} + c$$

$$c = 76.6 + 200 = 276.6$$

$$\Rightarrow \mathbf{r} = \frac{-1000}{t} + 276.6$$

At $t = 6$ s,

$$\mathbf{r} = \frac{-1000}{6} + 276.6 = 109.9$$

At $t = 6$ s, the displacement of P from O is 110 m (3s.f.).

8 a $x = 2t + k(t+1)^{-1}$

$v = \dfrac{dx}{dt} = 2 - k(t+1)^{-2} = 2 - \dfrac{k}{(t+1)^2}$

When $t = 0$, $v = 6$

$6 = 2 - \dfrac{k}{1^2} \Rightarrow k = -4$

b With $k = -4$

$x = 2t - \dfrac{4}{t+1}$

When $t = 0$, $x = 0 - \dfrac{4}{0+1} = -4$

The distance of P from O when $t = 0$ is 4 m.

c $v = 2 - 4(t+1)^{-2}$

$a = \dfrac{dv}{dt} = 8(t+1)^{-3} = \dfrac{8}{(t+1)^3}$

When $t = 3$

$a = \dfrac{8}{4^3} = \dfrac{1}{8}$

$F = ma$

$= 0.4 \times \dfrac{1}{8} = 0.05$

The magnitude of **F** when $t = 3$ is 0.05.

9 a When $t = \dfrac{1}{2}$

$x = 0.6\cos\left(\dfrac{\pi}{3} \times \dfrac{1}{2}\right)$

$= 0.6\cos\dfrac{\pi}{6}$

$= 0.6 \times \dfrac{\sqrt{3}}{2} = 0.3\sqrt{3}$

The distance of B from O when $t = \dfrac{1}{2}$ is $0.3\sqrt{3}$ m.

b $v = \dfrac{dx}{dt} = -0.6 \times \dfrac{\pi}{3}\sin\left(\dfrac{\pi t}{3}\right)$

The smallest value at which $v = 0$ is given by

$\dfrac{\pi t}{3} = \pi \Rightarrow t = 3$ s.

9 c $a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -0.6\left(\dfrac{\pi}{3}\right)^2 \cos\left(\dfrac{\pi t}{3}\right)$

When $t = 1$

$a = -0.6\left(\dfrac{\pi}{3}\right)^2 \cos\left(\dfrac{\pi}{3}\right) = -0.3289...$

The magnitude of the acceleration of B when $t = 1$ is $0.329\,\mathrm{m\,s^{-2}}$ (3 s.f.).

10 a $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = 4\mathrm{e}^{-0.5t} - 2t\mathrm{e}^{-0.5t}$

$a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = -2\mathrm{e}^{-0.5t} - 2\mathrm{e}^{-0.5t} + t\mathrm{e}^{-0.5t} = (t-4)\mathrm{e}^{-0.5t}$

When $t = \ln 4$

$$\begin{aligned} a &= (\ln 4 - 4)\mathrm{e}^{-0.5\ln 4} \\ &= (\ln 2^2 - 4)\mathrm{e}^{\ln 4^{-\frac{1}{2}}} \\ &= (2\ln 2 - 4)\mathrm{e}^{\ln\frac{1}{2}} \\ &= \frac{1}{2}(2\ln 2 - 4) \\ &= \ln 2 - 2 \end{aligned}$$

The acceleration of S when $t = \ln 4$ is $(\ln 2 - 2)\,\mathrm{m\,s^{-2}}$ in the direction of x increasing.

b For a maximum of x, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = v = 0$

$v = (4 - 2t)\mathrm{e}^{-0.5t} = 0 \Rightarrow t = 2$

When $t = 2$

$x = 4 \times 2\mathrm{e}^{-0.5\times 2} = 8\mathrm{e}^{-1}$

The greatest distance of S from O is $\dfrac{8}{\mathrm{e}}$ m.

11 a $\mathbf{v}_P = \dot{\mathbf{r}}_P = 6t\mathbf{i} + 2\mathbf{j}$

$\mathbf{v}_Q = \dot{\mathbf{r}}_Q = \mathbf{i} + 3t\mathbf{j}$

$\boxed{\dfrac{\mathrm{d}}{\mathrm{d}t}\big((t+6)\mathbf{i}\big) = 1\mathbf{i} = \mathbf{i}}$

The velocity of P at time t seconds is $(6t\mathbf{i} + 2\mathbf{j})\,\mathrm{m\,s^{-1}}$ and the velocity of Q is $(\mathbf{i} + 3t\mathbf{j})\,\mathrm{m\,s^{-1}}$

b When $t = 2$

$\mathbf{v}_P = 12\mathbf{i} + 2\mathbf{j}$

$|\mathbf{v}_P|^2 = 12^2 + 2^2 = 148 \Rightarrow \mathbf{v}_P = \sqrt{148} = 12.165...$

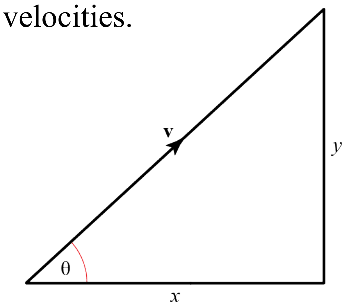
The speed of P when $t = 2$ is $12.2\,\mathrm{m\,s^{-1}}$ (3 s.f.).

11 c When P is moving parallel to Q

$$\frac{2}{6t} = \frac{3t}{1}$$
$$\Rightarrow 18t^2 = 2$$
$$\Rightarrow t^2 = \frac{1}{9}$$
$$t \geqslant 0,\ t = \frac{1}{3}$$

When the particles are moving parallel to each other, the angle each makes with **i** is the same.

If $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$, $\tan\theta = \frac{y}{x}$ must be the same for both velocities.



d **i**-components

$$3t^2 + 4 = t + 6$$
$$3t^2 - t - 2 = 0$$
$$(t-1)(3t+2) = 0$$
$$t = 1,\ -\frac{2}{3}$$

j-components

$$2t - \frac{1}{2} = \frac{3t^2}{2}$$
$$3t^2 - 4t + 1 = 0$$
$$(t-1)(3t-1) = 0$$
$$t = 1,\ \frac{1}{3}$$

For the particles to collide, both the **i** and **j** components of their position vectors must be the same for the same value of t. The appropriate method is to equate the **i** components and solve the resulting quadratic, and then do the same for **j** components. If one of the roots of the quadratics is the same, then the particles collide.

1 is a common root of the equations and, hence, P and Q collide at the point with position vector $\left(7\mathbf{i} + \frac{3}{2}\mathbf{j}\right)$ m.

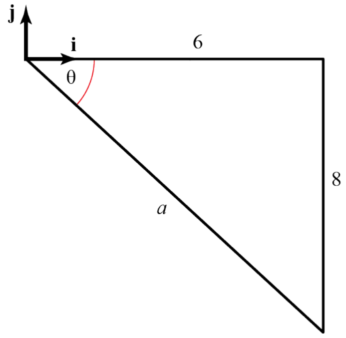
$t = 1$ can be substituted into either $\mathbf{r}_P$ or $\mathbf{r}_Q$ to find the position vector of the point where the particles collide.

12 a $\mathbf{v} = \dot{\mathbf{r}} = 6t\,\mathbf{i} - 8t\,\mathbf{j}$

$\mathbf{a} = \dot{\mathbf{v}} = 6\,\mathbf{i} - 8\,\mathbf{j}$

Acceleration does not depend on t, hence the acceleration is constant.

12 b



$$|\mathbf{a}|^2 = 6^2 + (-8)^2 = 100$$
$$\Rightarrow |\mathbf{a}| = 10$$

The magnitude of the acceleration is $10\,\mathrm{m\,s^{-2}}$

$$\tan\theta = \frac{8}{6} \Rightarrow \theta = 53.1°$$

The angle the acceleration makes with **j** is $90° + 53.1° = 143.1°$ (nearest 0.1°)

13 a $\mathbf{v} = \dot{\mathbf{r}} = -6\sin 3t\,\mathbf{i} - 6\cos 3t\,\mathbf{j}$

When $t = \frac{\pi}{6}$

$$\mathbf{v} = -6\sin\frac{\pi}{2}\mathbf{i} - 6\cos\frac{\pi}{2}\mathbf{j}$$
$$= -6\mathbf{i} - 0$$

The velocity of P when $t = \frac{\pi}{6}$ is $-6\mathbf{i}\ \mathrm{m\,s^{-1}}$

b $\mathbf{a} = \dot{\mathbf{v}} = -18\cos 3t\,\mathbf{i} + 18\sin 3t\,\mathbf{j}$

$$|\mathbf{a}|^2 = \left(-18\cos 3t\right)^2 + \left(18\sin 3t\right)^2$$
$$= 18^2\left(\cos^2 3t + \sin^2 3t\right) = 18^2$$
$$|\mathbf{a}| = 18$$

The magnitude of the acceleration is $18\,\mathrm{m\,s^{-2}}$, which is constant.

14 a $\mathbf{a} = \dot{\mathbf{v}} = 4c\,\mathbf{i} + 2(7-c)t\,\mathbf{j}$

$$\mathbf{F} = m\mathbf{a}$$
$$= 0.5\left(4c\,\mathbf{i} + 2(7-c)t\,\mathbf{j}\right)$$
$$= 2c\,\mathbf{i} + (7-c)t\,\mathbf{j}, \text{ as required}$$

b $t = 5 \Rightarrow \mathbf{F} = 2c\,\mathbf{i} + 5(7-c)\,\mathbf{j}$

$$|\mathbf{F}|^2 = 4c^2 + 25(7-c)^2 = 17^2$$
$$4c^2 + 1225 - 350c + 25c^2 = 289$$
$$29c^2 - 350c + 936 = 0$$
$$(c-4)(29c-234) = 0$$
$$c = 4,\ \frac{234}{29} \approx 8.07$$

15 a $\mathbf{v}=\int \mathbf{a}\mathrm{d}t=\int((8t^3-6t)\mathbf{i}+(8t-3)\mathbf{j})\mathrm{d}t$

$=(2t^4-3t^2)\mathbf{i}+(4t^2-3t)\mathbf{j}+C$

When $t=2$, $\mathbf{v}=16\mathbf{i}+3\mathbf{j}$

$16\mathbf{i}+3\mathbf{j}=20\mathbf{i}+10\mathbf{j}+C\Rightarrow C=-4\mathbf{i}-7\mathbf{j}$

$\mathbf{v}=(2t^4-3t^2-4)\mathbf{i}+(4t^2-3t-7)\mathbf{j}$

The velocity of P after t seconds is $\left((2t^4-3t^2-4)\mathbf{i}+(4t^2-3t-7)\mathbf{j}\right)\mathrm{m\,s}^{-1}$

b When P is moving parallel to **i**, the **j** component of the velocity is zero.

$4t^2-3t-7=0$

$(t+1)(4t-7)=0$

$t\geqslant 0\Rightarrow t=\frac{7}{4}$

16 $\mathbf{a}=\left(4t\mathbf{i}+5t^{-\frac{1}{2}}\mathbf{j}\right)$ ms^{-2}, $t=0$ s, $\mathbf{v}=10\mathbf{i}$ ms^{-1}; $t=5$ s, $|\mathbf{v}|=?$

$\mathbf{v}=\int \mathbf{a}\,\mathrm{d}t=\int\left(4t\mathbf{i}+5t^{-\frac{1}{2}}\mathbf{j}\right)\mathrm{d}t$

$=\frac{4t^2}{2}\mathbf{i}+\frac{5}{\frac{1}{2}}t^{\frac{1}{2}}\mathbf{j}+c$

$=2t^2\mathbf{i}+10t^{\frac{1}{2}}\mathbf{j}+c$

When $t=0$ s, $\mathbf{v}=10\,\mathbf{i}$ ms^{-1}

$10\mathbf{i}=0\mathbf{i}-0\mathbf{j}+c$

$c=10\mathbf{i}$

$\Rightarrow \mathbf{v}=(2t^2+10)\mathbf{i}+10t^{\frac{1}{2}}\mathbf{j}$

At $t=5$ s,

$\mathbf{v}=(50+10)\mathbf{i}+10\sqrt{5}\mathbf{j}$

$|\mathbf{v}|=\sqrt{60^2+\left(10\sqrt{5}\right)^2}=\sqrt{4100}$

$|\mathbf{v}|=10\sqrt{41}$

At $t=5$ s, the speed of the ball is $10\sqrt{41}$ ms^{-1}.

17 a $\mathbf{v}=\int \mathbf{a}\,\mathrm{d}t=\int 2t\,\mathbf{i}+3\mathbf{j}\,\mathrm{d}t$

$=t^2\mathbf{i}+3t\mathbf{j}+c$

When $t=0$ s, $v=3\mathbf{i}+13\mathbf{j}$

$3\mathbf{i}+13\mathbf{j}=0\mathbf{i}+0\mathbf{j}+c$

$c=3\mathbf{i}+13\mathbf{j}$

$\mathbf{v}=\left(t^2+3\right)\mathbf{i}+\left(3t+13\right)\mathbf{j}$

17 b When the train is moving NE, the coefficients of the **i** and **j** components are equal and positive.

$$t^2 + 3 = 3t + 13$$
$$t^2 - 3t - 10 = 0$$
$$(t-5)(t+2) = 0$$
$$t = 5, \ -2$$

Ignoring the negative root, as it denotes a time before the train was moving, the train is moving NE at $t = 5$ s (3s.f.).

Challenge

1 a $s(0) = 20$ m

b $$\frac{\mathrm{d}s}{\mathrm{d}t} = (20 - t^2) \times \frac{1}{2}(t+1)^{-\frac{1}{2}} - 2t(t+1)^{\frac{1}{2}}$$
$$= \frac{(20 - t^2) - 4t(t+1)}{2(t+1)^{\frac{1}{2}}}$$
$$= \frac{20 - 4t - 5t^2}{2\sqrt{t+1}}$$

Particle changes direction when $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 0 \Rightarrow$

$$20 - 4t - 5t^2 = 0$$
$t = 1.64$ s (ignoring negative root, since $t \geqslant 0$)

So particle changes direction exactly once, when $t = 1.64$ s

c Particle crosses O when $s = 0$

$$0 = (20 - t^2)\sqrt{t+1}$$
$$t = \sqrt{20}$$

At $t = \sqrt{20}$ s, $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{20 - 4\sqrt{20} - 5 \times 20}{2\sqrt{\sqrt{20}+1}}$

$$= \frac{-40 - 2\sqrt{20}}{\sqrt{\sqrt{20}+1}}$$
$$= -2\sqrt{20}\left(\sqrt{20}+1\right)^{\frac{1}{2}}$$

Challenge

2 a $\mathbf{v} = \dot{\mathbf{r}} = (6\omega\cos\omega t)\mathbf{i} - (4\omega\sin\omega t)\mathbf{j}$

$v^2 = |\mathbf{v}|^2 = 36\omega^2\cos^2\omega t + 16\omega^2\sin^2\omega t$

$= 36\omega^2\left(\frac{1}{2} + \frac{1}{2}\cos 2\omega t\right) + 16\omega^2\left(\frac{1}{2} - \frac{1}{2}\cos 2\omega t\right)$

$= 18\omega^2 + 18\omega^2\cos 2\omega t + 8\omega^2 - 8\omega^2\cos 2\omega t$

$= 26\omega^2 + 10\omega^2\cos 2\omega t$

$= 2\omega^2(13 + 5\cos 2\omega t)$, as required.

Use the double angle formulae $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 2\theta = 1 - 2\sin^2\theta$

b As $-1 \leqslant \cos 2\omega t \leqslant 1$

$2\omega^2(13-5) \leqslant 2\omega^2(13 + 5\cos 2\omega t) \leqslant 2\omega^2(13+5)$

$16\omega^2 \leqslant v^2 \leqslant 36\omega^2$

As $v > 0$ and $\omega > 0$, we can take the square root of each term and it will not change the inequality signs:

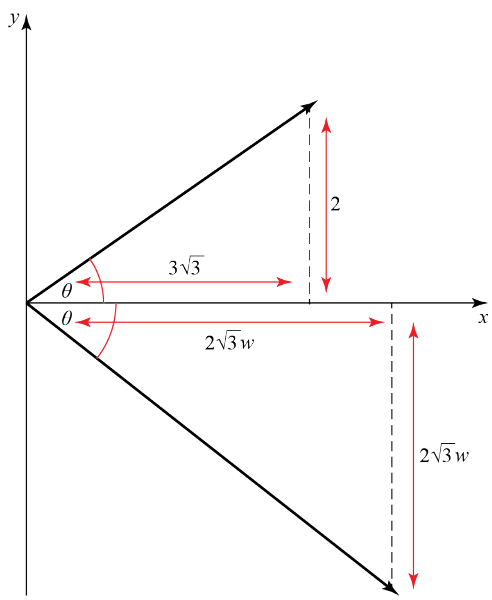
$4\omega \leqslant v \leqslant 6\omega$, as required.

c When $t = \dfrac{\pi}{3\omega}$

$\mathbf{r} = \left(6\sin\frac{\pi}{3}\right)\mathbf{i} + \left(4\cos\frac{\pi}{3}\right)\mathbf{j} = 3\sqrt{3}\,\mathbf{i} + 2\,\mathbf{j}$

$\dot{\mathbf{r}} = \left(6\omega\cos\frac{\pi}{3}\right)\mathbf{i} - \left(4\omega\sin\frac{\pi}{3}\right)\mathbf{j} = 3\omega\,\mathbf{i} - 2\sqrt{3}\omega\,\mathbf{j}$

Using $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$



A diagram is essential here. Once the diagram has been drawn, the problem reduces to basic trigonometry. You find the angles using the inverse tangent button on your calculator.

$\tan\theta = \dfrac{2}{3\sqrt{3}} \Rightarrow \theta = 0.3674...^c$

$\tan\phi = \dfrac{2\sqrt{3}\omega}{3\omega} = \dfrac{2\sqrt{3}}{3} \Rightarrow \phi = 0.8570...^c$

The angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ is

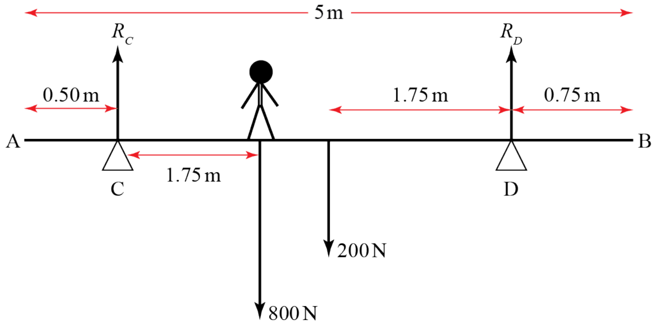
$\theta + \phi = 0.3674...^c + 0.8570...^c = 1.224...^c = 70.2°$ (3 s.f.)

Review exercise 2

1 a Let the reactions at C and D be R_C and R_D respectively.
Since the plank is in equilibrium, taking moments about D:
$(200 \times 1.75) + (800 \times 2.25) = R_C \times 3.75$

$$R_C = \frac{350 + 1800}{3.75} = 573.33...$$

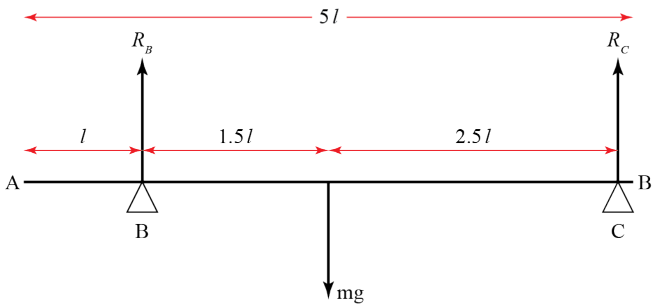
The reaction at C is 573 N (3 s.f.).



b Resolving vertically:
$$R_C + R_D = 200 + 800$$
$$R_D = 1000 - 573.33...$$
$$R_D = 426.66...$$
The reaction at C is 427 N (3 s.f.).

c By modelling the builder as a particle, we can assume all weight acts from a single point at his centre of gravity.

2 a Let the reactions at B and C be R_B and R_C respectively.
Since plank is in equilibrium, taking moments about C:
$$mg \times \frac{5}{2} l = R_B \times 4l$$
$$R_B = \frac{5mg}{2 \times 4}$$
$$R_B = \frac{5mg}{8} \text{ as required.}$$

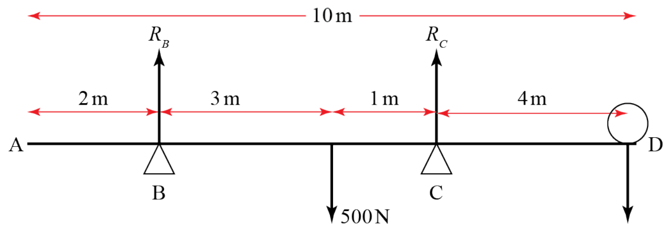


b Resolving vertically:
$$R_B + R_C = mg$$
$$\frac{5}{8}mg + R_C = mg \qquad \left(\text{Using } R_B = \frac{5mg}{8} \text{ from } \mathbf{a}\right)$$
$$R_c = mg\left(1 - \frac{5}{8}\right)$$
$$R_C = \frac{3mg}{8} \text{ as required.}$$

c i Assuming the plank is uniform allows us to assume the weight acts from its midpoint.

ii By assuming the plank is a rod, we ignore its width.

3 Let the reactions at B and C be R_B and R_C respectively.



a Let the weight placed at D be W.
As W is increased, the rod will begin to tip about C when $R_B = 0$.
Then, taking moments about C:

$$W \times 4 = 500 \times 1$$
$$W = 125$$

The largest weight that can be placed at D before the rod tips is 125 N.

b Let the weight now placed at A be W.
As W is increased, the rod will begin to tip about B when $R_C = 0$.
Then, taking moments about B:

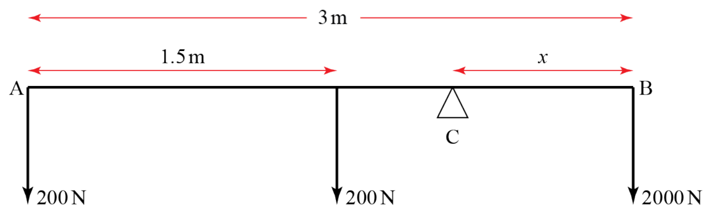
$$W \times 2 = 500 \times 3$$
$$W = 750$$

The largest weight that can be placed at A before the rod tips is 750 N.

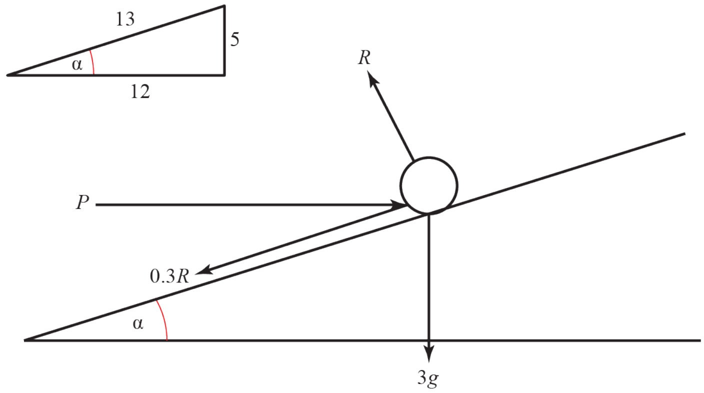
4 Let $CB = x$ m
Taking moments about C, since lever is in equilibrium:

$$2000x = 200(3 - x) + 200(1.5 - x)$$
$$2000x = (200 \times 4.5) - 400x$$
$$2400x = 900$$
$$x = 0.375$$

The length CB is 0.375 m.



5 Since the particle is moving at constant velocity, the forces acting on it are balanced.



$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$

R(↖):

$R = 3g\cos\alpha + P\sin\alpha$

$R = \frac{3g\times 12}{13} + \frac{5P}{13}$

$R = \frac{36g + 5P}{13}$

R(↗):

$P\cos\alpha = \mu R + 3g\sin\alpha$

$\frac{12}{13}P = \frac{1}{5}\left(\frac{36g+5P}{13}\right) + \frac{3g\times 5}{13}$

$12P = \frac{36g}{5} + P + 15g$

$11P = \left(\frac{36}{5} + 15\right)g$

$P = \left(\frac{36}{5} + 15\right)\times\frac{9.8}{11}$

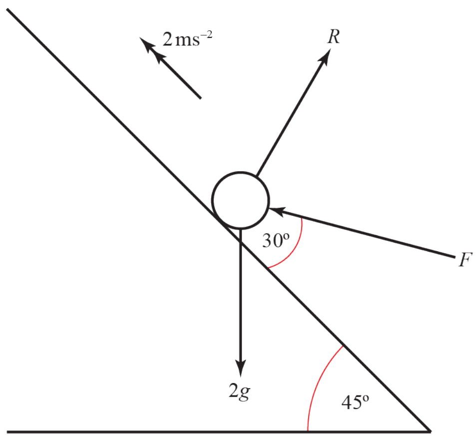
$= 19.778...$

P is 19.8 N (to 3s.f.).

6 $m = 2$ kg, $a = 2\ \text{ms}^{-2}$

Using Newton's second law of motion and resolving up the slope:

$$F = ma$$



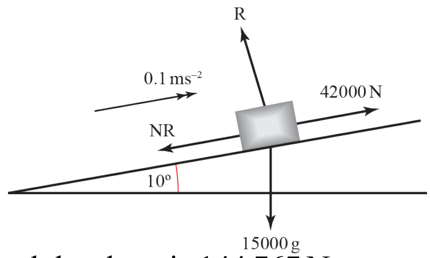
$F\cos 30^\circ - 2g\sin 45^\circ = 2\times 2$

$\frac{\sqrt{3}}{2}F - \frac{2g}{\sqrt{2}} = 4$

$\frac{\sqrt{3}}{2}F = 4 + \sqrt{2}g$

$F = \frac{2}{\sqrt{3}}\left(4 + \sqrt{2}g\right)$ as required.

7 $m = 15\,000$ kg, $a = 0.1$ ms^{-2}



a R(↖) :

$R = 15\,000g\cos 10^\circ$

$R = 15\,000 \times 9.8\cos 10^\circ = 144\,767$

To the nearest whole newton, the reaction between the container and the slope is 144 767 N.

b Using Newton's second law of motion and resolving up the slope:

$$F = ma$$

$$42\,000 - \mu R - 15\,000g\sin 10^\circ = 15\,000 \times 0.1$$

$$\mu \times 144\,767 = 42\,000 - 1500 - \left(15\,000 \times 9.8\sin 10^\circ\right)$$

$$\mu = \frac{40\,500 - 25\,526.2}{144\,767}$$

$$= 0.103\,433...$$

The coefficient of friction between the container and the slope is 0.103 (3s.f.).

c Using Newton's second law of motion and resolving down the slope after winch stops working:

$$F = ma$$

$$\mu R + 15\,000g\sin 10^\circ = 15\,000a$$

$$144\,767 \times 0.103\,433 + 15\,000g\sin 10^\circ = 15\,000a \quad \text{(using results from } \mathbf{a} \text{ and } \mathbf{b}\text{)}$$

$$a = \frac{40\,500}{15\,000}$$

$$= 2.7$$

So the container accelerates down the slope at 2.7 ms^{-2}

So: $u = -2$ ms^{-1}, $v = 0$ ms^{-1}, $a = 2.7$ ms^{-2}, $t = ?$

$v = u + at$

$0 = -2 + 2.7t$

$t = \dfrac{2}{2.7}$

$= 0.74074...$

The container takes 0.740 s (3s.f.) to come to rest.

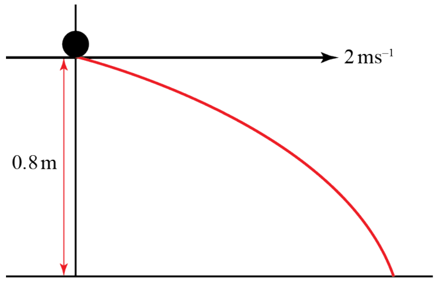
d Once the container comes to rest, the container will tend to move down the slope and hence the frictional force will act up the slope. The container will therefore move back down if the component of weight down the slope is greater than the frictional force; i.e. if

$mg\sin 10^\circ > \mu R$

$15\,000g\sin 10^\circ > 144\,767 \times 0.103\,433$

$25\,526 > 14\,974$

Since this inequality is true, the container will start to slide back down the slope.



8 a $R(\downarrow)$:

$s = 0.8$ m, $u = 0$ ms^{-1}, $a = 9.8$ ms^{-2}, $t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0.8 = 0 + \frac{9.8}{2}t^2$$

$$t = \sqrt{\frac{0.8}{4.9}} = 0.40406...$$

The ball reaches the ground after 0.404 s (3s.f.).

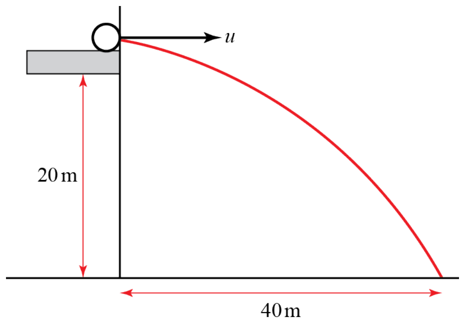
b $R(\rightarrow)$:

$v = 2$ ms^{-1}, $t = 0.404$ s, $s = ?$

$$s = vt$$

$$s = 2 \times 0.40406... = 0.80812...$$

The ball lands 0.808 m from the table edge (3s.f.).



9 a First resolve vertically to find time of flight, then resolve horizontally to find initial velocity.

$R(\downarrow)$:

$s = 20$ m, $u = 0$ ms^{-1}, $a = 9.8$ ms^{-2}, $t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$20 = 0 + \frac{9.8}{2}t^2$$

$$t = \sqrt{\frac{20}{4.9}} = \frac{10\sqrt{2}}{7}$$

$R(\rightarrow)$:

$t = \dfrac{10\sqrt{2}}{7}$ s, $s = 40.0$ m, $u = ?$

$$s = vt$$

$$40 = u \times \frac{10\sqrt{2}}{7}$$

$$u = \frac{4 \times 7}{\sqrt{2}} = 19.798...$$

The initial horizontal speed of the ball is 19.8 ms^{-1} (3s.f.).

b Assumptions made are that the ball behaves as a particle (i.e. that there is negligible air resistance) and that the acceleration due to gravity remains constant over the distance fallen.

10 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = 150\cos 10°$

$R(\uparrow)\ u_y = 150\sin 10°$

a $R(\uparrow)$

$u = 150\sin 10°,\ v = 0,\ a = -9.8,\ t = ?$

$v = u + at$

$0 = 150\sin 10° - 9.8t$

$$t = \frac{150\sin 10°}{9.8} = 2.657$$

The time taken to reach the projectile's highest point is 2.7 s (2 s.f.).

b By symmetry, the time of flight is $(2.657... \times 2)\text{s} = 5.315\text{s}$

[Note that you could also find the time of flight by resolving vertically with $s = 0$, but since you have already found *half* the time of flight in part **a**, it is simpler just to double this.]

Now find the range by resolving horizontally:

$R(\rightarrow)$:

$u = 150\cos 10°,\ t = 5.315,\ s = ?$

$s = ut$

$= 150\cos 10° \times 5.315$

$= 785.250$

The range of the projectile is 790 m (2 s.f.).

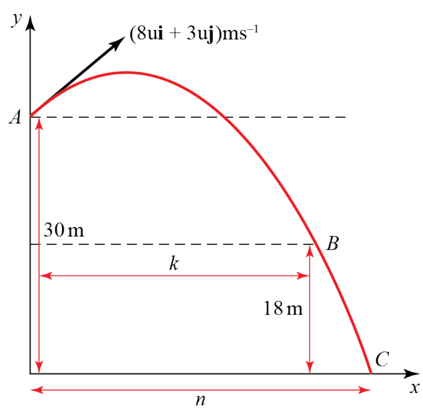
11 $\mathbf{u} = (8u\mathbf{i} + 3u\mathbf{j})\ \text{ms}^{-1},\ \mathbf{a} = -9.8\mathbf{j}\ \text{ms}^{-2},\ t = 3,\ \mathbf{s} = (k\mathbf{i} + 18\mathbf{j}) - 30\mathbf{j}$

$= (k\mathbf{i} - 12\mathbf{j})$

$$\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$k\mathbf{i} - 12\mathbf{j} = 3(8u\mathbf{i} + 3u\mathbf{j}) - \left(\frac{9}{2} \times 9.8\mathbf{j}\right)$$

$$k\mathbf{i} - 12\mathbf{j} = 24u\mathbf{i} + (9u - 44.1)\mathbf{j}$$



a Considering **j** components only:

$-12 = 9u - 44.1$

$$u = \frac{44.1 - 12}{9} = 3.5666...$$

The value of u is 3.6 (to 2s.f.).

b Considering **i** components only:

$k = 24u$

$k = 24 \times 3.567 = 85.6$

The value of k is 86 (to 2s.f.).

11 c At C, let $\mathbf{v} = \left(v_x\mathbf{i} - v_y\mathbf{j}\right)$ ms^{-1}

Now $v_x = 8 \times 3.567 = 28.533$ since there is no horizontal acceleration.

Considering the **j** components only:
$u = 3 \times 3.567,\ v = v_y,\ s = -30,\ a = -9.8$

$v^2 = u^2 + 2as$

$v_y{}^2 = \left(3 \times 3.6\right)^2 + \left(2 \times \left(-9.8\right) \times \left(-30\right)\right)$

$v_y = \sqrt{114.49 + 588}$

$v_y = 26.504$

$\tan\alpha = \dfrac{v_y}{v_x}$

$\tan\alpha = \dfrac{26.504}{28.533}$

$\alpha = 42.888$

At C, the velocity of P makes an angle of 43° (2s.f.) with the x-axis.

12 a Then $\mathbf{u} = \left(12\mathbf{i} + 24\mathbf{j}\right),\ \mathbf{a} = -9.8\mathbf{j},\ t = 3,\ \mathbf{s} = ?$

$$\begin{aligned}\mathbf{s} &= \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2 \\ &= 3 \times \left(12\mathbf{i} + 24\mathbf{j}\right) - 3^2 \times 4.9\mathbf{j} \\ &= 36\mathbf{i} + 27.9\mathbf{j}\end{aligned}$$

The position vector of P after 3 s is $(36\mathbf{i} + 27.9\mathbf{j})$ m

b $\mathbf{u} = \left(12\mathbf{i} + 24\mathbf{j}\right),\ \mathbf{a} = -9.8\mathbf{j},\ t = 3,\ \mathbf{v} = ?$

$$\begin{aligned}\mathbf{v} &= \mathbf{u} + \mathbf{a}t \\ &= \left(12\mathbf{i} + 24\mathbf{j}\right) - 3 \times 9.8\mathbf{j} \\ &= 12\mathbf{i} - 5.4\mathbf{j}\end{aligned}$$

Let the speed of P after 3 s be V m s^{-1}

$$\begin{aligned}V^2 &= 12^2 + (-5.4)^2 \\ &= 173.16 \\ V &= \sqrt{173.16} \\ &= 13.159\end{aligned}$$

The speed of P after 3 s is 13 m s^{-1} (2 s.f.).

13 Resolving the initial velocity horizontally and vertically

$R(\rightarrow)\ u_x = u\cos\alpha$

$R(\uparrow)\ u_y = u\sin\alpha$

a $R(\uparrow)$:

$u_y = u\sin\alpha,\ s = 0,\ a = -g,\ t = ?$

$$s = ut + \frac{1}{2}at^2$$

$$0 = u\sin\alpha t - \frac{1}{2}gt^2$$

$$0 = t(u\sin\alpha - \frac{1}{2}gt) \qquad (t = 0 \text{ corresponds to the point of projection})$$

$$\frac{1}{2}gt = u\sin\alpha$$

$$\Rightarrow t = \frac{2u\sin\alpha}{g}, \text{ as required}$$

b $R(\rightarrow)$:

$u_x = u\cos\alpha,\ t = \dfrac{2u\sin\alpha}{g},\ s = ?$

$$s = u\cos\alpha \times \frac{2u\sin\alpha}{g}$$

$$= \frac{u^2 \times 2\sin\alpha\cos\alpha}{g}$$

$$= \frac{u^2\sin 2\alpha}{g} \qquad (\text{Using } \sin 2\alpha = 2\sin\alpha\cos\alpha)$$

$$\therefore R = \frac{u^2\sin 2\alpha}{g}, \text{ as required}$$

c The greatest possible value of $\sin 2\alpha$ is 1, which is when $2\alpha = 90° \Rightarrow \alpha = 45°$.
Hence, for a fixed u, the greatest possible range is when $\alpha = 45°$.

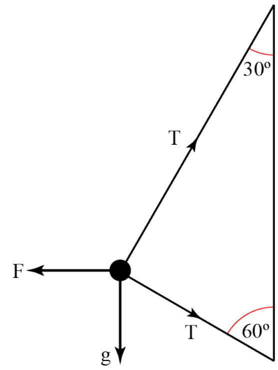
d $\dfrac{2u^2}{5g} = \dfrac{u^2\sin 2\alpha}{g} \Rightarrow \sin 2\alpha = \dfrac{2}{5}$

$2\alpha = 23.578°, 156.422°$

$\alpha = 11.79°,\ 78.21°$

The two possible angles of elevation are
$12°$ and $78°$ (nearest degree).

14 The system is in equilibrium.



a Resolving vertically:

$T\cos 60^\circ + g = T\cos 30^\circ$

$$\frac{T}{2} + g = \frac{T\sqrt{3}}{2}$$

$$2g = T\left(\sqrt{3}-1\right)$$

$$T = \frac{2g}{\sqrt{3}-1} \quad \text{as required.}$$

b Resolving horizontally:

$F = T\sin 60^\circ + T\sin 30^\circ$

$F = T\left(\sin 60^\circ + \sin 30^\circ\right)$

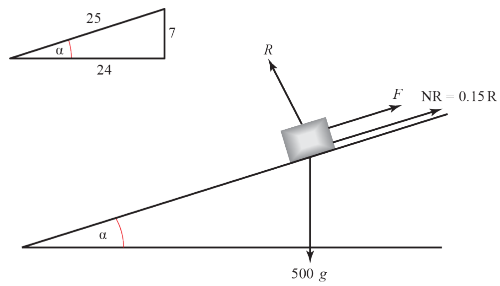
$$F = \left(\frac{2g}{\sqrt{3}-1}\right)\left(\frac{\sqrt{3}+1}{2}\right)$$

$$F = \left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)g \quad = 36.6 \text{ N (3 s.f.)}$$

c We model the bead as smooth in order to assume there is no friction between it and the string.

15 $\tan\alpha = \frac{7}{24} \Rightarrow \sin\alpha = \frac{7}{25}$ and $\cos\alpha = \frac{24}{25}$

The system is in equilibrium.



a R(↖):

$R = 500g\cos\alpha$

$$R = \frac{24}{25}\times 500g = 480g$$

The normal reaction of the hill on the crate is $480g$ N, as required.

b Minimum value of F occurs when the crate is on the point of sliding down the hill. Frictional force then acts up the hill.

R(↗):

$F + \mu R = 500g\sin\alpha$

$$F = \left(\frac{7}{25}\times 500g\right) - \left(\frac{3}{20}\times 480g\right) \qquad \left(\text{Using } \mu = \frac{3}{20}, \text{ and } R = 480g \text{ from } \mathbf{a}\right)$$

$$F = \left(140-72\right)g$$

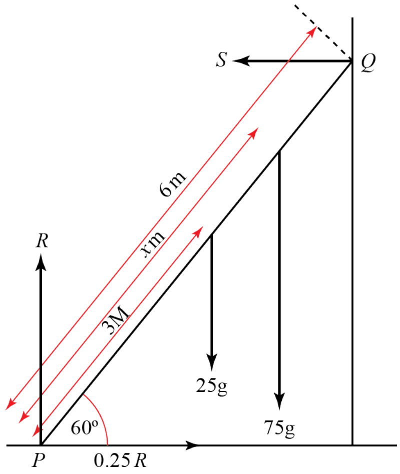
$$= 68g$$

The minimum value of F required to maintain equilibrium is $68g$ N.

16 Let:

R be the normal reaction of the floor on the ladder at P,
S be the normal reaction of the wall on the ladder at Q,
F be the friction between the floor and the ladder at P
x be the max. distance up the ladder from P that the builder can stand before the ladder begins to slip

$$R(\uparrow):\ R = 75g + 25g = 100g$$



$$R(\rightarrow): F = S$$

The ladder is in limiting equilibrium, so $F = \mu R$

Hence $S = \mu R$

$$= 0.25 \times 100g$$
$$= 25g$$

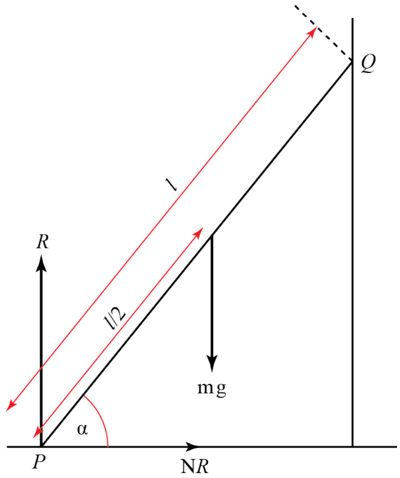
Taking moments about P:

$$S \times 6\sin 60^\circ = 75g \times x\cos 60^\circ + 25g \times 3\cos 60^\circ$$
$$25g \times 6\sin 60^\circ = 75(x+1)g\cos 60^\circ$$
$$x + 1 = \frac{25g \times 6\sin 60^\circ}{75g\cos 60^\circ}$$
$$x + 1 = 2\tan 60^\circ$$
$$x = 2\sqrt{3} - 1$$
$$x = 2.4641...$$

The maximum distance the builder can climb up the ladder before it slips is 2.46 m (3s.f.).

17 Let:

R be the normal reaction of the floor on the ladder at P,
S be the normal reaction of the wall on the ladder at Q,
F be the friction between the floor and the ladder at P



$$R(\uparrow):\ R = mg$$

$$R(\rightarrow): F = S$$

The ladder is in limiting equilibrium, so $F = \mu R$

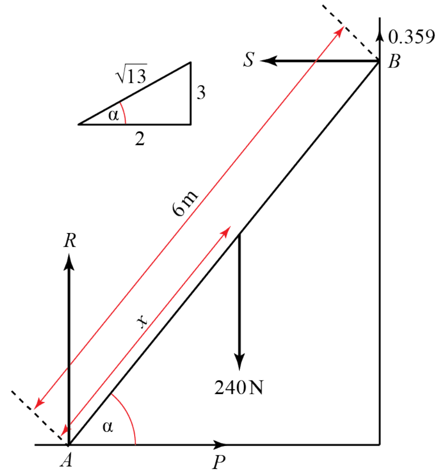
Hence $S = \mu R$

$$= \mu mg$$

Taking moments about P:

$$S \times l\sin\alpha = mg \times \frac{l}{2}\cos\alpha$$
$$\mu mgl\sin\alpha = \frac{mgl}{2}\cos\alpha$$
$$\mu = \frac{\cos\alpha}{2\sin\alpha} = \frac{1}{2\tan\alpha}$$

The coefficient of friction, μ, is given by $\dfrac{1}{2\tan\alpha}$.



18 $\tan\alpha = \frac{3}{2} \Rightarrow \sin\alpha = \frac{3}{\sqrt{13}}$ and $\cos\alpha = \frac{2}{\sqrt{13}}$

Let:
R be the normal reaction of the floor on the ladder at A,
S be the normal reaction of the wall on the ladder at B,
F be the friction between the wall and the ladder at B

$R(\rightarrow): P = S$

The ladder is in limiting equilibrium, so $F = \mu S$

$$= \mu P$$
$$= 0.3P$$

Taking moments about A:

$$240 \times 2\cos\alpha = (S \times 6\sin\alpha) + (F \times 6\cos\alpha)$$
$$240 \times 2\cos\alpha = (P \times 6\sin\alpha) + (0.3P \times 6\cos\alpha)$$
$$\frac{480 \times 2}{\sqrt{13}} = \frac{6P \times 3}{\sqrt{13}} + \frac{1.8P \times 2}{\sqrt{13}}$$
$$960 = (18 + 3.6)P$$
$$P = \frac{960}{21.6} = 44.444...$$

The minimum value of P is therefore 44.4 N (3 s.f.).

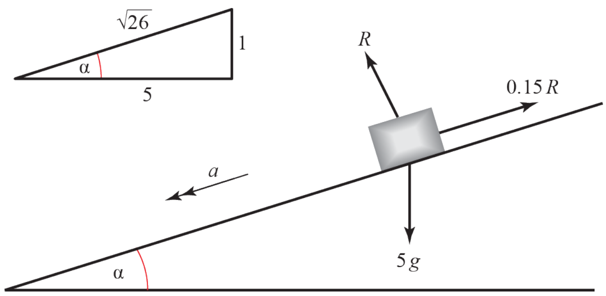
19 $\tan\alpha = \frac{1}{5} \Rightarrow \sin\alpha = \frac{1}{\sqrt{26}}$ and $\cos\alpha = \frac{5}{\sqrt{26}}$

Resolving at right angles to the hill:

$$R = 5g\cos\alpha$$
$$R = \frac{5g \times 5}{\sqrt{26}} = \frac{25g}{\sqrt{26}}$$

The sled slides down the hill; the frictional force therefore acts up the hill.
Resolving down the hill and using Newton's second law of motion:

$ma = F$



$$5a = 5g\sin\alpha - \mu R$$
$$5a = \frac{5g}{\sqrt{26}} - \frac{0.15 \times 25g}{\sqrt{26}}$$
$$5a = \left(\frac{5 - 3.75}{\sqrt{26}}\right)g$$
$$5a = \frac{5g}{4\sqrt{26}}$$
$$a = \frac{g}{4\sqrt{26}}$$

19 (Cont.)

Consider motion down the hill:

$u = 0,\ a = \dfrac{g}{4\sqrt{26}},\ s = 200,\ t = ?$

$s = ut + \dfrac{1}{2}at^2$

$200 = 0 + \left(\dfrac{1}{2} \times \dfrac{g}{4\sqrt{26}} t^2\right)$

$t^2 = \dfrac{200 \times 8\sqrt{26}}{g}$

$t = \sqrt{\dfrac{1600\sqrt{26}}{9.8}} = 28.853...$

To 3 s.f., the sled takes 28.9 s to travel 200 m down the hill.

20 At 10 am:

$\mathbf{r}_{P_0} = (400\mathbf{i} + 200\mathbf{j})$ km

$\mathbf{r}_{Q_0} = (500\mathbf{i} - 100\mathbf{j})$ km

$\mathbf{v}_P = (300\mathbf{i} + 250\mathbf{j})$ kmh^{-1}

$\mathbf{v}_Q = (600\mathbf{i} - 200\mathbf{j})$ kmh^{-1}

a $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t$

$\mathbf{r}_P = ((400 + 300t)\mathbf{i} + (200 + 250t)\mathbf{j})$km

$\mathbf{r}_Q = ((500 + 600t)\mathbf{i} - (100 + 200t)\mathbf{j})$km

b $\mathbf{r}_{QP} = \mathbf{r}_Q - \mathbf{r}_P$

$\mathbf{r}_{QP} = 500\mathbf{i} - 100\mathbf{j} - (400\mathbf{i} + 200\mathbf{j})$

$\mathbf{r}_{QP} = 100\mathbf{i} - 300\mathbf{j}$

c At noon, $t = 2$ h

$\mathbf{r}_P = (400 + (300 \times 2))\mathbf{i} + (200 + (250 \times 2))\mathbf{j}$

$\mathbf{r}_Q = (500 + (600 \times 2))\mathbf{i} - (100 + (200 \times 2))\mathbf{j}$

$\mathbf{r}_{QP} = 1700\mathbf{i} - 500\mathbf{j} - (1000\mathbf{i} + 700\mathbf{j})$

$\mathbf{r}_{QP} = 700\mathbf{i} - 1200\mathbf{j}$

$|\mathbf{r}_{QP}| = \sqrt{700^2 + 1200^2} = 1389.2...$

At noon, the two aeroplanes are 1390 km apart (3s.f.).

21 a $x = \left(3t - \dfrac{2k}{2t-1}\right)$m,

$$v = \frac{dx}{dt}$$

$$v = 3 + \frac{4k}{(2t-1)^2}$$

$$t = 0 \Rightarrow v = 10 \text{ ms}^{-1}$$

$$10 = 3 + \frac{4k}{-1^2}$$

$$k = \frac{10-3}{4} = \frac{7}{4} \text{ as required.}$$

b $t = 2$ s

$$x = (3 \times 2) - \frac{2 \times \frac{7}{4}}{(2 \times 2) - 1}$$

$$x = 6 - \frac{7}{2 \times 3} = \frac{29}{6}$$

After 2 s, P is $\dfrac{29}{6}$ m from O.

22 a $\mathbf{r} = \left(\left(\frac{1}{3}t^3 + 2t\right)\mathbf{i} + \left(\frac{1}{2}t^2 - 1\right)\mathbf{j}\right)$ m

$$\mathbf{v} = \dot{\mathbf{r}}$$

$$\mathbf{v} = (t^2 + 2)\mathbf{i} + t\mathbf{j}$$

b $t = 5$ s

$$\mathbf{v} = (5^2 + 2)\mathbf{i} + 5\mathbf{j}$$

$$\mathbf{v} = 27\mathbf{i} + 5\mathbf{j}$$

$$|\mathbf{v}| = \sqrt{27^2 + 5^2} = 27.459...$$

At $t = 5$ s, the speed of P is 27.5 ms^{-1} (3s.f.).

c

$$\mathbf{a} = \dot{\mathbf{v}}$$

$$\mathbf{a} = 2t\mathbf{i} + \mathbf{j}$$

$$t = 2s \Rightarrow$$

$$\mathbf{a} = 4\mathbf{i} + \mathbf{j}$$

$$|\mathbf{a}| = \sqrt{17} = 4.1231...$$

$$\tan\alpha = \frac{1}{4}$$

$$\alpha = 14.036...$$

At $t = 2$ s, P is accelerating at 4.12 ms^{-2} at an angle of 14.0° to the **i** vector (both values to 3s.f.).

23 a $\mathbf{r} = \left(\left(4t^2+1\right)\mathbf{i} + \left(2t^2-3\right)\mathbf{j}\right)$ m

$\mathbf{v} = \dot{\mathbf{r}}$

$\mathbf{v} = 8t\mathbf{i} + 4t\mathbf{j}$

$t = \ 3\,s \Rightarrow$

$\mathbf{v} = 24\mathbf{i} + 12\mathbf{j}$

At $t = 3$ s, the velocity of the particle is $\left(24\mathbf{i} + 12\mathbf{j}\right)$ ms^{-1}.

b $\mathbf{a} = \dot{\mathbf{v}}$

$\mathbf{a} = 8\mathbf{i} + 4\mathbf{j}$

Since all terms in this expression are independent of t, the acceleration is constant.

24 $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t = \int -2t\mathbf{i} + 3t^{\frac{1}{2}}\mathbf{j}$

$= -t^2\mathbf{i} + 3 \times \frac{2}{3}t^{\frac{3}{2}}\mathbf{j} + c$

$= -t^2\mathbf{i} + 2\sqrt{t^3}\,\mathbf{j} + c$

When $t = 0$, $\mathbf{r} = 2\mathbf{j}$ m $\Rightarrow$

$2\mathbf{j} = 0\mathbf{i} + 0\mathbf{j} + c \Rightarrow c = 2\mathbf{j}$

$\therefore \mathbf{r} = -t^2\mathbf{i} + 2\left(\sqrt{t^3} + 1\right)\mathbf{j}$

When $t = 4$ s

$\mathbf{r} = -16\mathbf{i} + 2\left(8+1\right)\mathbf{j}$

$\mathbf{r} = -16\mathbf{i} + 18\mathbf{j}$

$\left|\mathbf{r}\right| = \sqrt{16^2 + 18^2} = 24.083...$

After 4 s, P is 24 m from O (2s.f.).

25 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int \left(2t^2 - 3t^3\right)\mathbf{i} - 4\left(2t+1\right)\mathbf{j}\,\mathrm{d}t$

$\mathbf{v} = \left(t^2 - \frac{3}{4}t^4\right)\mathbf{i} - 4\left(t^2 + t\right)\mathbf{j} + c$

$t = 0 \Rightarrow \mathbf{v} = \left(3\mathbf{i} + \mathbf{j}\right)$ ms^{-1}

$3\mathbf{i} + \mathbf{j} = 0\mathbf{i} - 4(0)\mathbf{j} + c$

$c = 3\mathbf{i} + \mathbf{j}$

$\Rightarrow \mathbf{v} = \left(t^2 - \frac{3}{4}t^4 + 3\right)\mathbf{i} - \left(4t^2 + 4t - 1\right)\mathbf{j}$

25 b If P is moving in the direction of $\mathbf{i}$, the coefficient of $\mathbf{j}$ in the velocity vector is 0.

$$0 = 4t^2 + 4t - 1$$

$$t = \frac{-4 \pm \sqrt{16 - (4 \times 4 \times (-1))}}{8}$$

$$t = \frac{-1 \pm \sqrt{2}}{2}$$

The negative solution can be ignored as it is outside the range over which the equation applies.

P is moving in the direction of $\mathbf{i}$ after $\left(\frac{\sqrt{2}-1}{2}\right)$ s (0.207 s to 3 s.f.).

26 a $\mathbf{v} = \int \mathbf{a}\,\mathrm{d}t = \int (-4t\mathbf{i} - 2\mathbf{j})\,\mathrm{d}t$

$$\mathbf{v} = -2t^2\mathbf{i} - 2t\mathbf{j} + c$$

$$t = 0 \Rightarrow \mathbf{v} = 8\mathbf{i}\text{ ms}^{-1}$$

$$8\mathbf{i} = 0\mathbf{i} - 0\mathbf{j} + c$$

$$c = 8\mathbf{i}$$

$$\Rightarrow \mathbf{v} = 2(4 - t^2)\mathbf{i} - 2t\mathbf{j}$$

b When the windsurfer is moving due south, the coefficient of $\mathbf{i}$ in the velocity vector is 0.

$$0 = 2(4 - t^2)$$

$$t^2 = 4$$

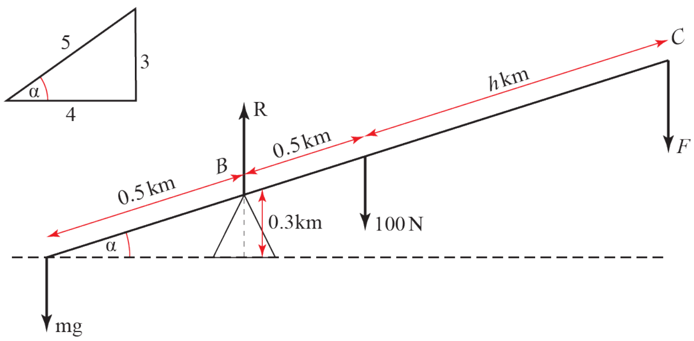
$$t = \pm 2$$

The negative solution can be ignored as it is before the time the windsurfer starts to move.
The windsurfer is moving due south after 2 s.

Challenge

1 The rod makes an angle of $\alpha°$ with the horizontal where

$$\sin\alpha = \frac{0.3}{0.5} = \frac{3}{5} \Rightarrow \cos\alpha = \frac{4}{5}$$



To lift the mass, total clockwise moments about B must exceed total anticlockwise moments about B:

$$(F \times 1.5k\cos\alpha) + (100 \times 0.5k\cos\alpha) > mg \times 0.5k\cos\alpha$$

$$1.5F + 50 > 0.5mg$$

$$\frac{3}{2}F > \frac{1}{2}mg - 50$$

$$F > \frac{1}{3}(mg - 100) \quad \text{as required.}$$

Challenge

2 $v = 3\sin kt + \cos kt \;\; \text{ms}^{-1}$

$a = \dot{v}$

$a = 3k\cos kt - \sin kt$

$t = 0 \Rightarrow a = 1.5 \text{ ms}^{-2}$

$\therefore 1.5 = 3k - 0$

$\therefore k = 0.5$

$s = \int v\,\mathrm{d}t = \int 3\sin kt + \cos kt \;\mathrm{d}t$

$= -\dfrac{3\cos kt}{k} + \dfrac{\sin kt}{k} + c$

$s = -6\cos\dfrac{t}{2} + 2\sin\dfrac{t}{2} + c$ (substituting $k = \dfrac{1}{2}$)

$t = 0 \Rightarrow s = 0$

$0 = -6 + 0 + c \Rightarrow c = 6$

$s = 2\left(3 - 3\cos\dfrac{t}{2} + \sin\dfrac{t}{2}\right)$

Maximum displacement occurs when $v = 0$

$0 = 3\sin kt + \cos kt$

$0 = 3\tan kt + 1$

$\tan kt = -\dfrac{1}{3}$

$0.5t = 161.565...$

$t = 323.13...$

Maximum displacement is therefore

$s = 2\left(3 - 3\cos(161.56...)^\circ + \sin(161.56...)^\circ\right)$

$s = 2(3 + 2.8460... + 0.31622..)$

$s = 12.324...$

The maximum displacement of 12.3 m first occurs at 323 s (both to 3s.f.).

Challenge

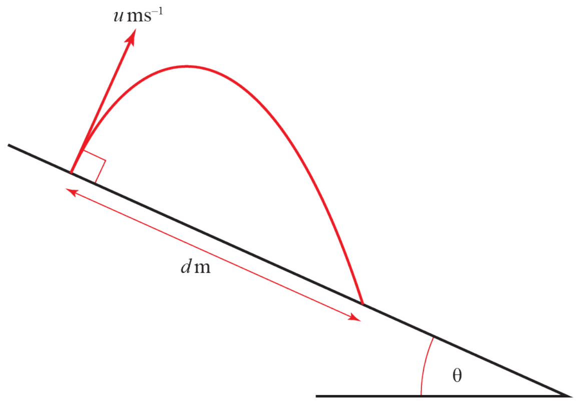
3 There is no change in the horizontal component of the velocity.

$R(\rightarrow)$:

$v = u_x = u\sin\theta,\ s = d\cos\theta,\ t = ?$

$s = vt$

$$t = \frac{s}{v} = \frac{d\cos\theta}{u\sin\theta}$$



$R(\uparrow)$

$$u = u_y = u\cos\theta,\ s = -d\sin\theta,\ a = -g,\ t = \frac{d\cos\theta}{u\sin\theta}$$

$$s = ut + \frac{1}{2}at^2$$

$$-d\sin\theta = u\cos\theta\left(\frac{d\cos\theta}{u\sin\theta}\right) - \frac{g}{2}\left(\frac{d\cos\theta}{u\sin\theta}\right)^2$$

$$-d\sin\theta = \frac{d\cos^2\theta}{\sin\theta} - \frac{gd^2\cos^2\theta}{2u^2\sin^2\theta}$$

$$-\frac{\sin\theta}{\cos\theta} = \frac{\cos\theta}{\sin\theta} - \frac{gd\cos\theta}{2u^2\sin^2\theta}$$

$$\frac{gd\cos\theta}{2u^2\sin^2\theta} = \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$

$$\frac{gd}{2u^2} = \frac{\sin^2\theta}{\cos\theta}\left(\frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}\right)$$

$$\frac{gd}{2u^2} = \tan\theta\left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right)$$

$$d = \frac{2u^2}{g}\tan\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta}\right)$$

$$d = \frac{2u^2}{g}\tan\theta\left(\frac{1}{\cos\theta}\right)$$

$$d = \frac{2u^2}{g}\tan\theta\sec\theta \quad \text{as required.}$$

Exam-style practice: Paper 3, Section A: Statistics

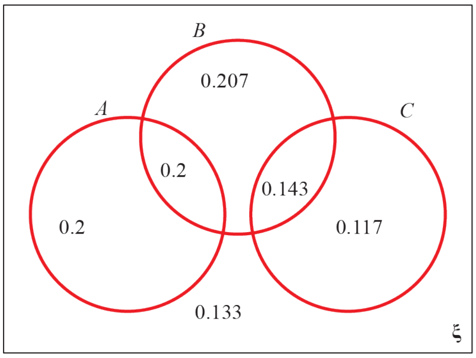
1 a Use the cumulative binomial distribution tables, with n = 40 and p = 0.52. Then $P(X \geqslant 22) = 1 - P(X \leqslant 21) = 1 - 0.5867 = 0.4133$ (4 s.f.).

b In order for the normal approximation to be used as an approximation to the binomial distribution the two conditions are: (i) n is large (>50); and (ii) p is close to 0.5.

c The two conditions for the normal approximation to be a valid approximation are satisfied. $\mu = np = 250 \times 0.52 = 130$ and $\sigma = \sqrt{np(1-p)} = \sqrt{130 \times 0.48} = \sqrt{62.4} = 7.90$ (3 s.f.). Therefore $B(250, 0.52) \approx N(130, 7 \cdot 9^2)$ so that $P(B \leqslant 120) \approx P(N \leqslant 120.5) = 0.1146$ (4 s.f.).

d If the engineer's claim is true, then the observed result had a less than 12% chance of occurring. This would mean that there would be insufficient evidence to reject her claim with a two-tailed hypothesis test at the 10% level. Though it does provide some doubt as to the validity of her claim.

2 a Since A and C are mutually exclusive, $P(A \cap C) = 0$ and their intersection need not be represented on the Venn diagram. Since B and C are independent, $P(B \cap C) = P(B) \times P(C) = 0.55 \times 0.26 = 0.143$. Using the remaining information in the question allows for the other regions to be labelled. Therefore the completed Venn diagram should be:



b $P(A) \times P(B) = 0.4 \times 0.55 = 0.22 \neq 0.2 = P(A \cap B)$ and so the events are not independent.

c $P(A \mid B') = \dfrac{P(A \cap B')}{P(B')} = \dfrac{0.2}{0.2 + 0.117 + 0.133} = \dfrac{0.2}{0.45} = 0.444$ (3 s.f.)

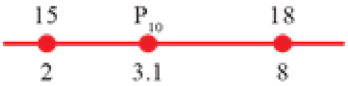
d $P(C \mid (A \cap B)') = \dfrac{P(C \cap (A \cap B)')}{P((A \cap B)')} = \dfrac{P(C)}{1 - P(A \cap B)} = \dfrac{0.26}{0.8} = 0.325$

3 a The variable t is continuous, since it can take any value between 12 and 26 (in degrees Celsius).

b Estimated mean 19.419; estimated standard deviation 2.814 (3 d.p.).

c Temperature is continuous and the data were given in a grouped frequency table.

3 d The 10th percentile is $\frac{31}{10} = 3.1$th value. Using linear interpolation:



$$\frac{P_{10}-15}{18-15}=\frac{3.1-2}{8-2} \Rightarrow \frac{P_{10}-15}{3}=\frac{1.1}{6} \Rightarrow P_{10}=3\times\frac{1.1}{6}+15=0.55+15=15.5$$

The 90th percentile is $\frac{9\times 31}{10} = 27.9$th value. Using linear interpolation:

$$\frac{P_{90}-22}{26-22}=\frac{27.9-26}{31-26} \Rightarrow \frac{P_{90}-22}{4}=\frac{1.9}{5} \Rightarrow P_{90}=4\times\frac{1.9}{5}+22=1.52+22=23.52$$

Therefore the 10th to 90th interpercentile range is $23.52-15.55=7.97$.

e Since the meteorologist believes that there is positive correlation, the hypotheses are
$H_0 : \rho = 0$
$H_1 : \rho > 0$
The sample size is 8, and so the critical value (for a one-tailed test) is 0.6215.
Since $r = 0.612 < 0.6215$, there is not sufficient evidence to reject H_0, and so there is not sufficient evidence, at the 5% significance level, to say that there is a positive correlation between the daily mean air temperature and the number of hours of sunshine.

4 a The value of 0.9998 is very close to 1, indicating that the plot of x against y is very close to being a linear relationship, and so the data should be well-modelled by an equation of the form $q = kt^n$.

b Rearranging the equation

$$y = 0.07601 + 2.1317x$$
$$\Rightarrow \log q = 0.07601 + 2.1317\log t$$
$$\Rightarrow q = 10^{0.07601+2.1317\log t} = 10^{0.07601}\times 10^{2.1317\log t}$$
$$\Rightarrow q = 10^{0.07601}\times 10^{\log t^{2.1317}} = 10^{0.07601}\times t^{2.1317}$$

Therefore $k = 10^{0.0761} = 1.19$ (3 s.f.) and $n = 2.1317$.

c It would not be sensible to use the model to predict the amount of substance produced when $t = 85^\circ\text{C}$, since this is considerably outside the range of the provided data (extrapolation).

5 **a** $P(Z < a) = 0.025 \Rightarrow a = -1.96$ and $P(Z > a) = 0.05 \Rightarrow a = 1.645$. Therefore, for the given distribution, $\frac{3.416-\mu}{\sigma} = -1.96$ and $\frac{4.858-\mu}{\sigma} = 1.645$. Rearranging these equations:
$\frac{3.416-\mu}{\sigma} = -1.96 \Rightarrow 3.416-\mu = -1.96\sigma$ and $\frac{4.858-\mu}{\sigma} = 1.645 \Rightarrow 4.858-\mu = 1.645\sigma$.
Now subtract the second equation from the first to obtain:
$4.858-\mu-(3.416-\mu) = 1.645\sigma-(-1.96\sigma) \Rightarrow 1.442 = 3.605\sigma \Rightarrow \sigma = 0.4$ and so, using the first equation, $3.416-\mu = -1.96\times 0.4 \Rightarrow \mu = 3.416+0.784 = 4.2$. Using these values within the normal distribution, $P(3.5 < X < 4.6) = P(4.6) - P(3.5) = 0.84134 - 0.04006 = 0.8013$ (4 s.f.) of the cats will be of the standard weight.

b Using the binomial distribution, $P(B \geqslant 10) = 1 - P(B \leqslant 9) = 1 - 0.0594 = 0.9406$ (4 s.f.).

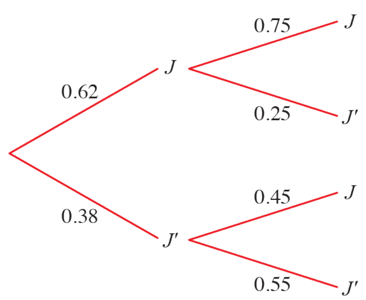
c Assume the mean is 4.5kg and standard deviation is 0.51. Then the sample $\bar{X}$ should be normally distributed with $\bar{X} \sim N\left(4.5, \frac{0.51^2}{12}\right)$. The hypothesis test should determine whether it is statistically significant, at the 10% level, that the mean is not 4.5kg. Therefore the test should be 2-tailed with

$H_0 : \mu = 4.5$
$H_1 : \mu \neq 4.5$

The critical region therefore consists of values greater than a where $P(\bar{X} > a) = 0.05$ and so $a = 4.742$ (4 s.f.) and values less than b where $P(\bar{X} < b) = 0.05$ and so $b = 4.258$ (4 s.f.).

Since the observed mean is 4.73 and $4.73 < a = 4.742$, there is not enough evidence, at the 10% significance level, to reject H_0 i.e. there is not sufficient evidence to say, at the 10% level, that the mean weight of all cats in the town is different from 4.5kg.

6 It is first worth displaying the information in a tree diagram. Let J denote the event that Jemima wins a game of tennis and J' be the event that Jemima loses a game of tennis. Since Jemima either wins or loses each game of tennis, $P(J) + P(J') = 1$. This allows for the other probabilities on the tree diagram to be filled in. Therefore the completed tree diagram should be:



The required probability is then:
$P(\text{wins both games} \mid \text{wins second game})$
$$= \frac{P(\text{wins both games})}{P(\text{wins second game})} = \frac{0.62\times 0.75}{0.62\times 0.75 + 0.38\times 0.45} = \frac{0.465}{0.465+0.171} = 0.731 \text{ (3 s.f.)}$$

Exam-style practice: Paper 3, Section B: Mechanics

7 $\mathbf{r} = \int \mathbf{v}\,\mathrm{d}t$

$= \int (2-6t^2)\mathbf{i} - t\mathbf{j}\,\mathrm{d}t$

$= \left(2t - \frac{6}{3}t^3\right)\mathbf{i} - \frac{t^2}{2}\mathbf{j} + c$

At $t = 1$ s, $\mathbf{r} = 5\mathbf{i}$ m $\Rightarrow 5\mathbf{i} = (2-2)\mathbf{i} - \frac{1}{2}\mathbf{j} + c$

$c = 5\mathbf{i} + \frac{1}{2}\mathbf{j}$

$\therefore \mathbf{r} = (2t - 2t^3 + 5)\mathbf{i} + \frac{1}{2}(1-t^2)\mathbf{j}$

When $t = 3$ s,

$\mathbf{r} = (6 - 54 + 5)\mathbf{i} + \frac{1}{2}(1-9)\mathbf{j}$

$\mathbf{r} = -43\mathbf{i} - 4\mathbf{j}$

$s = |\mathbf{r}| = \sqrt{43^2 + 4^2} = 43.185...$

At $t = 3$ s, P is 43.2 m from O (3 s.f.).

8 $R(\rightarrow)$: $u_x = 100\cos 30° = 50\sqrt{3}$

$R(\uparrow)$: $u_y = 100\cos 30° = 50$

a $R(\uparrow)$: $u_y = 50\text{ ms}^{-1}$, $s = 0$ m, $a = g = -9.8\text{ ms}^{-2}$, $t = ?$

$s = ut + \frac{1}{2}at^2$

$0 = 50t - 4.9t^2$

$4.9t^2 = 50t$

The solution $t = 0$ corresponds to the time the arrow is fired and can therefore be ignored.

$\therefore t = \frac{50}{4.9} = 10.204...$

The arrow reaches the ground after 10.2 s (3 s.f.).

b At maximum height, $v_y = 0$

$R(\uparrow)$: $u_y = 50\text{ ms}^{-1}$, $v_y = 0$ m, $a = g = -9.8\text{ ms}^{-2}$, $s = ?$

$v^2 = u^2 + 2as$

$0 = 50^2 - 19.6s$

$19.6s = 2500$

$s = \frac{2500}{19.6} = 127.55...$

The maximum height reached by the arrow is 128 m (3s.f.).

8 c At $t = 3$ s,

$R(\rightarrow)$: $v_x = u_x = 50\sqrt{3}$ ms^{-1} since horizontal speed remains constant.

$R(\uparrow)$: $u_y = 50$ ms^{-1}, $t = 3$ s, $a = g = -9.8$ ms^{-2}, $v_y = ?$

$$v = u + at$$

$$v_y = 50 - (3 \times 9.8) = 20.6$$

The speed at $t = 3$ s is given by:

$$v^2 = v_x^2 + v_y^2$$

$$v^2 = \left(50\sqrt{3}\right)^2 + (20.6)^2$$

$$v = \sqrt{7500 + 424.36} = 89.018...$$

The speed of the arrow after 3 s is 89.0 ms^{-1} (3 s.f).

9 a $\mathbf{u} = 2\mathbf{i}$ ms^{-1}, $t = 10$ s, $\mathbf{a} = 0.2\mathbf{i} - 0.8\mathbf{j}$ ms^{-2}, $\mathbf{r} = ?$

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

$$\mathbf{r} = 20\mathbf{i} + \frac{100}{2}(0.2\mathbf{i} - 0.8\mathbf{j})$$

$$\mathbf{r} = 20\mathbf{i} + 10\mathbf{i} - 40\mathbf{j}$$

After 10 s, the position vector of the cyclist is $(30\mathbf{i} - 40\mathbf{j})$ m.

b $s = |\mathbf{r}|$

$$s = \sqrt{30^2 + 40^2} = 50$$

After 10 s, the cyclist is 50 m from A.

c For $t > 10$ s, $\mathbf{v} = 5\mathbf{i}$ ms^{-1} and $\mathbf{a} = 0$

The position vector is now given by:

$$\mathbf{r} = (30\mathbf{i} - 40\mathbf{j}) + \mathbf{v}(t - 10)\mathbf{i}$$

$$\mathbf{r} = 30\mathbf{i} - 40\mathbf{j} + 5(t - 10)\mathbf{i}$$

$$\mathbf{r} = (5t - 20)\mathbf{i} - 40\mathbf{j}$$

The cyclist will be south-east of A when the coefficient of $\mathbf{i}$ is positive and coefficient of $\mathbf{j}$ is negative, but both have equal magnitude.

$$5t - 20 = 40$$

$$5t = 60$$

$$t = \frac{60}{5} = 12$$

The cyclist is directly south-east of A after 12 s.

9 d First, work out the position vector of B from A:

$\mathbf{r} = (5t - 20)\mathbf{i} - 40\mathbf{j}$

Cyclist reaches B when $t = 12 + 30 = 42$ s

$\mathbf{r} = ((5 \times 42) - 20)\mathbf{i} - 40\mathbf{j}$

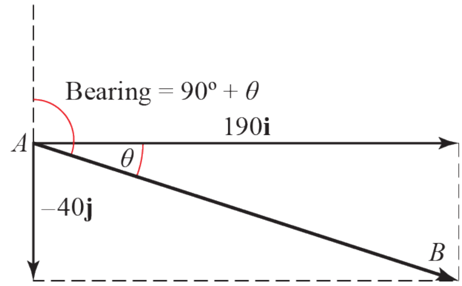
$\mathbf{r} = 190\mathbf{i} - 40\mathbf{j}$

Let θ be the acute angle between the horizontal and B (as shown in the diagram).

Then $\tan\theta = \dfrac{40}{190}$

$\theta = 11.888...$

To the nearest degree, the bearing of B from A is 90 +12 = 102º.



10 a Considering Q and using Newton's second law of motion:

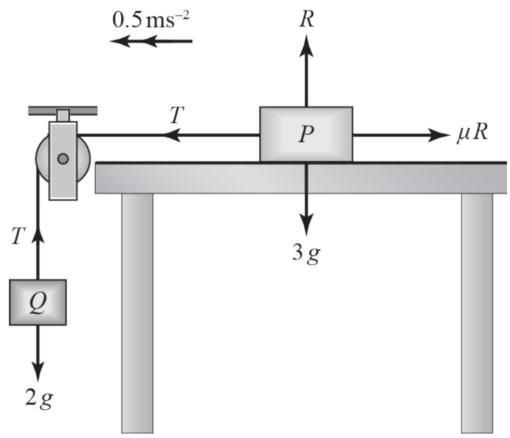
$a = 0.5$ ms^{-2}, $m = 2$ kg

$$F = ma$$

$$2g - T = 2 \times 0.5$$

$$T = (2 \times 9.8) - 1 = 18.6$$

The tension in the string immediately after the particles begin to move is 18.6 N.



b Considering P:

Resolving vertically $\Rightarrow R = 3g$

Resolving horizontally and using Newton's second law of motion with $a = 0.5$ ms^{-2} and $m = 3$ kg:

$$T - \mu R = 3 \times 0.5$$

$$3\mu g = T - 1.5$$

$$\mu = \frac{18.6 - 1.5}{3 \times 9.8} = 0.58163...$$

The coefficient of friction is 0.582 (3 s.f.), as required.

10 c Consider P before string breaks: $u = 0 \text{ ms}^{-1}$, $t = 2$ s, $a = 0.5 \text{ ms}^{-2}$, $v = ?$

$$v = u + at$$

$$v = 0 + (0.5 \times 2) = 1$$

After string breaks, the only force acting on P is a frictional force of magnitude $F = \mu R = 3\mu g$

Using Newton's Second Law for P,

$$F = ma$$

$$3\mu g = 3a$$

$$a = \mu g$$

$$a = 9.8 \times 0.58163...$$

$$= 5.7$$

The acceleration is in the opposite direction to the motion of P, hence

$u = 1 \text{ ms}^{-1}$, $v = 0 \text{ ms}^{-1}$, $a = -0.5 \text{ ms}^{-2}$, $t = ?$

$$v = u + at$$

$$0 = 1 - 5.7t$$

$$t = \frac{1}{5.7} = 0.17543...$$

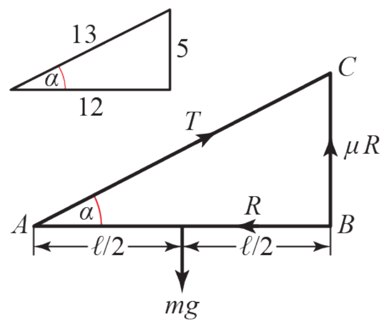
P takes 0.175 s (3 s.f.) to come to rest.

d The information that the string is inextensible has been used in assuming that the tension is the same in all parts of the string and that the acceleration of P and Q are identical while they are connected.

11 a The rod is in equilibrium so resultant force and moment are both zero.

$\tan\alpha = \frac{5}{12} \Rightarrow \sin\alpha = \frac{5}{13}$ and $\cos\alpha = \frac{12}{13}$

Taking moments about B:



$$mg\frac{l}{2} = (T\sin\alpha) \times l$$

$$T = \frac{mg}{2\sin\alpha}$$

$$T = \frac{mg}{2 \times \frac{5}{13}} = \frac{13mg}{10} \quad \text{as required.}$$

b Resolving horizontally:

$$R = T\cos\alpha$$

$$R = \frac{13mg}{10} \times \frac{12}{13} = \frac{6mg}{5}$$

Resolving vertically:

$$T\sin\alpha + \mu R = mg$$

$$\left(\frac{13mg}{10} \times \frac{5}{13}\right) + \mu\frac{6mg}{5} = mg$$

$$\frac{6}{5}\mu = 1 - \frac{1}{2}$$

$$\mu = \frac{5}{12}$$

The coefficient of friction between the rod and the wall is $\frac{5}{12}$.