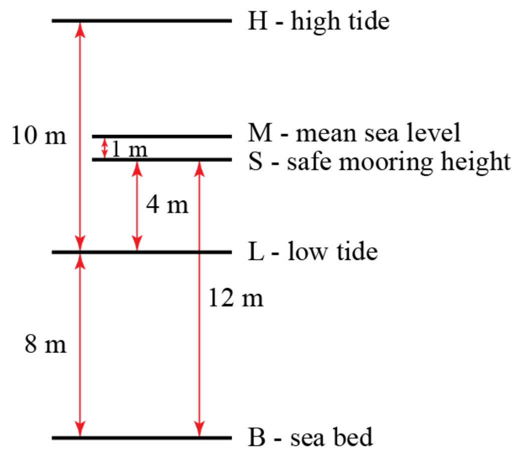
A Level Exam-style Practice Paper



1 a i The period is given by the time lapse between high tide and low tide which is 12.5 hours.

ii The amplitude is given by half the total displacement and so is 5 m.

b The 'safe mooring height' marker is 4 m above low tide, 1 m below mean sea level, the centre of the motion (see diagram).

$a = 5$ m, $x = -1$ m, $T = 12.5$ hours,

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{12.5} = \frac{4\pi}{25} \text{ rad hour}^{-1}$$

$$v^2 = \omega^2\left(a^2 - x^2\right)$$

$$v^2 = \left(\frac{4\pi}{25}\right)^2\left(5^2 - (-1)^2\right)$$

$$v^2 = \frac{384\pi^2}{625}$$

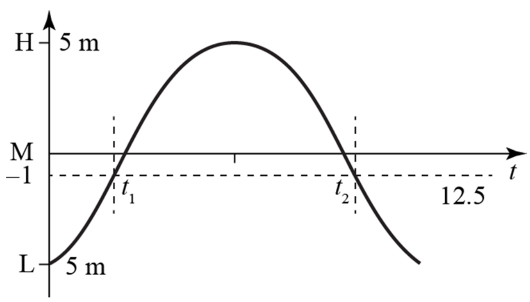
$$v = \frac{8\sqrt{6}}{25}\pi = 2.4624...$$

The water level is rising at a speed of 2.46 m hour^{-1} (3 s.f.) when it passes the marker.

c Taking the displacement to be zero at mean sea level we want the interval between the two times after low tide when $x = -1$ m. Taking low tide as $t = 0$, using $x = -a\cos\omega t$ allows us to work without a phase constant (see diagram) and the equation of motion becomes:

$$-1 = -5\cos\left(\frac{4\pi}{25}t\right)$$

This is first true t_1 hours after low tide, where:



$$\cos\left(\frac{4\pi}{25}t_1\right) = \frac{1}{5}$$

$$\frac{4\pi}{25}t_1 = 1.3694...$$

$$t_1 = \frac{25}{4\pi} \times 1.3694... = 2.7244...$$

Using symmetry (see diagram), the water falls to an unsafe depth t_2 hours after low tide where:

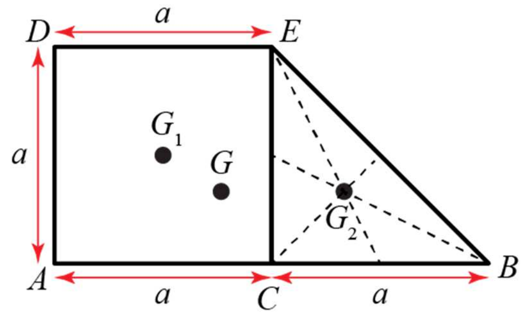
$$t_2 = T - t_1$$

$$t_2 = 12.5 - 2.7244... = 9.7755...$$

$$t_2 - t_1 = 9.7755... - 2.7244... = 7.0511...$$

So the total length of time for which boats can moor safely between two consecutive low tides is 7.05 hours (3 s.f.).

2 a $AD = EC = CB = a$
Since $AB = 2a$
$AC = DE = a$
Split the lamina along EC.
Both sections have equal mass (since the triangle has been folded over), call this m.
The total mass of the lamina is therefore $2m$.
The centre of mass of $ACED$, G_1 is $\frac{a}{2}$ from both AC and AD.



For the triangle. taking C as the origin, the coordinates of the vertices are:

$C:(0,0)$

$B:(a,0)$

$E:(0,a)$

So, with C as origin, the coordinates of the centre of mass of CBE, G_2, are $\left(\frac{0+a+0}{3}, \frac{0+0+a}{3}\right) = \left(\frac{a}{3}, \frac{a}{3}\right)$

The lamina can be replaced by two particles:

$ACED$ of mass m $\left(\frac{a}{2}, \frac{a}{2}\right)$ from A

CBE of mass m $\left(\frac{4a}{3}, \frac{a}{3}\right)$ from A

Using $\Sigma m_i r_i = r\Sigma m_i$, the centre of mass of the composite lamina has coordinates relative to A as the origin of:

$$m\begin{pmatrix}\frac{a}{2}\\ \frac{a}{2}\end{pmatrix} + m\begin{pmatrix}\frac{4a}{3}\\ \frac{a}{3}\end{pmatrix} = 2m\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

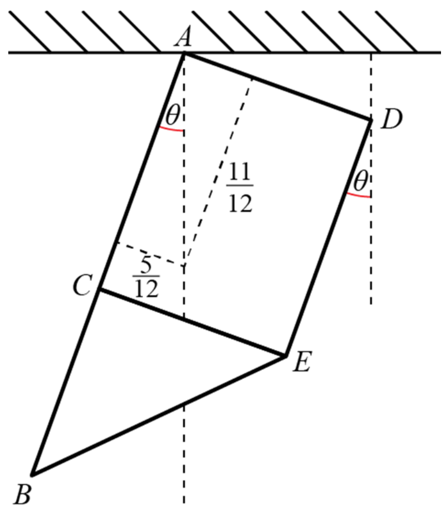
$$\begin{pmatrix}\frac{11a}{6}\\ \frac{5a}{6}\end{pmatrix} = 2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{11a}{12}\\ \frac{5a}{12}\end{pmatrix} = \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

G, The centre of mass of the lamina lies:

i $\frac{11a}{12}$ from AD.

ii $\frac{5a}{12}$ from AB.



b When suspended from A, the lamina hangs as shown, with G directly below A and AC (and therefore DE) making an angle of θ with the vertical where:

$\tan\theta = \frac{\frac{5a}{12}}{\frac{11a}{12}}$

$\tan\theta = \frac{5}{11}$

$\theta = 24.443...$

The edge DE makes an angle of 24° with the vertical (to the nearest whole degree).

3 a $a = -2\left(k^2 + v^2\right)\ \text{ms}^{-2}$

$$a = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -2\left(k^2 + v^2\right)$$

Separating variables and integrating:

$$\int \frac{v}{k^2 + v^2}\,\mathrm{d}v = -\int 2\,\mathrm{d}x$$

$$\frac{1}{2}\ln\left(k^2 + v^2\right) = -2x + B$$

At $t = 0$ s, $v = 2U\ \text{ms}^{-1}$

Use these initial conditions to find the constant of integration, B:

$$\frac{1}{2}\ln\left(k^2 + 4U^2\right) = -0 + B$$

$$B = \frac{1}{2}\ln\left(k^2 + 4U^2\right)$$

So displacement, x, is given by:

$$\frac{1}{2}\ln\left(k^2 + v^2\right) = -2x + \frac{1}{2}\ln\left(k^2 + 4U^2\right)$$

$$2x = \frac{1}{2}\ln\left(k^2 + 4U^2\right) - \frac{1}{2}\ln\left(k^2 + v^2\right)$$

$$x = \frac{1}{4}\ln\left(\frac{k^2 + 4U^2}{k^2 + v^2}\right)$$

When the particle reaches A, $x = OA$ and $v = U\ \text{ms}^{-1}$

Substituting these values gives:

$$OA = \frac{1}{4}\ln\left(\frac{k^2 + 4U^2}{k^2 + U^2}\right)$$ m, as required.

b Using the relationship $a = \frac{\mathrm{d}v}{\mathrm{d}t}$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2\left(k^2 + v^2\right)$$

Separating variables and integrating:

$$\int_U^{2U} \frac{1}{k^2 + v^2}\,\mathrm{d}v = -\int_0^t 2\,\mathrm{d}t$$

$$\left[\frac{1}{k}\arctan\frac{v}{k}\right]_U^{2U} = -\left[2t\right]_0^t$$

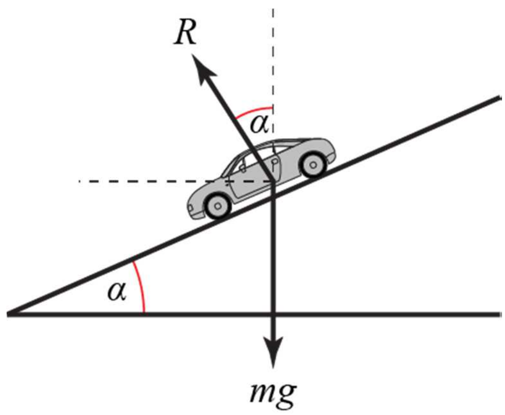
$$\frac{1}{k}\arctan\frac{2U}{k} - \frac{1}{k}\arctan\frac{U}{k} = -2t$$

$$t = \frac{1}{2k}\left(\arctan\frac{U}{k} - \arctan\frac{2U}{k}\right)$$

4 $r = 306\text{ m},\ \tan\alpha = \frac{3}{4} \Rightarrow \sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$

a Let the mass of the car be m and the normal reaction force be R
Resolving vertically:

$$mg = R\cos\alpha$$
$$\Rightarrow R = \frac{mg}{\cos\alpha}$$



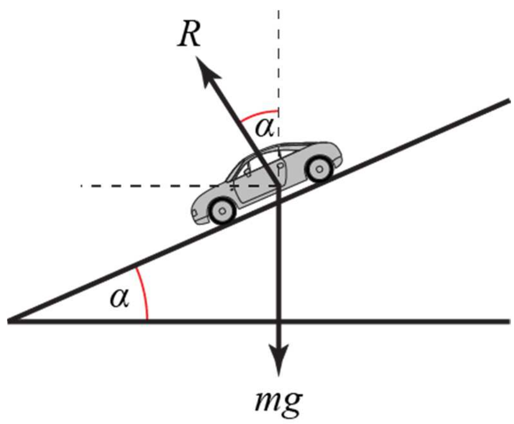
Resolving horizontally:

$$\frac{mv^2}{r} = R\sin\alpha$$
$$\frac{mv^2}{r} = \frac{mg\sin\alpha}{\cos\alpha}$$
$$\frac{v^2}{306} = g\tan\alpha$$
$$v^2 = 306 \times 9.8 \times \frac{3}{4}$$
$$v = 47.424...$$

The car travels at a speed of 47.4 ms^{-1} (to 3 s.f.).

b The frictional force, $F = \mu R,\ \mu = 0.5$
If the car is about to slide up the slope, F acts down the slope.
Resolving vertically:

$$mg + \mu R\sin\alpha = R\cos\alpha$$
$$R\cos\alpha - \mu R\sin\alpha = mg$$
$$R = \frac{mg}{\cos\alpha - \mu\sin\alpha}$$
$$R = \frac{mg}{\frac{4}{5} - \left(\frac{1}{2} \times \frac{3}{5}\right)} = 2mg$$



Resolving horizontally:

$$\frac{mv^2}{r} = R\sin\alpha + \mu R\cos\alpha$$
$$\frac{mv^2}{r} = 2mg\left(\sin\alpha + \mu\cos\alpha\right)$$
$$\frac{v^2}{306} = 2g\left(\frac{3}{5} + \left(\frac{1}{2} \times \frac{4}{5}\right)\right)$$
$$v^2 = 306 \times 2 \times 9.8 \times 1$$
$$v = 77.444...$$

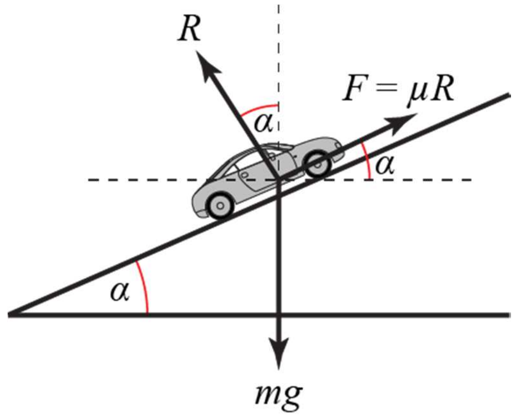
4 b continued

If the car is about to slide down the slope, F acts up the slope.

Resolving vertically:

$$mg = R\cos\alpha + \mu R\sin\alpha$$

$$R = \frac{mg}{\frac{4}{5}+\left(\frac{1}{2}\times\frac{3}{5}\right)} = \frac{10mg}{11}$$



Resolving horizontally:

$$\frac{mv^2}{r} = R\sin\alpha - \mu R\cos\alpha$$

$$\frac{mv^2}{r} = \frac{10mg}{11}(\sin\alpha - \mu\cos\alpha)$$

$$\frac{v^2}{306} = \frac{10g}{11}\left(\frac{3}{5}-\left(\frac{1}{2}\times\frac{4}{5}\right)\right)$$

$$v^2 = 306\times\frac{10}{11}\times 9.8\times\frac{1}{5}$$

$$v = 23.350...$$

If the car is not to skid up or down the slope, the speed must remain between 23.4 ms^{-1} and 77.4 ms^{-1} (to 3 s.f.).

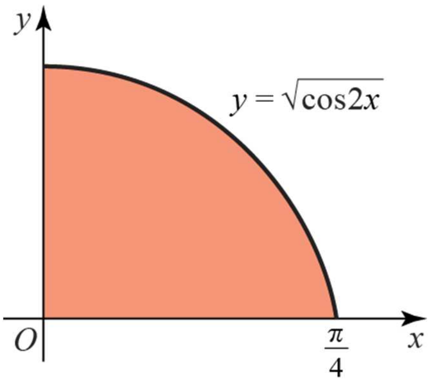
5 a $y = \sqrt{\cos 2x}$

$$V = \pi\int y^2\,dx$$

$$V = \pi\int_0^{\frac{\pi}{4}} \cos 2x\,dx$$

$$V = \frac{\pi}{2}[\sin 2x]_0^{\frac{\pi}{4}}$$

$$V = \frac{\pi}{2}(1-0)$$



The volume of the solid is $\frac{\pi}{2}$ m^3, as required.

5 b Centre of mass:

$$\bar{x} = \frac{\int y^2 x \, dx}{\int y^2 \, dx}$$

$$\bar{x} = \frac{\int_0^{\frac{\pi}{4}} x \cos 2x \, dx}{\int_0^{\frac{\pi}{4}} \cos 2x \, dx}$$

Using integration by parts to find $\int x \cos 2x \, dx$

$$\int \big(f(x)g(x)\big) dx = f(x)\int g(x)\,dx - \int\left(\frac{d}{dx} f(x) \int g(x)\,dx\right)dx$$

$$\int x\cos 2x\,dx = x\int \cos 2x\,dx - \int\left(\frac{d}{dx}x\int \cos 2x\,dx\right)dx$$

$$= x\frac{\sin 2x}{2} - \int\left(1\frac{\sin 2x}{2}\right)dx$$

$$= \frac{x}{2}\sin 2x - \frac{1}{2}\int \sin 2x\,dx$$

$$= \frac{x}{2}\sin 2x - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= \tfrac{1}{4}(2x\sin x + \cos 2x)$$

Therefore, centre of mass is at:

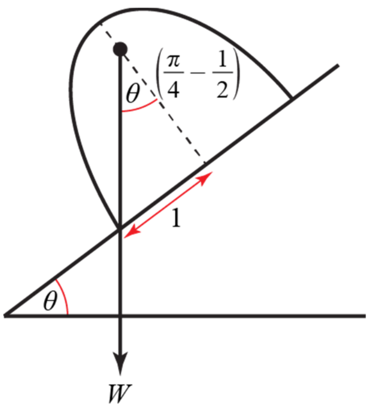
$$\bar{x} = \frac{\frac{1}{4}\left[2x\sin 2x + \cos 2x\right]_0^{\frac{\pi}{4}}}{\frac{1}{2}\left[\sin 2x\right]_0^{\frac{\pi}{4}}}$$

$$\bar{x} = \frac{\frac{1}{4}\left(\frac{\pi}{2}(1+0) - (0+1)\right)}{\frac{1}{2}(1-0)}$$

$$\bar{x} = \frac{1}{2}\left(\frac{\pi}{2} - 1\right)$$

The x coordinate of the centre of mass is $\frac{\pi}{4} - \frac{1}{2}$

c When the solid is about to topple, the weight acts through the base (see diagram).
The radius of the base is the value of y when $x = 0$



$$y_0 = \sqrt{\cos(2 \times 0)}$$

$$y_0 = \sqrt{1} = 1$$

$$\tan\theta = \frac{1}{\frac{\pi}{4} - \frac{1}{2}}$$

$$\tan\theta = \frac{4}{\pi - 2} = 3.5038...$$

$$\theta = 74.071...$$

The solid is on the point of toppling when the plane is inclined at an angle of 74° (to the nearest whole degree) to the horizontal.

6 The starting position is a distance $a\cos 60° = \frac{1}{2}a$ beneath O (see diagram).

Taking this as the zero level for potential energy, the total energy of the particle is the kinetic energy in this position.

$$\text{K.E.} = \frac{1}{2}mv^2$$

$$\text{K.E.} = \frac{1}{2}m \times 3ga$$

So total energy $= \frac{3mga}{2}$

If the tension in the string is T and the angle with the upward vertical is θ, resolving towards the centre of the circle gives:

$$T + mg\cos\theta = \frac{mv^2}{r} = \frac{mv^2}{a}$$

When string first becomes slack, $T = 0$ so this becomes:

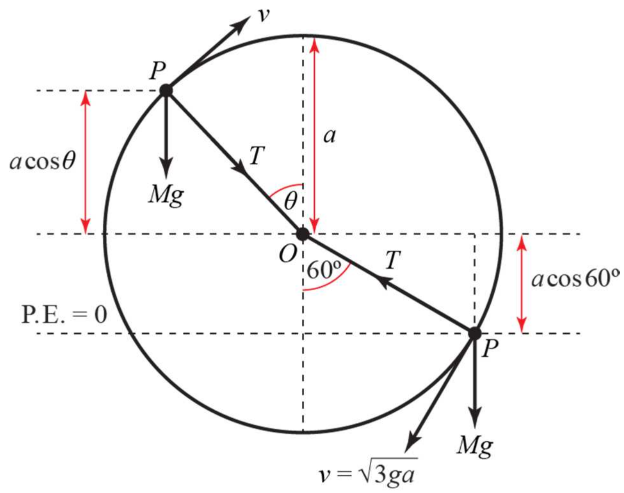
$$mg\cos\theta = \frac{mv^2}{a}$$

$$\Rightarrow ag\cos\theta = v^2 \qquad \textbf{(1)}$$

This happens a distance $a\cos\theta$ above O (see diagram), so the potential energy at this point is:

$$\text{P.E.} = mgh = mg(a\cos\theta + a\cos 60^\circ)$$

Since total energy remains constant:



$$K.E = \frac{3mga}{2} - P.E.$$

$$\frac{1}{2}mv^2 = \frac{3mga}{2} - mg\left(a\cos\theta + \frac{1}{2}a\right)$$

$$v^2 = 3ga - 2ga\cos\theta - ga$$

$$= 2ga - 2ga\cos\theta$$

Substituting from **(1)** for v^2 gives:

$$ga\cos\theta = 2ga - 2ga\cos\theta$$

$$\cos\theta = 2 - 2\cos\theta$$

$$3\cos\theta = 2$$

$$\cos\theta = \frac{2}{3}$$

$$\theta = 48.189...$$

The string first becomes slack when OP makes an angle of 48° with the upward vertical (to nearest whole degree) and P therefore does not reach the top of the circle.