AS Level Exam-style Practice Paper

1 a $a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{144 - v^2}{48}\,\mathrm{ms}^{-2}$

$\frac{1}{144 - v^2}\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{1}{48}$

$\int \frac{1}{144 - v^2}\,\mathrm{d}v = \int \frac{1}{48}\,\mathrm{d}t$

Rearranging gives:

$\frac{1}{144 - v^2} = \frac{1}{(12+v)(12-v)} = \frac{1}{24}\left(\frac{1}{12+v} + \frac{1}{12-v}\right)$

So $\frac{1}{24}\int \frac{1}{12+v} + \frac{1}{12-v}\,\mathrm{d}v = \int \frac{1}{48}\,\mathrm{d}t$

$\frac{1}{24}\ln(12+v) - \frac{1}{24}\ln(12-v) = \frac{t}{48} + C$

$\frac{1}{24}\ln\left(\frac{12+v}{12-v}\right) = \frac{t}{48} + C$

$\ln\left(\frac{12+v}{12-v}\right) = \frac{t}{2} + 24C$

$\left(\frac{12+v}{12-v}\right) = \mathrm{e}^{\frac{t}{2}+24C} = D\mathrm{e}^{\frac{t}{2}}$ where $D = \mathrm{e}^{24C}$

When $t = 0$ s, $v = 0\,\mathrm{ms}^{-1}$, so $\frac{12}{12} = De^0 \Rightarrow D = 1$

So $\left(\frac{12+v}{12-v}\right) = \mathrm{e}^{\frac{t}{2}}$

$12 + v = 12\mathrm{e}^{\frac{t}{2}} - v\mathrm{e}^{\frac{t}{2}}$

$v(\mathrm{e}^{\frac{t}{2}} - 1) = 12(\mathrm{e}^{\frac{t}{2}} - 1)$

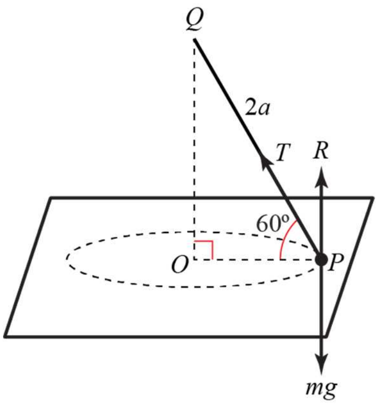
$v = \frac{12(\mathrm{e}^{\frac{t}{2}} - 1)}{\mathrm{e}^{\frac{t}{2}} + 1}$

b For all real t, $e^{\frac{t}{2}} + 1 > e^{\frac{t}{2}} - 1$

Therefore $\frac{e^{\frac{t}{2}} - 1}{e^{\frac{t}{2}} + 1} < 1$ and $\left|\frac{12(e^{\frac{t}{2}} - 1)}{e^{\frac{t}{2}} + 1}\right| < 12$

So the speed of the particle cannot be greater than $12\,\mathrm{ms}^{-1}$

2 a Let the tension in the string be T, the normal reaction between P and the table be R and the angular speed be $\omega = \sqrt{\frac{kg}{4a}}$



Resolving horizontally:

$R(\leftarrow)$: $T\cos 60° = ma = m\omega^2 r$ using $F = ma$ and where r is the radius of the circular motion

By geometry, $r = 2a\cos 60°$ so substituting for r and ω this gives:

$$T\cos 60° = m \times \frac{kg}{4a} \times 2a\cos 60°$$

$$T = \frac{mkg}{2}$$ as required

b Resolving vertically:

$R(\uparrow)$: $T\sin 60° + R = mg$

$$\Rightarrow R = mg - \left(\frac{mkg}{2} \times \frac{\sqrt{3}}{2}\right) = mg\left(1 - \frac{\sqrt{3}k}{4}\right)$$

c For the particle to remain in contact with the table, $R > 0$. Therefore

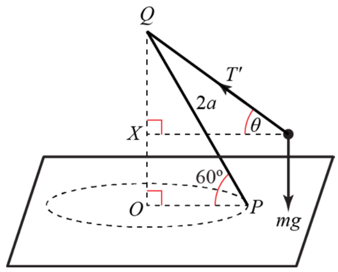
$$1 - \frac{\sqrt{3}k}{4} > 0$$

$$\frac{\sqrt{3}k}{4} < 1$$

$$\sqrt{3}k < 4$$

$$k < \frac{4}{\sqrt{3}}$$

2 d Let the new tension in the string be T', the angle the string now makes with the horizontal be θ, and the angular speed be $\omega' = \sqrt{\frac{3g}{a}}$



So $QO = 2a\sin 60° = \sqrt{3}a$ and $QX = 2a\sin\theta$

Resolving horizontally:

$R(\leftarrow):\ T'\cos\theta = m\omega^2 r = m \times \frac{3g}{a} \times 2a\cos\theta$

$\Rightarrow T' = 6mg$

Resolving vertically:

$T'\sin\theta = mg$, so $6mg\sin\theta = mg$ substituting for T'

$\Rightarrow \sin\theta = \frac{1}{6}$

Substituting into $QX = 2a\sin\theta$ gives $QX = \frac{a}{3}$

Therefore $QX : QO = \frac{a}{3} : \sqrt{3}a = 1 : 3\sqrt{3}$ as required

3 a The rods are uniform, so the centre of mass of each rod is at its midpoint. The particles are treated as point masses.

Let $\bar{x}$ be the centre of mass of the loaded framework is from AD and $\bar{y}$ the centre of mass of the loaded framework is 3.25a from AB. Then using $\Sigma m_i r_i = r\Sigma m_i$, taking A as the origin, AB and AD as x- and y-axes respectively, and working round the figure in the order: AB, B, BC, C, CD, DA:

$$5m\begin{pmatrix}2.5a\\0\end{pmatrix} + 2m\begin{pmatrix}5a\\0\end{pmatrix} + 2m\begin{pmatrix}5a\\0\end{pmatrix} + 4m\begin{pmatrix}5a\\2a\end{pmatrix} + 5m\begin{pmatrix}2.5a\\2a\end{pmatrix} + 2m\begin{pmatrix}0\\a\end{pmatrix} = (5+2+2+4+5+2)m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}12.5a\\0\end{pmatrix} + \begin{pmatrix}10a\\0\end{pmatrix} + \begin{pmatrix}10a\\2a\end{pmatrix} + \begin{pmatrix}20a\\8a\end{pmatrix} + \begin{pmatrix}12.5a\\10a\end{pmatrix} + \begin{pmatrix}0\\2a\end{pmatrix} = 20\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

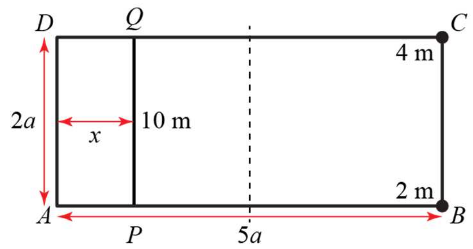
$$\begin{pmatrix}65a\\22a\end{pmatrix} = 20\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

So $\bar{x} = \frac{65a}{20} = 3.25a$ and $\bar{y} = \frac{65a}{20} = 1.1a$ The centre of mass of the framework lies:

i The centre of mass of the loaded framework is 1.1a from AB.

ii The centre of mass of the loaded framework is 3.25a from AD.

3 b Let the distance AP be x.



Treating the existing framework as a single object, taking the same axes as before and letting the centre of mass of the loaded framework including the rod PQ:

$$20m\begin{pmatrix}3.25a\\1.1a\end{pmatrix}+10m\begin{pmatrix}x\\a\end{pmatrix}=30m\begin{pmatrix}2.5a\\z\end{pmatrix}$$

$$\begin{pmatrix}65a+10x\\32a\end{pmatrix}=\begin{pmatrix}75a\\30z\end{pmatrix}$$

$$10x=75a-65a$$

$$x=a$$

The distance AP is a

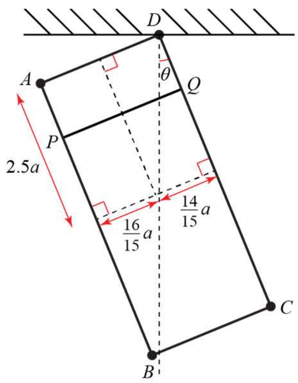
c Using part **b**, the distance from AB to the centre of mass also changes when PQ is added.

$$\begin{pmatrix}65a+10x\\32a\end{pmatrix}=\begin{pmatrix}75a\\30z\end{pmatrix}$$

$$32a=30z$$

$$z=\frac{32a}{30}=\frac{16a}{15}$$

The framework therefore hangs with is centre of mass directly below D and CD making an angle of θ with the vertical.



So $\tan\theta=\dfrac{2-\dfrac{16}{15}}{2.5}=\dfrac{2}{5}\times\dfrac{14}{15}=\dfrac{28}{75}$

$\Rightarrow \theta=20.5°$ (3 s.f.)

The rod DC makes an angle of 20.5° (to 3 s.f.) with the vertical.

d The assumption that the rods are uniform allows the system to be modelled by assuming that the entire weight of each rod acts through its midpoint.