Review exercise 2

1 $a = \frac{dv}{dt} = e^{2t}$

$v = \int e^{2t} dt = \frac{1}{2}e^{2t} + A$

When $t = 0, v = 0$

$0 = \frac{1}{2} + A \Rightarrow A = -\frac{1}{2}$

Hence $v = \frac{1}{2}(e^{2t} - 1)$, as required.

> To find the acceleration, integrate the velocity with respect to time. Remember to include a constant of integration.

2 a $a = \frac{dv}{dt} = \frac{1}{2}e^{-\frac{1}{6}t}$

$v = \int \frac{1}{2}e^{-\frac{1}{6}t} dt = -3e^{-\frac{1}{6}t} + A$

When $t = 0, v = 10$

$10 = -3 + A \Rightarrow A = 13$

Hence $v = 13 - 3e^{-\frac{1}{6}t}$

> Using $\int e^{kt} dt = \frac{1}{k}e^{kt} A$, then
>
> $\int \frac{1}{2}e^{-\frac{1}{6}t} dt = \frac{1}{2 \times \left(-\frac{1}{6}\right)}e^{-\frac{1}{6}t} + A = -\frac{1}{\frac{1}{3}}e^{-\frac{1}{6}t} + A$
>
> $= -3e^{-\frac{1}{6}t} + A$

b When $t = 3$

$v = 13 - 3e^{-\frac{1}{2}} = 11.180...$

The speed of P when $t = 3$ is 11.2 m s^{-1} (3 s.f.)

c As $t \to \infty, e^{-\frac{1}{6}t} \to 0$ and $v \to 13$.

The limiting value of v is 13.

> As t gets large, $e^{-\frac{1}{6}t}$ gets very small. For example, if t = 120, then $e^{-\frac{1}{6}t} \approx 2.06 \times 10^{-9}$. In this question, as t gets larger, v gets closer and closer to 13 and so 13 is the limiting value of v.

3 a $a = \frac{dv}{dt} = 2\sin\frac{1}{2}t$

$v = \int 2\sin\frac{1}{2}t \ dt = -4\cos\frac{1}{2}t + A$

When $t = 0, v = 4$

$4 = -4 + A \Rightarrow A = 8$

Hence $v = 8 - 4\cos\frac{1}{2}t$

> Using the formula $\int \sin at \, dt = -\frac{1}{a}\cos at + A$,
>
> $\int 2\sin\frac{1}{2}t \, dt = -\frac{2}{\frac{1}{2}}\cos\frac{1}{2}t + A = -4\cos\frac{1}{2}t + A$

3 b The distance, s metres, travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is given by

$$s = \int_0^{\frac{\pi}{2}} \left(8 - 4\cos\frac{1}{2}t\right)\mathrm{d}t$$
$$= \left[8t - 8\sin\frac{1}{2}t\right]_0^{\frac{\pi}{2}}$$
$$= 4\pi - 8\sin\frac{\pi}{4} = 4\pi - \frac{8}{\sqrt{2}}$$
$$= 4\pi - 4\sqrt{2} = 4(\pi - \sqrt{2})$$

The change in the displacement of P between any two times, say t_1 and t_2, can be found by calculating the definite integral of the velocity between the limits t_1 and t_2. If P has not turned around, this will also give the distance travelled by P.
The particle in this question does turn around when $\frac{\mathrm{d}v}{\mathrm{d}t} = a = 0$ but that does not happen until $t = 2\pi$, so P does not turn round in the interval $0 \leqslant t \leqslant \frac{\pi}{2}$.

The distance travelled by P between the times $t = 0$ and $t = \frac{\pi}{2}$ is $4(\pi - \sqrt{2})$ m.

4 a $a = \frac{\mathrm{d}v}{\mathrm{d}t} = -4\mathrm{e}^{-2t}$

$v = -\int 4\mathrm{e}^{-2t}\,\mathrm{d}t = 2\mathrm{e}^{-2t} + A$

At $t = 0, v = 1$

$1 = 2 + A \Rightarrow A = -1$

Hence $v = 2\mathrm{e}^{-2t} - 1$

b P is instantaneously at rest when $v = 0$.

$0 = 2\mathrm{e}^{-2t} - 1$

To find the speed when P is instantaneously at rest, you will need to know the value of t when $v = 0$.

$\mathrm{e}^{-2t} = \frac{1}{2} \Rightarrow \mathrm{e}^{2t} = 2$

Take natural logarithms of both sides of this equation and use the property that, for any x, $\ln(\mathrm{e}^x) = x$

$2t = \ln 2 \Rightarrow t = \frac{1}{2}\ln 2$

$x = \int v\,\mathrm{d}t = \int (2\mathrm{e}^{-2t} - 1)\mathrm{d}t$

$= -\mathrm{e}^{-2t} - t + B$

When $t = 0, x = 0$

$0 = -1 - 0 + B \Rightarrow B = 1$

Using $\mathrm{e}^0 = 1$. It is a common error to obtain $B = 0$ by carelessly writing $\mathrm{e}^0 = 0$.

Hence $x = 1 - \mathrm{e}^{-2t} - t$

When $t = \frac{1}{2}\ln 2$

$x = 1 - \mathrm{e}^{-2(\frac{1}{2}\ln 2)} - \frac{1}{2}\ln 2 = 1 - \mathrm{e}^{-\ln 2} - \frac{1}{2}\ln 2$

Using the law of logarithms, $\ln a^x = n\ln a$ with $n = -1$, $-\ln 2 = (-1)\ln 2 = \ln 2^{-1} = \ln\frac{1}{2}$. Then as for any x, $\mathrm{e}^{\ln x} = x, \mathrm{e}^{\ln\frac{1}{2}} = \frac{1}{2}$.

$= 1 - \mathrm{e}^{\ln\frac{1}{2}} - \frac{1}{2}\ln 2 = 1 - \frac{1}{2} - \frac{1}{2}\ln 2$

$= \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$

The distance from P from O when P comes to instantaneous rest is $\frac{1}{2}(1 - \ln 2)$ m.

5 a $a = \frac{dv}{dt} = -\frac{3}{\sqrt{(t+4)}} = -3(t-4)^{-\frac{1}{2}}$

As the acceleration is towards O, $\frac{dv}{dt}$, which is always measured in the direction of x increasing, is negative.

$v = -3\int (t+4)^{-\frac{1}{2}} dt = \frac{-3(t+4)^{\frac{1}{2}}}{\frac{1}{2}} + A = A - 6(t+4)^{\frac{1}{2}}$

When $t = 0, v = 18$

$18 = A - 6 \times 2 \Rightarrow A = 30$

Hence

$v = 30 - 6(t+4)^{\frac{1}{2}}$

The velocity of P is $\left[30 - 6\sqrt{(t+4)}\right]$ m s^{-1}, as required.

b $0 = 30 - 6(t+4)^{\frac{1}{2}}$

$(t+4)^{\frac{1}{2}} = 5 \Rightarrow t + 4 = 25 \Rightarrow t = 21$

There are three steps needed to solve part **b**. First you must find the value of t for which P is instantaneously at rest; that is when $v = 0$. You must also find x in terms of t by integrating the expression you proved in part **a**. Finally you substitute your value of t into your expression for x. It is a characteristic of harder questions at this level that you often have to construct for yourself the steps needed to solve a problem.

$v = \frac{dx}{dt} = 30 - 6(t+4)^{\frac{1}{2}}$

$x = \int \left(30 - 6(t+4)^{\frac{1}{2}}\right) dt = 30t - \frac{6(t+4)^{\frac{3}{2}}}{\frac{3}{2}} + B$

$= 30t - 4(t+4)^{\frac{3}{2}} + B$

When $t = 0, x = 0$

$0 = 0 - 4 \times 4^{\frac{3}{2}} + B$

$B = 4 \times 4^{\frac{3}{2}} = 4 \times 8 = 32$

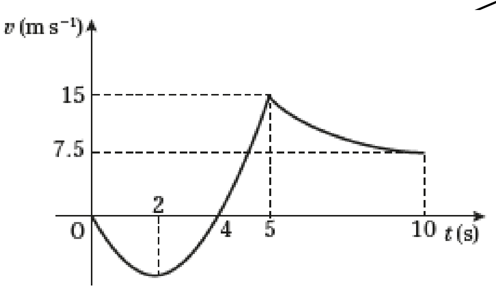
Hence $x = 30t - 4(t+4)^{\frac{3}{2}} + 32$

When $t = 21$

$x = 30 \times 21 - 4(25)^{\frac{3}{2}} + 32 = 630 - 500 + 32 = 162$

The distance of P from O when P comes to instantaneous rest is 162 m.

6 a



In the interval $0 \leqslant t \leqslant 5$, the graph is part of a parabola which meets the t-axis at the origin and where $t = 4$. In the interval $5 < t \leqslant 10$, the graph is a segment of a hyperbola joining $(5, 15)$ to $(10, 7.5)$.

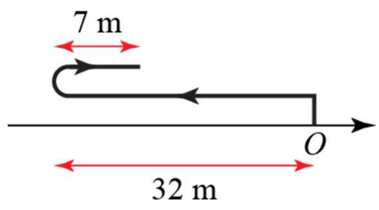
6 b The set of values of t for which the acceleration is positive is $2 < t < 5$.

The acceleration is positive when the velocity–time graph has a positive gradient. By the symmetry of a parabola, the graph has a minimum when $t = 2$ and the set of values of t for which the gradient is positive can be written down by inspecting the graph.

c
$$\int_0^4 3t(t-4)\,dt = \int_0^4 (3t^2 - 12t)\,dt$$
$$= \left[t^3 - 6t^2\right]_0^4$$
$$= (64 - 96) - 0 = -32$$
$$\int_4^5 3t(t-4)\,dt = \int_4^5 (3t^2 - 12t)\,dt$$
$$= \left[t^3 - 6t^2\right]_4^5$$
$$= (125 - 150) - (64 - 96)$$
$$= 7$$

The distance travelled by P in the interval $0 \leqslant t \leqslant 5$ is $(32 + 7)\text{m} = 39\,\text{m}$.

Taking the direction of v increasing as positive, for the first 4 seconds the particle travels 32 m in the negative direction. In the next second, it travels 7 m in the positive direction. So in 5 seconds, it travels a total of (32 + 7)m ending at a point which is (32–7)m from O in the negative direction.



d For $t > 5$
$$x = \int v\,dt = \int 75t^{-1}\,dt$$
$$= 75\ln t + A$$

At time $t = 5$, the particle is $(32 - 7)\text{m} = 25\text{m}$ from O in the negative direction.

So when $t = 5, x = -25$

$$-25 = 75\ln 5 + A \Rightarrow A = -75\ln 5 - 25$$

Using the law of logarithms
$$\ln a - \ln b = \ln\left(\frac{a}{b}\right),$$
$$75\ln t - 75\ln 5 = 75(\ln t - \ln 5) = 75\ln\left(\frac{t}{5}\right).$$

Hence
$$x = 75\ln t - 75\ln 5 - 25 = 75\ln\left(\frac{t}{5}\right) - 25$$

At $x = 0$
$$0 = 75\ln\left(\frac{t}{5}\right) - 25 \Rightarrow \ln\left(\frac{t}{5}\right) = \frac{1}{3}$$
$$\frac{t}{5} = e^{\frac{1}{3}} \Rightarrow t = 5e^{\frac{1}{3}} = 6.98\ (3\ \text{s.f.})$$

You solve this equation for t by taking exponentials of both sides of the equation and using $e^{\ln\left(\frac{t}{5}\right)} = \frac{t}{5}$.

7 a $v = \dfrac{1}{p+qt} = (p+qt)^{-1}$

$a = \dfrac{\mathrm{d}v}{\mathrm{d}t} = (-1)q(p+qt)^{-2}$

$= -\dfrac{q}{(p+qt)^2} = -qv^2$

The deceleration is the negative of the acceleration. If $d = kv^n$, for any constant k, then d is proportional to v^n.

So the deceleration is proportional to the square of the speed.

The square of the speed and the square of the velocity are identical because, for example, $(-20)^2 = 20^2$.

b When $t = 0, a = -0.75$ and $v = 20$

$a = -qv^2$

$-0.75 = -q \times 20^2$

$q = \dfrac{3}{4} \times \dfrac{1}{20^2} = \dfrac{3}{1600}$

The exact decimal answers $p = 0.05$ and $q = 0.001875$ are also acceptable.

$v = \dfrac{1}{p+qt}$

When $t = 0, v = 20$

$20 = \dfrac{1}{p} \Rightarrow p = \dfrac{1}{20}$

$p = \dfrac{1}{20}, q = \dfrac{3}{1600}$

c $v = \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{p+qt}$

$x = \displaystyle\int \frac{1}{p+qt}\,\mathrm{d}t = \frac{1}{q}\ln(p+qt) + A$

$= \dfrac{1600}{3}\ln\left(\dfrac{1}{20} + \dfrac{3}{1600}t\right) + A$

When $t = 0, x = 0$

$0 = \dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right) + A \Rightarrow A = -\dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right)$

Hence $x = \dfrac{1600}{3}\ln\left(\dfrac{1}{20} + \dfrac{3}{1600}t\right) - \dfrac{1600}{3}\ln\left(\dfrac{1}{20}\right)$

This expression can be simplified using the law of logarithms $\ln a - \ln b = \ln\dfrac{a}{b}$.

However, as the question specifies no particular form for the answer, an unsimplified answer or an answer with decimals would be accepted.

$= \dfrac{1600}{3}\ln\left(\dfrac{\dfrac{1}{20} + \dfrac{3}{1600}t}{\dfrac{1}{20}}\right)$

$x = \dfrac{1600}{3}\ln\left(1 + \dfrac{3}{80}t\right)$

8 $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x$

$\frac{1}{2}v^2 = \int 4x\,dx = 2x^2 + A$

$v^2 = 4x^2 + B$, where $B = 2A$

At $x = 2, v = 4$

$16 = 16 + B \Rightarrow B = 0$

Hence

$v^2 = 4x^2$

When the acceleration is a function of the displacement, x metres, you write $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ and integrate both sides of the equation with respect to x.

Even when, as here, the constant of integration is 0, it is essential for you to show how this follows from the information given in the question to gain full marks.

9 $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 1 - \frac{4}{x^2} = 1 - 4x^{-2}$

$\frac{1}{2}v^2 = \int (1 - 4x^{-2})dx$

$= x - \frac{4x^{-1}}{-1} + A = x + \frac{4}{x} + A$

$v^2 = 2x + \frac{8}{x} + B$, where $B = 2A$

At $x = 1, v = 3\sqrt{2}$

$18 = 2 + 8 + B \Rightarrow B = 8$

Hence $v^2 = 2x + \frac{8}{x} + 8$

At $x = \frac{3}{2}$

$v^2 = 2 \times \frac{3}{2} + 8 \times \frac{2}{3} + 8 = 11 + \frac{16}{3} = \frac{49}{3}$, as required.

Multiplying the equation $\frac{1}{2}v^2 = 2x + \frac{4}{x} + A$ throughout by 2.
Twice one arbitrary constant is another arbitrary constant.

You use the information that at $x = 1$, $v = 3\sqrt{2}$ to evaluate the constant of integration B. You then substitute $x = \frac{3}{2}$ into the resulting equation and show that $v^2 = \frac{49}{3}$.

10 $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 5 + 3\sin 3x$

$\frac{1}{2}v^2 = \int (5 + 3\sin 3x)dx = 5x - \cos 3x + A$

$v^2 = 10x - 2\cos 3x + B$, where $B = 2A$

At $x = 0, v = 4$

$16 = 0 - 2 + B \Rightarrow B = 18$

Hence $v^2 = 10x - 2\cos 3x + 18$

At $x = 6$

$v^2 = 60 - 2\cos 18 + 18 = 76.679...$

$v = \sqrt{76.679...} = 8.756...$

The speed of P at $x = 6$ is 8.76 m s^{-1} (3 s.f.)

You use the information that at $x = 0$, $v = 4$ to evaluate the constant of integration B. You then substitute $x = 6$ into the resulting equation and use your calculator to find v.

When calculus has been used, it is assumed that all angles are measured in radians and you must make sure that your calculator is in the correct mode.

11 a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{4k^2}{(x+1)^2} = 4k^2(x+1)^{-2}$

$\frac{1}{2}v^2 = \int 4k^2(x+1)^{-2}\,dx = \frac{4k^2(x+1)^{-1}}{-1} + A$

$v^2 = B - \frac{8k^2}{x+1}$, where $B = 2A$

At $x = 1, v = 0$

$0 = B - \frac{8k^2}{2} \Rightarrow B = 4k^2$

Hence $v^2 = 4k^2 - \frac{8k^2}{x+1} = 4k^2\left(1 - \frac{2}{x+1}\right)$

b $v = 2k\sqrt{\left(1 - \frac{2}{x+1}\right)}$

As P is moving on the positive x-axis in the direction of x increasing, you need not consider the possibility of a negative square root.

As x is positive, $1 - \frac{1}{x+1} < 1$

As x is positive, $\frac{1}{1+x}$ is positive and one minus a positive number must be less than one.

Hence $v < 2k$ and v cannot exceed $2k$.

12 a At the maximum value of v, $\frac{dv}{dt} = 0$.

As $a = \frac{dv}{dt}$, the maximum speed of P occurs when $a = \frac{1}{12}(30 - x) = 0 \Rightarrow x = 30$.

b $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{12}(30 - x)$

$\frac{1}{2}v^2 = \int \frac{1}{12}(30 - x)dx = \int\left(\frac{5}{2} - \frac{x}{12}\right)dx$

$= \frac{5x}{2} - \frac{x^2}{24} + A$

$v^2 = 5x - \frac{x^2}{12} + B$, where $B = 2A$

Multiplying the equation $\frac{1}{2}v^2 = \frac{5x}{2} - \frac{x^2}{24} + A$ throughout by 2. Twice one arbitrary constant is another arbitrary constant.

At $x = 30, v = 10$

$100 = 5 \times 30 - \frac{900}{12} + B$

$B = 100 + \frac{900}{12} - 150 = 25$

Hence $v^2 = 5x - \frac{x^2}{12} + 25$

An alternative form of this answer, completing the square, is $v^2 = 100 - \frac{1}{12}(30 - x)^2$. This confirms that the speed has a maximum at $x = 30$.

13 a $u = 0, a = 20, v = 6, s = ?$

$v^2 = u^2 + 2as$

$36 = 0 + 2 \times 20 \times s$

$s = \frac{36}{40} = 0.9$

For the first model, the distance moved by P while accelerating from rest to 6 m s^{-1} is 0.9 m.

The model in part **a** is that of constant acceleration, which you studied in module M1. The specification for M3 includes 'a knowledge of the specifications for M1 and M2 and their prerequisites and associated formulae ... is assumed and may be tested'.

b $a = p - qx$

At $x = 0, a = 20$

$20 = p - 0 \Rightarrow p = 20$

Hence $a = 20 - qx$

At $x = 2, a = 12$

$12 = 20 - 2q \Rightarrow q = \frac{20 - 12}{2} = 4$

$p = 20, q = 4,$ as required.

The initial acceleration is 20 m s^{-2}. This applies to all parts of the question. Additionally in part **b**, you are given that $a = 12$ when $x = 2$. The two conditions enable you to find the two unknowns p and q.

c $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 20 - 4x$

$\frac{1}{2}v^2 = \int (20 - 4x)dx = 20x - 2x^2 + A$

$v^2 = 40x - 4x^2 + B,$ where $B = 2A$

At $x = 0, v = 0$

$0 = 0 - 0 + B \Rightarrow B = 0$

Hence $v^2 = 40x - 4x^2$

When $v = 6$

$36 = 40x - 4x^2 \Rightarrow 4x^2 - 40x + 36 = 0$

$x^2 - 10x + 9 = (x - 1)(x - 9) = 0$

$x = 1, 9$

Divide this equation throughout by 4 and factorise.

The distance moved by P in first attaining a speed of 6 m s^{-1} is 1 m.

Comparing this with result in part **a**, the revised model predicts that P moves a little further before reaching the speed of 6 m s^{-1}.

14 a $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{72}{(2x+1)^2} = 72(2x+1)^{-2}$

$\frac{1}{2}v^2 = \int 72(2x+1)^{-2}\,dx = \frac{72(2x+1)^{-1}}{2\times(-1)} + A$

$= A - \frac{36}{2x+1}$

$v^2 = B - \frac{72}{2x+1}$, where $B = 2A$

At $x = 1, v = -6$

$36 = B - \frac{72}{3} \Rightarrow B = 60$

Hence $v^2 = 60 - \frac{72}{2x+1}$

To integrate $(2x+1)^{-2}$, you can use the formula $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + A$.

b At a minimum value of x, $\frac{dx}{dt} = 0$

and so $v = 0$.

In this question, it is not practical to find x in terms of t. However, to find the minimum value of x, this is not necessary. The minimum is a stationary value and at a stationary value $\frac{dx}{dt}$, which is v, is zero

Substituting $v = 0$ into the result of part **a**

$0 = 60 - \frac{72}{2x+1} \quad \Rightarrow 60 = \frac{72}{2x+1}$

$2x+1 = \frac{72}{60} = 1.2 \Rightarrow x = \frac{1.2-1}{2} = 0.1$

The minimum distance of P from O is 0.1 m.

15 a $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{k}{2x^2} + \frac{k}{4c^2} = \frac{k}{2}x^{-2} + \frac{k}{4c^2}$

$$\frac{1}{2}v^2 = \int\left(\frac{k}{2}x^{-2} + \frac{k}{4c^2}\right)\mathrm{d}x = \frac{k}{2}\times\frac{x^{-1}}{-1} + \frac{kx}{4c^2} + A$$

$$= -\frac{k}{2x} + \frac{kx}{4c^2} + A$$

$$v^2 = -\frac{k}{x} + \frac{kx}{2c^2} + B, \text{where } B = 2A$$

At $x = 2c, v = -\sqrt{\left(\frac{k}{c}\right)}$

$$\frac{k}{c} = -\frac{k}{2c} + \frac{2kc}{2c^2} + B$$

$$B = \frac{k}{c} + \frac{k}{2c} - \frac{k}{c} = \frac{k}{2c}$$

Hence $v^2 = -\frac{k}{x} + \frac{kx}{2c^2} + \frac{k}{2c}$

To show that the particle comes to rest at A, you use integration to obtain v^2 in terms of x, and then substitute $x = c$ into your expression and show that $v = 0$.

At $x = c$

$$v^2 = -\frac{k}{c} + \frac{kc}{2c^2} + \frac{k}{2c} = -\frac{k}{c} + \frac{k}{2c} + \frac{k}{2c} = 0$$

The particle comes to instantaneous rest where $x = c$.

b $a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = \frac{k}{2x^2} - \frac{k}{4c^2} = \frac{k}{2}x^{-2} - \frac{k}{4c^2}$

$$\frac{1}{2}v^2 = \int\left(\frac{k}{2}x^{-2} - \frac{k}{4c^2}\right)\mathrm{d}x = \frac{k}{2}\times\frac{x^{-1}}{-1} - \frac{kx}{4c^2} + C$$

$$= -\frac{k}{2x} - \frac{kx}{4c^2} + C$$

$$v^2 = -\frac{k}{x} - \frac{kx}{2c^2} + D, \text{where } D = 2C$$

At $x = c, v = 0$

Although the acceleration changes at B you can assume that the velocity is continuous and that the final velocity, 0, in part **a** is the initial velocity in part **b**.

$$0 = -\frac{k}{c} - \frac{k}{2c} + D \Rightarrow D = \frac{3k}{2c}$$

Hence

$$v^2 = -\frac{k}{x} - \frac{kx}{2c^2} + \frac{3k}{2c} = \frac{-2kc^2 - kx^2 + 3kcx}{2c^2x}$$

$$= -\frac{k}{2c^2x}(x^2 - 2cx + 2c^2) = -\frac{k}{2c^2x}(x - c)(x - 2c)$$

There are a number of different ways of completing this question. The solution shown here puts all of the terms on the right of the equation over a common denominator and factorises the resulting expression.

When $v = 0, (x - c)(x - 2c) = 0$

$x = c, 2c$

$x = c$ corresponds to the point B and $x = 2c$ corresponds to the point A.

After leaving B, the particle next comes to rest at A, where $x = 2c$.

16 a $\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{v^2}{10}$

Separating the variables and integrating:

$$-\int_{12}^{6} \frac{\mathrm{d}v}{v^2} = \int_{0}^{T} \frac{\mathrm{d}t}{10}$$
$$\left[\frac{1}{v}\right]_{12}^{6} = \frac{T}{10}$$
$$\frac{1}{6} - \frac{1}{12} = \frac{T}{10}$$
$$\frac{1}{12} = \frac{T}{10}$$
$$T = \frac{5}{6}$$

b $v\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{v^2}{10}$

Separating the variables, setting $X = OA$ and integrating:

$$-\int_{12}^{6} \frac{\mathrm{d}v}{v} = \int_{0}^{X} \frac{\mathrm{d}x}{10}$$
$$\left[-\ln v\right]_{12}^{6} = \frac{X}{10}$$
$$\ln 12 - \ln 6 = \frac{X}{10}$$
$$\ln 2 = \frac{X}{10}$$
$$X = OA = 10\ln 2$$

17 a $v\frac{\mathrm{d}v}{\mathrm{d}x} = -(k + v)$

Separating the variables, setting $X = OA$ and integrating:

$$\int_{U}^{0} \frac{v}{k+v}\,\mathrm{d}v = -\int_{0}^{X} \mathrm{d}x$$
$$\int_{U}^{0} \left(1 - \frac{k}{k+v}\right)\mathrm{d}v = -\int_{0}^{X} \mathrm{d}x$$
$$\left[v - k\ln(k+v)\right]_{U}^{0} = -X$$
$$(0 - k\ln k) - (U - k\ln(k+U)) = -X$$
$$X = U - k\ln(k+U) + k\ln k$$
$$X = OA = U + k\ln\left(\frac{k}{k+U}\right)\text{m}$$

17 b $\frac{\mathrm{d}v}{\mathrm{d}t} = -(k+v)$

Separating the variables and integrating:

$$\int_U^0 \frac{\mathrm{d}v}{k+v} = -\int_0^T \mathrm{d}t$$

$$\left[\ln(k+v)\right]_U^0 = -T$$

$$\ln k - \ln(k+U) = -T$$

$$T = \ln(k+U) - \ln k$$

$$T = \ln\left(\frac{k+U}{k}\right) \text{ s}$$

18 a $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{169-v^2}{100}$

Separating the variables and integrating:

$$\int\frac{\mathrm{d}v}{169-v^2}=\int\frac{\mathrm{d}t}{100}$$
$$\int\frac{\mathrm{d}v}{(13+v)(13-v)}=\int\frac{\mathrm{d}t}{100}$$

Using partial fractions,
$$\frac{1}{(13+v)(13-v)}=\frac{\frac{1}{26}}{13+v}+\frac{\frac{1}{26}}{13-v}$$

So $\int\left(\frac{\frac{1}{26}}{13+v}+\frac{\frac{1}{26}}{13-v}\right)\mathrm{d}v=\int\frac{\mathrm{d}t}{100}$
$$\frac{1}{26}\int\left(\frac{1}{13+v}+\frac{1}{13-v}\right)\mathrm{d}v=\int\frac{\mathrm{d}t}{100}$$
$$\frac{1}{26}\left(\ln(13+v)-\ln(13-v)\right)=\frac{t}{100}+c$$

When $v=0, t=0$, therefore $c=0$

$$\frac{1}{26}\left(\ln(13+v)-\ln(13-v)\right)=\frac{t}{100}$$
$$\frac{1}{26}\ln\left(\frac{13+v}{13-v}\right)=\frac{t}{100}$$
$$\ln\left(\frac{13+v}{13-v}\right)=\frac{26t}{100}$$
$$\frac{13+v}{13-v}=e^{\frac{13t}{50}}$$
$$13+v=13e^{\frac{13t}{50}}-ve^{\frac{13t}{50}}$$
$$v+ve^{\frac{13t}{50}}=13e^{\frac{13t}{50}}-13$$
$$v\left(1+e^{\frac{13t}{50}}\right)=13\left(e^{\frac{13t}{50}}-1\right)$$
$$v=\frac{13\left(e^{\frac{13t}{50}}-1\right)}{\left(e^{\frac{13t}{50}}+1\right)}\ \mathrm{m\,s^{-1}}$$

18 b Since $\frac{\left(e^{\frac{13t}{50}}-1\right)}{\left(e^{\frac{13t}{50}}+1\right)}<1$ for $t\geq 0$,

$v<13$

Therefore the speed of the van cannot exceed $13\ \text{m s}^{-1}$

19 a $\frac{\mathrm{d}v}{\mathrm{d}t}=6-\frac{v}{5}=\frac{30-v}{5}$

Separating the variables and integrating:

$$\int\frac{\mathrm{d}v}{30-v}=\int\frac{\mathrm{d}t}{5}$$

$$-\ln(30-v)=\frac{t}{5}+c$$

When $t=0, v=0,$ therefore $c=-\ln 30$

$$\ln 30-\ln(30-v)=\frac{t}{5}$$

$$\ln\left(\frac{30}{30-v}\right)=\frac{t}{5}$$

$$\frac{30}{30-v}=e^{\frac{t}{5}}$$

$$30=30e^{\frac{t}{5}}-ve^{\frac{t}{5}}$$

$$ve^{\frac{t}{5}}=30e^{\frac{t}{5}}-30$$

$$v=\frac{30\left(e^{\frac{t}{5}}-1\right)}{e^{\frac{t}{5}}}$$

b $v=30-30e^{-\frac{t}{5}}$

As $t\to\infty$, $v\to 30\ \text{m s}^{-1}$

20 a $\frac{\mathrm{d}v}{\mathrm{d}t} = -\left(a^2 + v^2\right)$

Separating the variables and integrating:

$$\int_{20}^{12} \frac{\mathrm{d}v}{a^2 + v^2} = -\int_{0}^{T} \mathrm{d}t$$

$$\left[\frac{1}{a}\arctan\left(\frac{v}{a}\right)\right]_{20}^{12} = -T$$

$$\frac{1}{a}\arctan\left(\frac{12}{a}\right) - \frac{1}{a}\arctan\left(\frac{20}{a}\right) = -T$$

$$T = \frac{1}{a}\arctan\left(\frac{20}{a}\right) - \frac{1}{a}\arctan\left(\frac{12}{a}\right)$$

b $v\frac{\mathrm{d}v}{\mathrm{d}x} = -\left(a^2 + v^2\right)$

Separating the variables and integrating:

$$\int_{20}^{12} \frac{v}{a^2 + v^2}\,\mathrm{d}v = -\int_{0}^{X} \mathrm{d}x$$

$$\frac{1}{2}\left[\ln\left(a^2 + v^2\right)\right]_{20}^{12} = -X$$

$$\frac{1}{2}\ln\left(a^2 + 144\right) - \frac{1}{2}\ln\left(a^2 + 400\right) = -X$$

$$X = \frac{1}{2}\ln\left(a^2 + 400\right) - \frac{1}{2}\ln\left(a^2 + 144\right)$$

$$X = \frac{1}{2}\ln\left(\frac{a^2 + 400}{a^2 + 144}\right)$$

21

$$\mathbf{F} = m\mathbf{a}$$

$$\frac{5}{x+1} = 0.2a$$

$$a = \frac{5}{0.2(x+1)} = \frac{25}{x+1}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{25}{x+1}$$

$$\frac{1}{2}v^2 = \int \frac{25}{x+1}\,dx = 25\ln(x+1) + A$$

$$v^2 = 50\ln(x+1) + B, \text{ where } B = 2A$$

At $x = 0, v = 5$

$25 = 50\ln 1 + B \Rightarrow B = 25$

Hence $v^2 = 50\ln(x+1) + 25$

When $v = 15$

$$15^2 = 50\ln(x+1) + 25$$

$$\ln(x+1) = \frac{225-25}{50} = 4$$

$$x+1 = e^4$$

$$x = e^4 - 1 = 53.6 \text{ (3 s.f.)}$$

You need to remember that $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

You integrate both sides of this equation with respect to x. It is important to include a constant of integration. Most questions include information which enables you to find the constant.

You use the information that $v = 5$ when $x = 0$ to evaluate the constant.

Take exponentials of both sides of this equation and use $e^{\ln(x+1)} = x+1$.

22 a

$$\mathbf{F} = m\mathbf{a}$$

$$2e^{-0.1x} = 2.5a$$

$$a = \frac{2}{2.5}e^{-0.1x} = 0.8e^{-0.1x}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 0.8e^{-0.1x}$$

$$\frac{1}{2}v^2 = \int 0.8e^{-0.1x}\,dx = \frac{0.8e^{-0.1x}}{-0.1} + A$$

$$= -8e^{-0.1x} + A$$

$$v^2 = B - 16e^{-0.1x}, \text{ where } B = 2A$$

When $x = 0, v = 2$

$4 = B - 16 \Rightarrow B = 20$

Hence $v^2 = 20 - 16e^{-0.1x}$

Using $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

Twice one arbitrary constant is another arbitrary constant.

At $x = 0, e^{-0.1x} = e^0 = 1$.

b When $v = 4$

$$16 = 20 - 16e^{-0.1x}$$

$$e^{-0.1x} = \frac{20-16}{16} = \frac{1}{4}$$

$$-0.1x = \ln\left(\frac{1}{4}\right) = -\ln 4$$

$$x = 10\ln 4$$

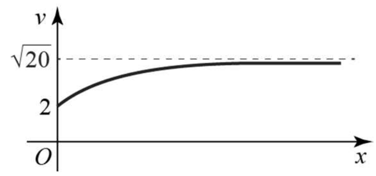
Take logarithms of both sides of this equation and use $\ln(e^{-0.1x}) = -0.1x$.

22 c For all x, $e^{-0.1x} > 0$

So $v^2 = 20 - 16e^{-0.1x} < 20$

and hence $v < \sqrt{20}$

The speed of P does not exceed $\sqrt{20}\ \text{m s}^{-1}$.



This sketch, of v against x, shows that as x increases, the velocity of P gets closer to $\sqrt{20}\ \text{m s}^{-1}$ but never reaches it. $\sqrt{20}\ \text{m s}^{-1}$ is the **terminal** or **limiting speed** of P.

23 a $\mathbf{F} = m\mathbf{a}$

$$\frac{1}{10}x(4-3x) = 0.2a$$

$$a = \frac{1}{0.2\times 10}x(4-3x) = \frac{1}{2}x(4-3x) = 2x - \frac{3x^2}{2}$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x - \frac{3x^2}{2}$$

Integrate both sides of this equation with respect to x. Remember to include a constant of integration.

$$\frac{1}{2}v^2 = \int\left(2x - \frac{3x^2}{2}\right)dx = x^2 - \frac{x^3}{2} + A$$

$$v^2 = 2x^2 - x^3 + B, \text{where } B = 2A$$

At $x = 6, v = 0$

The car comes to instantaneous rest when $x = 6$. So $v = 0$ at $x = 6$.

$0 = 2\times 36 - 216 + B \Rightarrow B = 144$

Hence $v^2 = 2x^2 - x^3 + 144$

b When $x = 0$

$v^2 = 144 \Rightarrow v = \pm 12$

Both signs are possible as the car could pass through O in either direction when $t = 0$. However, in either case, the speed of the car, which is the magnitude of the velocity, is 12 m s^{-1}.

The initial speed of the car is 12 m s^{-1}.

24 a $\mathbf{F} = m\mathbf{a}$

$3e^{-0.5t} = 0.6a$

$a = \frac{3e^{-0.5t}}{0.6} = 5e^{-0.5t}$

$\frac{dv}{dt} = 5e^{-0.5t}$

When the acceleration is a function of time, you use $a = \frac{dv}{dt}$. When the acceleration is a function of distance, you can use $a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$.

$v = \int 5e^{-0.5t}\ dt = \frac{5e^{-0.5t}}{-0.5} + A = A - 10e^{-0.5t}$

When $t = 0, v = 2$

$2 = A - 10 \Rightarrow A = 12$

Hence $v = 12 - 10e^{-0.5t}$

When $v = 8$

$8 = 12 - 10e^{-0.5t} \Rightarrow e^{-0.5t} = \frac{12-8}{10} = \frac{2}{5}$

$-0.5t = \ln\left(\frac{2}{5}\right)$

$t = -2\ln\left(\frac{2}{5}\right) = 2\ln\left(\frac{5}{2}\right)$

No particular form of the answer is asked for in the question and an approximate answer, such as $t = 1.83$, would be accepted.

b $v = \frac{dx}{dt} = 12 - 10e^{-0.5t}$

$x = \int (12 - 10e^{-0.5t})\ dt = 12t + 20e^{-0.5t} + B$

When $t = 0, x = 0$

$0 = 0 + 20 + B \Rightarrow B = -20$

Using $e^0 = 1$. Carelessly writing $e^0 = 0$ is a common error.

$x = 12t + 20e^{-0.5t} - 20$

When $t = 2$

$x = 24 + 20e^{-1} - 20 = 4 + 20e^{-1}$

The distance of P from O when $t = 2$ is $(4 + 20e^{-1})$ m.

25 a $\mathbf{F} = m\mathbf{a}$

$$\frac{48\,000}{(t+2)^2} = 800a$$

$$a = \frac{dv}{dt} = \frac{60}{(t+2)^2} = 60(t+2)^{-2}$$

When the acceleration is a function of t, the velocity can be found by writing $a = \frac{dv}{dt}$ and integrating with respect to t.

$$v = \int 60(t+2)^{-2}\,dt = \frac{60(t+2)^{-1}}{-1} + A$$

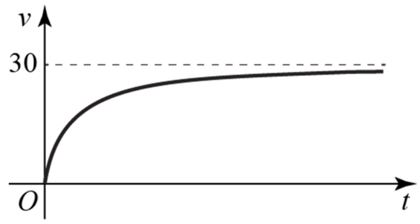
$$= A - \frac{60}{t+2}$$

When $t = 0, v = 0$

$$0 = A - \frac{60}{2} \Rightarrow A = 30$$

Hence $v = 30 - \frac{60}{t+2}$

As $t \to \infty, \frac{60}{t+2} \to 0$ and $v \to 30$



As the value of t increases, the value of $\frac{60}{t+2}$ decreases and so $30 - \frac{60}{t+2}$ gets closer to 30. The graph of v against t approaches $v = 30$ as an asymptote.

As t increases, the car approaches a limiting speed of 30 m s^{-1}.

b The distance moved in the first 6 s is given by

$$x = \int_0^6 \left(30 - \frac{60}{t+2}\right) dt$$

$$= \left[30t - 60\ln(t+2)\right]_0^6$$

$$= (180 - 60\ln 8) - (0 - 60\ln 2)$$

$$= 180 - 60\ln 2^3 + 60\ln 2$$

$$= 180 - 180\ln 2 + 60\ln 2$$

$$= 180 - 120\ln 2$$

The car is always travelling in the same direction. It does not turn round and so the distance moved in the interval $0 \leqslant t \leqslant 6$ can be found by evaluating the definite integral $\int_0^6 v\,dt$.

The distance moved by the car in the first 6 s of its motion is (180 – 120 ln 2) m.

26 a

$$\mathbf{F} = m\mathbf{a}$$

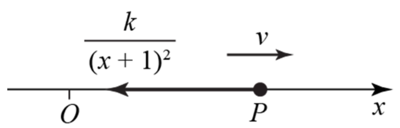
$$-\frac{k}{(x+1)^2} f = \frac{1}{3} a$$

$$a = -\frac{3k}{(x+1)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -3k(x+1)^{-2}$$

$$\frac{1}{2}v^2 = \frac{-3k(x+1)^{-1}}{-1} + A = \frac{3k}{x+1} + A$$

$$v^2 = \frac{6k}{x+1} + B, \text{where } B = 2A$$



The particle is moving along the positive x-axis and the force is directed toward O. So the force is in the direction of x decreasing and force has a negative sign in this equation.

At $x = 1, v = 4$

$$16 = \frac{6k}{2} + B \Rightarrow 3k + B = 16 \qquad \textbf{(1)}$$

At $x = 8, v = \sqrt{2}$

$$2 = \frac{6k}{9} + B \Rightarrow \frac{2}{3}k + B = 2 \qquad \textbf{(2)}$$

The information in the question gives you the values of v at two values of x and you use the information to obtain two simultaneous equations, which you solve.

(1) − (2)

$$3k - \frac{2}{3}k = \frac{7}{3}k = 14$$

$$k = 14 \times \frac{3}{7} = 6$$

b Substituting $k = 6$ into **(1)**

$$18 + B = 16 \Rightarrow B = -2$$

Hence $v^2 = \frac{36}{x+1} - 2$

To find the value of x for which P comes to rest, substitute $v = 0$ into this equation and solve for x.

When $v = 0$

$$0 = \frac{36}{x+1} - 2 \Rightarrow 2(x+1) = 36$$

$$2x + 2 = 36 \Rightarrow x = \frac{36-2}{2} = 17$$

The distance of P from O when P first comes to instantaneous rest is 17 m.

27 a Applying $F = ma$:

$$0.7g - 2.1v = 0.7\frac{dv}{dt}$$

$$g - 3v = \frac{dv}{dt}$$

Separating the variables and integrating:

$$\int \frac{dv}{g-3v} = \int dt$$

$$-\frac{1}{3}\ln(g-3v) = t + c$$

When $t = 0, v = 0,$ therefore $c = -\frac{1}{3}\ln g$

$$\frac{1}{3}\ln g - \frac{1}{3}\ln(g-3v) = t$$

$$\frac{1}{3}\ln\left(\frac{g}{g-3v}\right) = t$$

$$\ln\left(\frac{g}{g-3v}\right) = 3t$$

$$\frac{g}{g-3v} = e^{3t}$$

$$g = e^{3t}(g-3v)$$

$$g = ge^{3t} - 3ve^{3t}$$

$$3ve^{3t} = ge^{3t} - g$$

$$3ve^{3t} = g(e^{3t}-1)$$

$$v = \frac{g(e^{3t}-1)}{3e^{3t}} \text{ m s}^{-1}$$

b $$\frac{dx}{dt} = \frac{g(e^{3t}-1)}{3e^{3t}}$$

$$\frac{dx}{dt} = \frac{g}{3} - \frac{ge^{-3t}}{3}$$

$$x = \int_0^2 \left(\frac{g}{3} - \frac{ge^{-3t}}{3}\right) dt$$

$$x = \left[\frac{gt}{3} + \frac{ge^{-3t}}{9}\right]_0^2$$

$$x = \frac{2g}{3} + \frac{ge^{-6}}{9} - \frac{g}{9}$$

$$x = \frac{5g}{9} + \frac{ge^{-6}}{9}$$

$$x = \frac{g}{9}(5 + e^{-6}) \text{ m}$$

28 Applying $F = ma$:

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = -mg - \frac{mv}{k}$$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -g - \frac{v}{k} = \frac{-gk - v}{k}$$

Separating the variables and integrating:

$$\int_U^0 \frac{v}{gk + v}\,\mathrm{d}v = -\int_0^X \frac{\mathrm{d}x}{k}$$

$$\int_U^0 \left(1 - \frac{gk}{gk + v}\right)\mathrm{d}v = -\int_0^X \frac{\mathrm{d}x}{k}$$

$$\left[v - gk\ln(gk + v)\right]_U^0 = -\frac{X}{k}$$

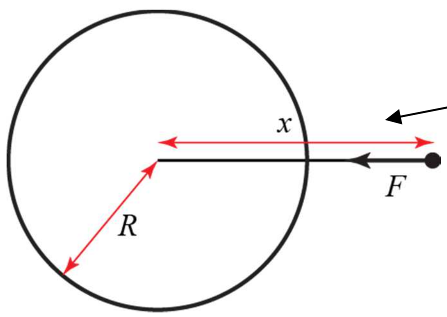
$$0 - gk\ln k - \left(U - gk\ln(gk + U)\right) = -\frac{X}{k}$$

$$\frac{X}{k} = U - gk\ln(gk + U) + gk\ln gk$$

$$\frac{X}{k} = U + gk\ln\left(\frac{gk}{gk + U}\right)$$

$$X = Uk + gk^2\ln\left(\frac{gk}{gk + U}\right)$$

29 a



The gravitational force is directed towards the centre of the Earth and so is in the direction of x decreasing.

As F is proportional to $\frac{1}{x^2}$, $F = -\frac{k}{x^2}$

At $x = R$, $F = -mg$

$-mg = -\frac{k}{R^2} \Rightarrow k = mgR^2$

Hence $F = -\frac{mgR^2}{x^2}$

You introduce a constant of proportionality k and use the fact, that the force due to gravity at the surface of the Earth is known to have magnitude mg, to find k.

The magnitude of the force is $\frac{mgR^2}{x^2}$, as required.

29 b $\mathbf{F} = m\mathbf{a}$

$$-\frac{mgR^2}{x^2} = ma$$

In this equation, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$$a = -\frac{gR^2}{x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -gR^2x^{-2}$$

$$\frac{1}{2}v^2 = -\int gR^2x^{-2}\ \mathrm{d}x = -\frac{gR^2x^{-1}}{-1} + A$$

$$v^2 = \frac{2gR^2}{x} + B, \text{where } B = 2A$$

At $x = R, v^2 = \frac{3}{2}gR$

The question gives the velocity of the particle as it is fired from the surface of the Earth. That is the velocity when $x = R$, the radius of the Earth.

$$\frac{3}{2}gR = \frac{2gR^2}{R} + B \Rightarrow B = \frac{3}{2}gR - 2gR = -\frac{1}{2}gR$$

Hence $v^2 = \frac{2gR^2}{x} - \frac{1}{2}gR$

When $x = 3R$

When the particle is at a height of $2R$ above the surface of the Earth, it is $2R + R = 3R$ from the centre of the Earth.

$$v^2 = \frac{2gR^2}{3R} - \frac{1}{2}gR = \frac{gR}{6} \Rightarrow v = \sqrt{\left(\frac{gR}{6}\right)}$$

The speed of the particle when it is $2R$ above the surface of the Earth is $\sqrt{\left(\frac{gR}{6}\right)}$.

30 a $\mathbf{F} = m\mathbf{a}$

$$-\frac{cm}{x^2} = ma$$

In the equation of motion, the force due to gravity has a negative sign as it acts in a direction which decreases the distance, x, of the particle from the centre of the Earth.

$$a = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}v^2\right) = -\frac{c}{x^2} = -cx^{-2}$$

$$\frac{1}{2}v^2 = -\int cx^{-2}\ \mathrm{d}x = \frac{c}{x} + A$$

Using $\int x^{-2}\ \mathrm{d}x = \frac{x^{-1}}{-1} + k = -\frac{1}{x} + k.$

$$v^2 = \frac{2c}{x} + B, \text{where } B = 2A$$

When $x = R, v = U$

$$U^2 = \frac{2c}{R} + B \Rightarrow B = U^2 - \frac{2c}{R}$$

Hence $v^2 = \frac{2c}{x} + U^2 - \frac{2c}{R}$

$$= U^2 + 2c\left(\frac{1}{x} - \frac{1}{R}\right), \text{ as required}$$

30 b At $x = R, v = U$ and the kinetic energy of P is $\frac{1}{2}mU^2$.

At $x = 2R$, using the result of part **a**

$$v^2 = U^2 + 2c\left(\frac{1}{2R} - \frac{1}{R}\right) = U^2 + 2c\left(-\frac{1}{2R}\right)$$

$$v^2 = U^2 - \frac{c}{R}$$

and the kinetic energy of P is $\frac{1}{2}mv^2 = \frac{1}{2}m\left(U^2 - \frac{c}{R}\right)$

(kinetic energy at $x = 2R$) $= \frac{1}{2}$(kinetic energy at $x = R$)

$$\frac{1}{2}m\left(U^2 - \frac{c}{R}\right) = \frac{1}{2}\left(\frac{1}{2}mU^2\right)$$

$$U^2 - \frac{c}{R} = \frac{1}{2}U^2$$

$$\frac{1}{2}U^2 = \frac{c}{R}$$

$$c = \frac{1}{2}RU^2$$

Divide this equation throughout by $\frac{1}{2}m$ and then make c the subject of the formula.

31 a $\mathbf{F} = m\mathbf{a}$

$$-\frac{k}{x^2} = ma \quad \textbf{(1)}$$

$$a = -\frac{k}{mx^2}$$

At $x = R, a = -g$

$$-g = \frac{k}{mR^2}$$

$$k = mgR^2$$

Substituting $k = mgR^2$ into **(1)**

$$-\frac{mgR^2}{x^2} = ma$$

$$a = v\frac{dv}{dx} = -\frac{gR^2}{x^2}$$, as required.

The force is negative in equation **(1)** as the force on P due to gravity is directed towards the centre of the Earth and that is the direction of x decreasing.

You know that the acceleration due to gravity at the surface of the Earth is g and that the direction of the acceleration is towards the centre of the Earth. Substituting $a = -g$ into **(1)** gives you k in terms of m, g and R.

$a = v\frac{dv}{dx}$ is one of the alternative forms of the acceleration.

$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = v\frac{dv}{dx}$ and you must pick out the form of a which you need in any particular question.

31 b Separating the variables in the printed answer to part **a** and integrating

$$\int v\,\mathrm{d}v = -\int \frac{gR^2}{x^2}\mathrm{d}x = -\int gR^2x^{-2}\mathrm{d}x$$

$$\frac{1}{2}v^2 = \frac{-gR^2x^{-1}}{-1} + A$$

$$v^2 = \frac{2gR^2}{x} + B, \text{where } B = 2A$$

At $x = R, v = U$

The projectile is fired from a point on the Earth's surface with speed U. This gives you that at $x = R, v = U$.

$$U^2 = \frac{2gR^2}{R} + B \Rightarrow B = U^2 - 2gR$$

Hence $v^2 = \dfrac{2gR^2}{x} + U^2 - 2gR$

When $v = 0, x = X$

$$0 = \frac{2gR^2}{X} + U^2 - 2gR$$

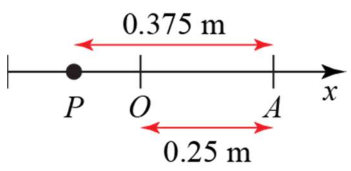
Multiply this equation throughout by X and then make X the subject of the formula.

$$0 = 2gR^2 + U^2X - 2gRX$$

$$X(2gR - U^2) = 2gR^2$$

$$X = \frac{2gR^2}{2gR - U^2}$$

32



The motion is simple harmonic with amplitude, a m, given by $a = 0.25$.

At P, $x = 0.25 - 0.375 = -0.125$

When $AP = 0.375$ m, P is 0.125 m from the centre of oscillation O. It is the other side of O from A and, if OA is taken as the direction of x increasing, the displacement of P is 0.125 m.

The Period is 2 s.

Hence $T = \dfrac{2\pi}{\omega} = 2 \Rightarrow \omega = \pi$

$$x = a\cos\omega t$$

If the time, t seconds, is measured from the time when the velocity is zero, that is when the distance of P from O is the amplitude, then $x = a\cos\omega t$ is the appropriate formula connecting the displacement with the time.

$$-0.125 = 0.25\cos\pi t$$

$$\cos\pi t = -\frac{1}{2}$$

The smallest positive value of t is given by

$$\pi t = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$t = \frac{2}{3}$$

33 a If P completes 5 oscillations in one second,

then P takes $\frac{1}{5}$ s to complete one oscillation.

$T = \frac{2\pi}{\omega} = \frac{1}{5} \Rightarrow \omega = 10\pi$

The acceleration $\ddot{x}$ m s^{-2} is given by

$\ddot{x} = -\omega^2 x$

In many other topics in Mechanics, it is usual to use the symbol a for the acceleration. With simple harmonic motion, a is often used for the amplitude and it is sensible to use another symbol. Here the calculus symbol $\ddot{x}$, for the acceleration is used.

The amplitude of the oscillation is

$\frac{0.2}{2}\text{m} = 0.1\text{m}$

The amplitude is half of the distance between the extreme points of the oscillation.

The greatest magnitude of the acceleration is given by

$|\ddot{x}| = \omega^2 a = (10\pi)^2 \times 0.1 = 10\pi^2$

The maximum magnitude of the force is given by

$|\mathbf{F}| = |m\mathbf{a}| = m\omega^2 a$

$= 0.2 \times 10\pi^2 = 2\pi^2 = 19.739...$

The greatest magnitude of the acceleration, and hence the force of greatest magnitude, occurs when the displacement is the amplitude.

The magnitude of the greatest force exerted on P is 19.7 N (3 s.f)

b $\omega = 10\pi, a = 2 \times 0.1 = 0.2$

$v^2 = \omega^2(a^2 - x^2)$

$= 100\pi^2(0.2^2 - 0.1^2) = 3\pi^2$

$v = \sqrt{3}\pi = 5.441...$

The blow is struck when P is 0.1 m from the centre of oscillation. So $x = 0.1$.

The speed of P immediately after it has been struck is 5.44 m s^{-1} (3 s.f).

34 a $T = \frac{2\pi}{\omega} = \pi \Rightarrow \omega = 2$

When $x = 0.5, v = 2.4$

$$v^2 = \omega^2(a^2 - x^2)$$
$$2.4^2 = 2^2(a^2 - 0.5^2)$$
$$a^2 - 0.25 = \frac{2.4^2}{2^2} = 1.44$$
$$a^2 = 1.69 \Rightarrow a = 1.3$$

The amplitude of the motion is 1.3 m.

b The maximum speed is given by

$v = \omega a = 2 \times 1.3 = 2.6$

The maximum speed of P during its motion is 2.6 m s^{-1}.

As $v^2 = \omega^2(a^2 - x^2)$ and x^2 is positive for all x, the greatest value of v^2 is at $x = 0$. So the greatest value of v^2 is $\omega^2 a^2$ and the greatest value of the speed is ωa.

c The maximum magnitude of the acceleration is given by

$|\ddot{x}| = |\omega^2 a| = 4 \times 1.3 = 5.2$

The maximum magnitude of the acceleration is 5.2 m s^{-2}.

The acceleration is given by $\ddot{x} = \omega^2 x$ and this has the greatest size when x is the amplitude.

d $x = a\cos\omega t$

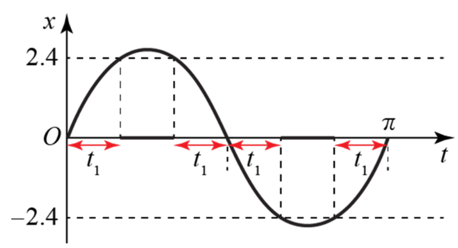
Differentiating with respect to t

$$\dot{x} = -a\omega\sin\omega t$$
$$|\dot{x}| = |a\omega\sin\omega t|$$
$$2.4 = 1.3 \times 2\sin 2t_1 = 2.6\sin 2t_1$$
$$\sin 2t_1 = \frac{12}{13}$$
$$2t_1 = \arcsin\left(\frac{12}{13}\right) = 1.176...$$
$$t_1 = 0.588...$$

The required time is given by

$$T' = \pi - 4t_1 = \pi - 4 \times 0.588...$$
$$= 0.789...$$

The time for which the speed is greater than 2.4 m is 0.79 s (2 d.p.).



This diagram illustrates that if t_1 is the smallest positive solution of $2.6\sin 2t_1 = 2.4$, the time for which the speed is greater than 2.4 is $\pi - 4t_1$.

35 a The period of motion is 6 s.

$T = \frac{2\pi}{\omega}$

$6 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{3}$

> The time taken to move from A to B is half of a complete oscillation. So the period is $2 \times 3\text{s} = 6\,\text{s}$.

Measuring the time, t seconds, from an instant when C is at Q and the displacement from the centre of the oscillation, O say

$x = a\cos\omega t$

> The formula for the displacement in terms of time when time is measured from the instant when a particle is at the amplitude is $x = a\cos\omega t$. If time is measured from the instant when a particle is at the centre of oscillation, then the formula is $x = a\sin\,\omega t$. In part **a**, it is more efficient to use the formula with the cosine but in part **c**, the formula with the sine gives you a quicker solution.

The amplitude is $2L$ and $\omega = \frac{\pi}{3}$

After 0.75 s, C is at P.

$x = 2L\cos\left(\frac{\pi}{3} \times 0.75\right) = 2L\cos\frac{\pi}{4}$

$= 2L \times \frac{1}{\sqrt{2}} = \sqrt{2}L$

$b = a - x = 2L - \sqrt{2}L$

$= (2 - \sqrt{2})L$, as required.

> From part **a**, when C is at P, its displacement from the centre of oscillation is $\sqrt{2}L$ so $x = \sqrt{2}L$.

b The speed of C at P is given by

$v^2 = \omega^2(a^2 - x^2)$

$= \left(\frac{\pi}{3}\right)^2 ((2L)^2 - (\sqrt{2}L)^2)$

$= \left(\frac{\pi}{3}\right)^2 (4L^2 - 2L^2) = 2\left(\frac{\pi}{3}\right)^2 L^2$

$v = \frac{\sqrt{2}\pi L}{3}$

c If the window is centred the displacement of Q from the centre of oscillation is given by

$x = \frac{b}{2} = \frac{3-\sqrt{2}}{2}L$

Measuring time, t seconds, from the centre of oscillation, at Q

$x = a\sin\omega t$

$\frac{2-\sqrt{2}}{2}L = 2L\sin\left(\frac{\pi}{3}t\right)$

$\sin\left(\frac{\pi}{3}t\right) = \frac{2-\sqrt{2}}{4} = 0.146\,446...$

$\frac{\pi}{3}t = 0.146\,975... \Rightarrow t = 0.140\,35...$

> In all SHM questions, it is assumed that angles are measured in radians. It is important that you make sure your calculator is in the correct mode.

The time from P to Q is given by

$T' = 2t = 0.2807...$

The time taken for C to pass from P to Q is 0.28 (2 d.p.).

36 a At C $v^2 = \omega^2(a^2 - x^2)$

$0^2 = \omega^2(a^2 - 1.2^2) \Rightarrow a = 1.2$

At A $v^2 = \omega^2(a^2 - x^2)$

$$\left(\frac{3}{10}\sqrt{3}\right)^2 = \omega^2(1.2^2 - 0.6^2)$$

$$\frac{27}{100} = \omega^2 \times 1.08$$

$$\omega^2 = \frac{27}{108} = \frac{1}{4} \Rightarrow \omega = \frac{1}{2}$$

You show that the information is the question is consistent with SHM by taking the information you have been given about two of the points and using it to find the values of a and ω. You then confirm these values are correct using the information about the third point. As the information about C gives you a directly, it is sensible to start with that point.

Checking $a = 1.2$ and $\omega = \frac{1}{2}$ at B

$$v^2 = \omega^2(a^2 - x^2)$$

$$= \frac{1}{4}(1.2^2 - 0.8^2) = 0.2 = \frac{1}{5}$$

$$v = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} = \frac{1}{5}\sqrt{5}$$

Using $a = 1.2$ and $\omega = \frac{1}{2}$, you find the speed of P at B.

This calculation confirms the speed of P given in the question and you deduce that the information is consistent with P performing simple harmonic motion.

This is consistent with the information in the question. The information is consistent with P performing SHM with centre O.

b At O, $x = 0$. Using $v^2 = \omega^2(a^2 - x^2)$

$$= \frac{1}{4}(1.2^2 - 0^2) = 0.36$$

$$v = \sqrt{0.36} = 0.6$$

The speed of P at O is 0.6 m s^{-1}, as required.

c At A $\ddot{x} = -\omega^2 x = -\frac{1}{4} \times 0.6 = -0.15$

The magnitude of the acceleration at A is 0.15 m s^{-2}.

d At A $x = a\sin\omega t$

$$0.6 = 1.2\sin\frac{1}{2}t_1 \Rightarrow \sin\frac{1}{2}t_1 = \frac{1}{2}$$

$$\frac{1}{2}t_1 = \frac{\pi}{6} \Rightarrow t_1 = \frac{\pi}{3}$$

In this question, as you need to find the difference between the times at which P is at A and B, it does not matter which of the formulae $x = a\sin\omega t$ or $x = a\cos\omega t$ you use. If you use the formula with cosine, you obtain $\frac{\pi}{3}$ s and 1.459455... s as the times. The difference between these times is again 0.412 (3 s.f).

At B $x = a\sin\omega t$

$$0.8 = 1.2\sin\frac{1}{2}t_2 \Rightarrow \sin\frac{1}{2}t_2 = \frac{2}{3}$$

$$\frac{1}{2}t_2 = 0.729\,727... \Rightarrow t_2 = 1.459\,455...$$

$$t_2 - t_1 = 1.459\,455... - \frac{\pi}{3} = 0.412\,257...$$

The time taken to move directly from A to B is 0.412 s (3 s.f).

37 a $a = \frac{10-4}{2} = 3$

The period of motion is 12 hours.

The 6 hours, from 1100 to 1700, are half of a complete oscillation.

$T = \frac{2\pi}{\omega} = 12 \Rightarrow \omega = \frac{\pi}{6}$

At 1600 hours, $t = 5$

$x = a\cos\omega t$

$= 3\cos\frac{5\pi}{6} = 3 \times \left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{2}$

$v^2 = \omega^2(a^2 - x^2)$

$= \left(\frac{\pi}{6}\right)^2\left(3^2 - \left(-\frac{3\sqrt{3}}{2}\right)^2\right) = \left(\frac{\pi}{6}\right)^2\left(9 - \frac{27}{4}\right)$

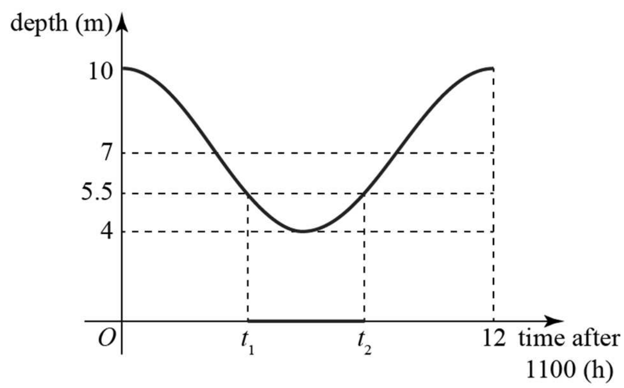
$= \left(\frac{\pi}{6}\right)^2 \times \frac{9}{4}$

$v = (-)\frac{\pi}{6} \times \frac{3}{2} = \frac{\pi}{4}$

The formulae for simple harmonic motion can be used with any consistent set of units. Here metres and hours are used.

At 1600, the water level is falling at a rate of $\frac{\pi}{4}$ m h^{-1}.

b



You find the times, here labelled t_1 and t_2, where the water is at a depth of 5.5 m. The diagram shows that the total time for which the depth of the water is less that 5.5 m is the difference between these times.

5.5 m is 1.5 m below the centre of oscillation

The centre of the oscillation is at a depth of $\frac{10-4}{2}$ m = 7m.

$x = a\cos\omega t$

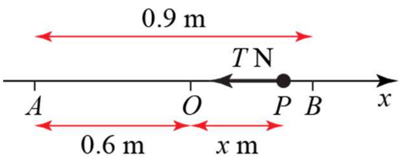
$-1.5 = 3\cos\left(\frac{\pi}{6}t\right) \Rightarrow \cos\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$

$\frac{\pi}{6}t = \frac{2\pi}{3}, \frac{4\pi}{3} \Rightarrow t = 4, 8$

The time for which the depth of water in the harbour is less than 5.5 m is

(8 – 4) hours = 4 hours.

38 a



Let the piston be modelled by the particle P.
Let O be the point where $AO = 0.6$ m
When P is at a general point in its motion, let $OP = x$ metres and the forces of the spring on P be T newtons.

Displacements in simple harmonic questions are usually measured from the centre of the motion. At the centre, the acceleration of the particle is zero and the forces on the particle are in equilibrium. In this question, the point of equilibrium, O, is where the spring is at its natural length. No horizontal forces will then be acting on the particle.

Hooke's Law

$$T = \frac{\lambda x}{l} = \frac{48x}{0.6} = 80x$$

$$R(\rightarrow)\mathbf{F} = m\mathbf{a}$$

$$-T = 0.2\ddot{x}$$

$$-80x = 0.2\ddot{x}$$

$$\ddot{x} = -400x = -20^2 x$$

When x is positive, the tension in the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

To show that P is moving with simple harmonic motion, you have to show that, at a general point of its motion, the equation of motion of P has the form $\ddot{x} = -kx$, where k is a positive constant. In this case $k = \omega^2 = 100$.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 20$. The period, T seconds, is given by

$$T = \frac{2\pi}{20} = \frac{\pi}{10} \text{ s, as required.}$$

b $a = 0.3,\ \omega = 20$

The maximum speed is given by

$$v = a\omega = 0.3 \times 20 = 6$$

The maximum speed is 6 m s^{-1}.

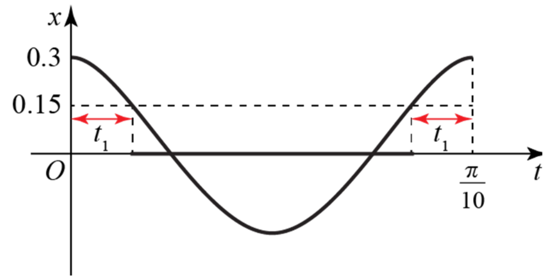
c When the length of the spring is 0.75 m

$$x = 0.75 - 0.6 = 0.15$$

$$x = a\cos\omega t$$

$$0.15 = 0.3\cos 20t_1 \Rightarrow \cos 20t_1 = \frac{1}{2}$$

$$20t_1 = \frac{\pi}{3} \Rightarrow t_1 = \frac{\pi}{60}$$



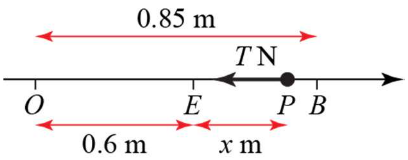
When the length of the spring is less than 0.75 m, the extension of the spring, x m, is less than 0.15 m. This sketch shows you that if the first time where the extension is 0.15 m is t_1 s, the length of time for which the extension is less than 0.15 m is $\left(\frac{\pi}{10} - 2t_1\right)$s.

The total time for which the length of the spring is less than 0.75 m is given by

$$T' = T - 2t_1 = \frac{\pi}{10} - 2 \times \frac{\pi}{60} = \frac{\pi}{15} \text{ s}$$

The length of time for which the length of the spring is less than 0.75 m is $\frac{\pi}{15}$s.

39 a



As you will use Newton's second law in this question, it is safer to use base SI units. So convert the distances in cm to m.

Let E be the point where $OE = 0.6$ m.

When P is at a general point in its motion, let $EP = x$ metres and the force of the spring on P be T newtons.

Hooke's Law

$$T = \frac{\lambda x}{l} = \frac{12x}{0.6} = 20x$$

$$\text{R}(\rightarrow) \quad \mathbf{F} = m\mathbf{a}$$

$$-T = 0.8\ddot{x}$$

$$-20x = 0.8\ddot{x}$$

$$\ddot{x} = -25x = -5^2 x$$

When x is positive, the tension is the string is acting in the direction of x decreasing, so T has a negative sign in this equation.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, this is simple harmonic motion with $\omega = 5$. The period, T seconds, is given by

$T = \frac{2\pi}{\omega} = \frac{2\pi}{5}$ s as required.

b The amplitude of the motion is 0.25 m.

The maximum magnitude of the acceleration is given by

$$|\ddot{x}| = |\omega^2 a| = 25 \times 0.25 = 6.25$$

The maximum magnitude of the acceleration is 6.25 m s^{-2}.

The acceleration is given by $\ddot{x} = -\omega^2 x$ and this has the greatest size when x is the amplitude.

c $x = a\cos\omega t$

$x = -a\omega\sin\omega t$

At $t = 2$

$\dot{x} = -0.25 \times 5\sin(5 \times 2) = -1.25\sin 10$

$= +0.680\,026\ldots$

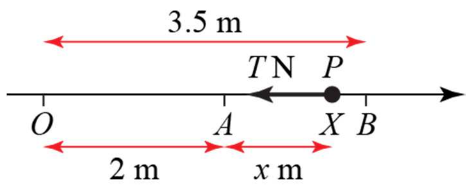
The speed of P as it passes through C is 0.68 m s^{-1} (2 s.f.).

You can drive an equation connecting velocity with time by differentiating $x = a\cos\omega t$ with respect to t. You obtain $v = \dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = -a\omega\sin\omega t$. This equation is particularly useful when you are asked about the direction of motion of a particle. As the v is squared in $v^2 = \omega^2(a^2 - x^2)$, values of v found using this formula have an ambiguous ± sign.

d As the sign of $\dot{x}$ in part **c** is positive, P is travelling in the direction of x increasing as it passes through C.

As it passes through C, P is moving away from O towards B.

40 a



When P is at the point X, where $AX = x$ m, let the tension in the spring be

Hooke's law

$$T = \frac{\lambda x}{l} = \frac{21.6 \times x}{2} = 10.8x$$

$$R(\rightarrow)\mathbf{F} = m\mathbf{a}$$

$$-T = 0.3\ddot{x}$$

$$-10.8x = 0.3\ddot{x}$$

$$\ddot{x} = -36x = -6^2 x$$

When x is positive, the tension in the spring is acting in the direction of x decreasing, so T has a negative sign in the equation of motion.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, P is performing simple harmonic motion about A with $\omega = 6$.

The period of motion Ts is given by $T = \frac{2\pi}{\omega} = \frac{2\pi}{6} = \frac{\pi}{3}$ s as required.

b At A, $x = 0$

$$v^2 = \omega^2(a^2 - x^2) = 36(1.5^2 - 0^2) = 81$$

$$v = \sqrt{81} = 9$$

The speed of P at A is 9 m s^{-1}.

c At C, $x = \frac{1.5}{2} = 0.75$

$$x = a\cos\omega t$$

$$0.75 = 1.5\cos 6t$$

$$\cos 6t = \frac{1}{2} \Rightarrow 6t = \frac{\pi}{3} \Rightarrow t = \frac{\pi}{18}$$

The time when P first reaches C is the smallest positive value of t for which this equation is true. In principle, in simple harmonic motion, P will reach this point infinitely many times.

P reaches C for the first time after $\frac{\pi}{18}$ s.

40 d Before impact, the linear momentum of P is
$m_1u = 0.3\times 9\text{ N s} = 2.7\text{ N s}$

Let the velocity of the combined particle R immediately after impact be $U\text{ m s}^{-1}$.

After impact, the linear momentum of R is
$m_2U = 0.5U\text{ N s}$

Conservation of linear momentum
$0.5U = 2.7 \Rightarrow U = 5.4$

Conservation of linear momentum is an M1 topic and is assumed, and can be tested, in any of the subsequent Mechanics modules.

For R

$R(\rightarrow)\quad -T = 0.5\ddot{x}$

$-10.8x = 0.5\ddot{x}$

$\ddot{x} = -21.6x$

When R is at X, Hooke's law gives $T = 10.8x$, exactly as in part **a**. There is no need to repeat the working in part **d**.

Comparing with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x$, after the impact R is performing simple harmonic motion about A with $\omega^2 = 21.6$.

$v = U = a\omega$

$5.4 = a\sqrt{21.6}$

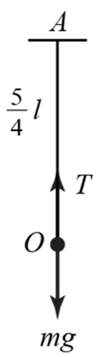
$a = \dfrac{5.4}{\sqrt{21.6}} = 1.161\ldots$

As R is performing simple harmonic motion about A, the speed of R immediately after the impact is the maximum speed of R during its motion. The maximum speed during simple harmonic motion is given by $v = a\omega$.

The amplitude of the motion is
1.16 m (3 s.f.)

No accuracy is specified in the question and the accurate answer, $\dfrac{3\sqrt{15}}{10}$ m, or any reasonable approximation would be accepted.

41 a



At the equilibrium level

$R(\uparrow)\quad T = mg \qquad$ **(1)**

Hooke's law

$T = \dfrac{\lambda e}{l} = \dfrac{\lambda \times \frac{1}{4}l}{l} = \dfrac{\lambda}{4} \qquad$ **(2)**

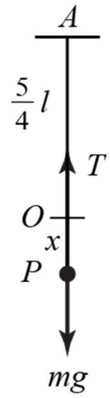
The information you have been given at the equilibrium level enables you to obtain two equations for the tension T, one using Hooke's law and a second by resolving vertically. Eliminating T between the two equations gives you an equation for the modulus of elasticity λ.

Combining **(1)** and **(2)**

$\dfrac{\lambda}{4} = mg \Rightarrow \lambda = 4mg$

The coefficient of elasticity is $4mg$.

41 b



Hooke's law

$$T = \frac{\lambda e}{l} = \frac{4mg\left(\frac{1}{4}l + x\right)}{l}$$

$$= \frac{mgl + 4mgx}{l} = mg + \frac{4mgx}{l} \quad \textbf{(3)}$$

When $AP = \frac{5}{4}l + x$, the extension is $\left(\frac{5}{4}l + x\right) - l = \frac{1}{4}l + x$.

Newton's second law

R(↓) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m\frac{d^2x}{dt^2} \quad \textbf{(4)}$$

You take the forces in the direction of x increasing. The weight tends to increase the value of x, so mg is positive. The tension tends to decrease the value of x so, in this equation, T has a negative sign.

Substituting **(3)** into **(4)**

$$mg - \left(mg + \frac{4mgx}{l}\right) = m\frac{d^2x}{dt^2}$$

$$-\frac{4mgx}{l} = m\frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} = -\frac{4gx}{l}, \text{ as required.}$$

41 c Comparing the result of part **b** with the standard formula $\ddot{x} = -\omega^2 x$, while the string is taut, P moves with SHM about O,

with $\omega^2 = \frac{4g}{l}$.

When P is released from the point where $AP = \frac{7}{4}l$, the amplitude, a, is given by

$a = AP - AO = \frac{7}{4}l - \frac{5}{4}l = \frac{1}{2}l$

At B, $x = -\frac{1}{4}l$

$v^2 = \omega^2(a^2 - x^2)$

$= \frac{4g}{l}\left(\left(\frac{1}{2}l\right)^2 - \left(-\frac{1}{4}l\right)^2\right)$

$= \frac{4g}{l}\left(\frac{1}{4}l^2 - \frac{1}{16}l^2\right) = \frac{4g}{l} \times \frac{3l^2}{16} = \frac{3gl}{4}$

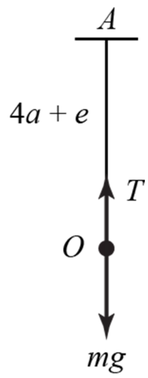
$v = \frac{1}{2}\sqrt{(3gl)}$

The speed of P at B is $\frac{1}{2}\sqrt{(3gl)}$.

> Alternately, you can use conservation of energy between the point of release and B to find the velocity at B.
>
> Loss in elastic energy = gain in kinetic energy + gain in potential energy
>
> $\frac{4mg\left(\frac{3}{4}l\right)^2}{2l} = \frac{1}{2}mv^2 + mg \times \frac{3}{4}l.$
>
> This, of course, leads to the same answer.

d First P moves freely under gravity until it returns to B.
Then it moves with simple harmonic motion about O.

42 a



> A particle attached to one end of an elastic string will oscillate about the equilibrium position. When solving problems about vertical oscillations, you often have to begin by finding the point of equilibrium. In this case, the oscillations later in the question have centre O.

At the equilibrium level, let $AO = 4a + e$, where e is the extension of the string.
Hooke's law

$T = \frac{\lambda e}{l} = \frac{8mge}{4a} = \frac{2mge}{a}$

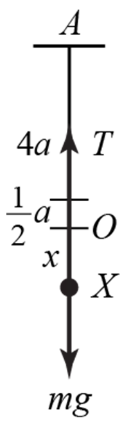
$R(\uparrow) T = mg$

Hence

$mg = \frac{2mge}{a} \Rightarrow e = \frac{a}{2}$

$AO = 4a + e = 4a + \frac{a}{2} = \frac{9a}{2}$

42 b



When P is at a general point, X say, of its motion, let $OX = x$.

To show that P is moving with simple harmonic motion, you have to show that, at a general point in its motion, the equation of motion of P has the form $\ddot{x} = -\omega^2 x$, where ω is a positive constant.

At this point, the extension of the string is $\frac{1}{2}a + x$

Hooke's law

$$T = \frac{\lambda \times \text{extension}}{\text{natural length}} = \frac{8mg\left(\frac{1}{2}a + x\right)}{4a}$$

$$= \frac{4mga + 8mgx}{4a} = mg + \frac{2mgx}{a} \quad \textbf{(1)}$$

Hooke's law and Newton's second law give you two equations from which you eliminate the tension, T.

Newton's second law

R($\downarrow$) $\mathbf{F} = m\mathbf{a}$

$$mg - T = m\ddot{x} \quad \textbf{(2)}$$

Substituting **(1)** into **(2)**

$$mg - \left(mg + \frac{2mgx}{a}\right) = m\ddot{x}$$

$$-\frac{2mgx}{a} = m\ddot{x}$$

$$\ddot{x} = -\frac{2g}{a}x$$

Comparing this equation with the standard formula for simple harmonic motion, $\ddot{x} = -\omega^2 x, P$ moves with simple harmonic motion about O and

$$\omega = \sqrt{\left(\frac{2g}{a}\right)}.$$

The period of motion T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{a}{2g}\right)} = \pi\sqrt{\left(\frac{2a}{g}\right)},$$ as required.

42 c The maximum speed is given by $v = a\omega$

$$\frac{1}{2}\sqrt{(ga)} = d\sqrt{\left(\frac{2a}{a}\right)}$$

As the particle is pulled down a distance d from the equilibrium position and released from rest, d is the amplitude of the motion.

$$\frac{1}{4}ga = d^2 \times \frac{2g}{a}$$

Squaring both sides of the previous line.

$$d^2 = \frac{1}{8}a^2$$

$$d = \frac{1}{2\sqrt{2}}a$$

d As $a > \frac{1}{2}a$, the string will become slack during its motion. The subsequent motion of P will be partly under gravity, partly simple harmonic motion.

Challenge

1 a $\frac{\mathrm{d}v}{\mathrm{d}t} = -e^{2kv}$

Separating the variables and integrating:

$$\int_u^0 e^{-2kv}\,\mathrm{d}v = -\int_0^T \mathrm{d}t$$

$$\left[-\frac{1}{2k}e^{-2kv}\right]_u^0 = -T$$

$$-\frac{1}{2k} + \frac{1}{2k}e^{-2ku} = -T$$

$$T = \frac{1}{2k} - \frac{1}{2k}e^{-2ku}$$

$$T = \frac{1 - e^{-2ku}}{2k}$$

$$T = \frac{1}{2k}\left(1 - e^{-2ku}\right)$$

$$T = \frac{1}{2k}\left(\frac{e^{2ku} - 1}{e^{2ku}}\right)$$

Challenge

1 b $v\frac{\mathrm{d}v}{\mathrm{d}x} = -e^{2kv}$

Separating the variables and integrating:

$$\int_u^0 ve^{-2kv}\mathrm{d}v = -\int_0^X \mathrm{d}x$$

$$\left[v\left(-\frac{1}{2k}e^{-2kv}\right)\right]_u^0 - \int_u^0 -\frac{1}{2k}e^{-2kv}\mathrm{d}v = -X$$

$$\left[-\frac{v}{2k}e^{-2kv} - \frac{1}{4k^2}e^{-2kv}\right]_u^0 = -X$$

$$\left(0 - \frac{1}{4k^2}\right) - \left(-\frac{u}{2k}e^{-2ku} - \frac{1}{4k^2}e^{-2ku}\right) = -X$$

$$X = \frac{1}{4k^2} - \left(\frac{U}{2k}e^{-2ku} + \frac{1}{4k^2}e^{-2ku}\right)$$

$$X = \frac{1}{4k^2} - \left(\frac{2kue^{-2ku}}{4k^2} + \frac{e^{-2ku}}{4k^2}\right)$$

$$X = \frac{1}{4k^2} - \left(\frac{2kue^{-2ku} + e^{-2ku}}{4k^2}\right)$$

$$X = \left(\frac{1}{4k^2} - \frac{1+2ku}{4k^2e^{2ku}}\right) \text{ m}$$

2 $a = 8x\frac{\mathrm{d}x}{\mathrm{d}t}$

$v\frac{\mathrm{d}v}{\mathrm{d}x} = 8xv$

$\frac{\mathrm{d}v}{\mathrm{d}x} = 8x$

$v = \int 8x\mathrm{d}x$

$v = 4x^2 + c$

When $t = 0, x = 0$ and $v = -k$, so $c = -k$

$v = 4x^2 - k$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 4x^2 - k$

The displacement x is maximum when $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$

i.e. $4x^2 - k = 0$

$x^2 = \frac{k}{4}$

$x = \frac{\sqrt{k}}{2}$

Therefore the distance of the particle from the origin never exceeds $\frac{\sqrt{k}}{2}$.

Challenge

3 Applying $F = ma : mv\dfrac{\mathrm{d}v}{\mathrm{d}x} = -mg - mgkv^2$

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -g - kgv^2$$

Separating the variables and integrating:

$$\int_U^{\frac{U}{2}} \frac{v}{g + kgv^2}\,\mathrm{d}v = -\int_0^X \mathrm{d}x$$

$$\left[\frac{1}{2gk}\ln\left(g + gkv^2\right)\right]_U^{\frac{U}{2}} = -X$$

$$\frac{1}{2gk}\ln\left(g + \frac{kgU^2}{4}\right) - \frac{1}{2gk}\ln\left(g + kgU^2\right) = -X$$

$$X = \frac{1}{2gk}\ln\left(g + kgU^2\right) - \frac{1}{2gk}\ln\left(g + \frac{kgU^2}{4}\right)$$

$$X = \frac{1}{2gk}\ln\left(\frac{g + kgU^2}{g + \frac{kgU^2}{4}}\right)$$

$$X = \frac{1}{2gk}\ln\left(\frac{4 + 4kU^2}{4 + kU^2}\right)$$